Stochastic dynamics of density fluctuations in a gravitating medium

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We examine the evolution of density inhomogeneities in a gravitating medium, taking into account the influence of surrounding inhomogeneities. It is shown that the collective stochastic gravitational field due to density fluctuations can significantly change the rate of evolution of density differences: in the spatially flat Friedmann model, $\delta \sim t^{8/3}$. An estimate is derived for the spatial correlation scale of the fluctuations.

1. We consider the cosmological problem of the origin of the observed distribution of matter. Let there be density inhomogeneities in a uniform, isotropic Friedmann model of the universe filled with a medium obeying Pascal's principle. The probability of thermal fluctuations in density[1] is

$$\omega \propto \exp(-R_{min}/kT), \tag{1}$$

where T is the temperature, and $R_{\min}$ is the minimum work required to produce the density difference δ reversibly in the gravitating medium, with $R_{\min} \sim \delta^2$ (Ref. 2). It was shown in a classic paper by Lifshitz[3] that those density differences which exceed the scale set by thermal processes in a medium are unstable. These large-scale fluctuations are nonstationary, they have an amplitude distribution different from (1), and they comprise the nuclei of a new distribution of matter, which is neither uniform nor isotropic. A description of the development of such a nucleus ought to take into account the effect of surrounding fluctuations within the gravitational interaction radius ct (horizon), where t is the cosmological time in the model. Let there be in fact two neighboring spherical inhomogeneities of mass $m_1 + \delta m_1$ and $m_2 + \delta m_2$. In the Newtonian field of the mass m_1, the fluctuations δm_1 experience an accelerating force $Gm_1\delta m_1/r^2$ directed toward the center of m_1; r is the radius of the inhomogeneity, and G is the gravitational constant. The total mass $m_1 + \delta m_1$ experiences an accelerating force in the field of the mass $m_2 + \delta m_2$:

$$G(m_1+\delta m_1)(m_2+\delta m_2)/R^2,$$

where $R > r$ is the distance between the centers of the inhomogeneities. The net accelerating force on δm_1 is (to first order in δm)

$$b\delta m_1 = -G\frac{m_1\delta m_1}{r^2} + G\frac{m_2(m_1+\delta m_1)}{R^2} + G\frac{m_1\delta m_2}{R^2}, \tag{2}$$

where b is the acceleration. In a uniform and isotropic model, the net effect of the masses m_2 is zero, so that after averaging over all m_2 the second term on the right-hand side of Eq. (2) vanishes. The first term is responsible for self-gravitation of the inhomogeneity $m_1 + \delta m_2$, and the third takes into account the change in state of motion of m_1 in the presence of δm_2. As the model expands, the number of fluctuation inside the horizon varies, and thus so does their amplitude distribution, so the net effect of the fluctuations δm_2 is not zero. This collective interaction results in large-scale density inhomogeneities evolving more or less in concord.

This effect is similar to one which is well known in the general theory of relativity,[4,5] involving local changes in a coordinate system during accelerated motion of gravitating fluctuations. If we describe the dynamics of density fluctuations relative to a uniform and isotropic coordinate system, the effect may be considered a manifestation of the collective stochastic gravitational field of the fluctuations.

We analyze below, in the spatially flat Friedmann model, the stochastic dynamics of density fluctuations on spatial scales which do not exceed the horizon. We assume that the model is filled by a medium with negligible pressure (dust). This makes it possible to ignore thermal effects (fluctuation-induced density inhomogeneities, acoustic waves), and to take the distribution (1) as a starting point.

In linear perturbation theory, the evolution of density fluctuations does not depend on the behavior of other fluctuations, and is governed by determinative equations. By inserting the appropriate Langevin sources into these equations, one can proceed beyond the scope of perturbation theory (see Ref. 6, for example), and take the collective field into consideration. The statistical characteristics of Langevin sources are given by a nonstationary amplitude fluctuation distribution function.

The existence of collective fields leads to a change in the rate of growth of density differences in a uniform and isotropic model. According to one of the linear modes derived in Ref. 3, the excess density $\delta\rho$ falls off more slowly ($\delta\rho \sim t^{-4/3}$) than the density of the expanding background, for which $\rho \sim t^{-2}$, so the density difference grows with time: $\delta \equiv \delta\rho/\rho \sim t^{2/3}$. Nearby fluctuations can slow down the rate at which $\delta\rho$ decreases, and the density difference will then grow more efficiently than in linear perturbation theory. Quantitatively, the effect is important for small fluctuations, which still participate in the overall expansion of the model, and are naturally described in an expanding frame of reference. Collective field processes which have been retarded can be neglected at the nonlinear stage of development of density perturbations.

2. We describe the development of density inhomogeneities relative to a background model with a synchronous metric:

$$ds^2 = a^2(\eta)(d\eta^2 - dx^2 - dy^2 - dz^2) = g_{ij}dx^i dx^j,$$

where the scale factor $\theta \sim \eta^2$, η is the conformal time, $\theta d\eta = cdt$, $\eta \sim t^{1/3}$, Roman indices are summed from 0 to 3, and Greek indices from 1 to 3. The motion of matter within the frame of reference is governed by the field equations

$$\varkappa T_{ij} = G_{ij}, \tag{3}$$

where $\varkappa$ is the Einstein gravitational constant, T_{ij} is the ener-

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gy-momentum tensor of the medium, and G_{ij} is the Einstein tensor.

Consider two nearby fluctuations, $\delta G_{ij}(\eta, x^\alpha - \bar{x}^\alpha)$ and $\delta \tilde{G}_{ij}(\eta, x^\alpha - \tilde{x}^\alpha)$, in the neighborhood of the points $\bar{x}^\alpha$ and $\tilde{x}^\alpha$ respectively. The superposition principle holds for small perturbations, so that in the reference frame of (3) the stochastic dynamics of density fluctuations in the vicinity of the coordinate origin $\bar{x}^\alpha$ are given by

$$\varkappa[\delta T_{ij}(\eta, x^\alpha)+\delta \tilde{T}_{ij}(\eta, x^\alpha)]=\delta G_{ij}(\eta, x^\alpha)+\delta \tilde{G}_{ij}(\eta, x^\alpha-\tilde{x}^\alpha), \tag{4}$$

where $\delta T_{ij}(\eta, x^\alpha)$ is the fluctuation corresponding to $\delta G_{ij}(\eta, x^\alpha)$ in the energy-momentum tensor of the medium, and $\delta \tilde{T}_{ij}(\eta, x^\alpha)$ is the increment to $\delta T_{ij}(\eta, x^\alpha)$, corresponding to the value of the tensor $\delta \tilde{G}_{ij}(\eta, x^\alpha - \tilde{x}^\alpha)$ in the neighborhood of $\bar{x}^\alpha = 0$. The second terms on both sides of Eq. (4) play the role of Langevin sources.

For a dusty medium, Eqs. (4) break up into two groups of equations. The first defines the connection of the density differences and of the velocity perturbations with the scalar perturbations of the metric [see Eqs. (12) and (13) below].

The subject of our analysis is the second group of equations, which describes the dynamics of scalar perturbations of the metric. Following the development in Ref. 3, we represent scalar perturbations as a set of harmonic oscillators $\exp(in_\alpha x^\alpha)$ with amplitudes

$$(\mu_n, \lambda_n)_\beta{}^\alpha = {}^1/_3 \mu_n \delta_\beta{}^\alpha + \left({}^1/_3 \delta_\beta{}^\alpha - \frac{n^\alpha n_\beta}{n^2}\right)\lambda_n,$$

where n^α is a wave vector in the space $\{x^\alpha\}$, and $n^2 = n^\alpha n_\alpha$. We then obtain from the equations (4) for the isotropic and anisotropic amplitudes μ_n and λ_n

$$\begin{aligned} &\ddot{\mu}_n + \frac{4}{\eta}\dot{\mu}_n + \frac{n^2}{3}(\mu_n+\lambda_n) \\ &= -\exp(-in_\alpha \tilde{x}^\alpha)\left[\ddot{\tilde{\mu}}_n + \frac{4}{\eta}\dot{\tilde{\mu}}_n + \frac{n^2}{3}(\tilde{\mu}_n+\tilde{\lambda}_n)\right], \\ &\ddot{\lambda}_n + \frac{4}{\eta}\dot{\lambda}_n - \frac{n^2}{3}(\mu_n+\lambda_n) \\ &= -\exp(-in_\alpha \tilde{x}^\alpha)\left[\ddot{\tilde{\lambda}}_n + \frac{4}{\eta}\dot{\tilde{\lambda}}_n - \frac{n^2}{3}(\tilde{\mu}_n+\tilde{\lambda}_n)\right], \end{aligned} \tag{5}$$

where the dots signify differentiation with respect to η, and the amplitudes $\tilde{\mu}_n$ and $\tilde{\lambda}_n$ pertain to the neighboring oscillator $\exp[in_\alpha(x^\alpha - \tilde{x}^\alpha)]$ localized in the vicinity of the point $\tilde{x}^\alpha$.

In the orthogonal space of the variables $\{\mu_n, \lambda_n\}$, we introduce the distribution function $f(\mu_n, \lambda_n, \tilde{\mu}_n, \tilde{\lambda}_n, \tilde{x}^\alpha, \eta)$ for the amplitudes μ_n and λ_n (using the approach developed in Ref. 7), under the condition that there is a neighboring oscillator with amplitudes $(\tilde{\mu}_n, \tilde{\lambda}_n)$. The continuity equation for f is

$$\dot{f} + \frac{\partial}{\partial \mu_n}(\dot{\mu}_n f) + \frac{\partial}{\partial \lambda_n}(\dot{\lambda}_n f) = 0. \tag{6}$$

Let $W(\tilde{x}^\alpha, \eta)$ be the probability that the oscillator (μ_n, λ_n) has a neighbor $(\tilde{\mu}_n, \tilde{\lambda}_n)$. From the formula for the total probability, we find the equation for the amplitude distribution function,

$$F(\mu_n, \lambda_n, \eta) = \int f(\mu_n, \lambda_n, |\tilde{\mu}_n, \tilde{\lambda}_n, \tilde{x}^\alpha, \eta) \times F(\tilde{\mu}_n, \tilde{\lambda}_n, \eta) W(\tilde{x}^\alpha, \eta)\, d\tilde{\mu}_n\, d\tilde{\lambda}_n\, d^3\tilde{x}, \tag{7}$$

where the integration over $\tilde{x}^\alpha$ is carried out in the flat space, and the integrations over $\tilde{\mu}_n$ and $\tilde{\lambda}_n$ span all allowable values: $|\mu_n| \leqslant \mu^* \ll 1$, $|\lambda_n| \leqslant \lambda^* \ll 1$. We derive the equation for W in the following way. On the one hand, the number of oscillators is $N = 1 + \int W d^3\tilde{x}$, and on the other, $N = \int F d\mu_n d\lambda_n$, so the equation for W takes the form

$$1 + \int W\, d^3\tilde{x} = \int F\, d\mu_n\, d\lambda_n. \tag{8}$$

Equations (4)–(8) describe the self-consistent problem of the stochastic dynamics of density fluctuations at spatial scales within the event horizon. At larger scales, fluctuations are independent of one another, and evolve in accordance with the Lifshitz equations.[3]

At the initial time η_0, let the right-hand sides in (5) equal zero; for the physical modes, then

$$\begin{aligned} \lambda_n &= C_1(1+{}^2/_{15}n^2\eta^2) - C_2/\eta^3 + (\Phi_1-\Phi_2)\exp(-in_\alpha\tilde{x}^\alpha), \\ \mu_n &= C_1(1-{}^2/_{15}n^2\eta^2) + C_2/\eta^3 + (\Phi_1+\Phi_2)\exp(-in_\alpha\tilde{x}^\alpha), \end{aligned} \tag{9}$$

where $2\Phi_1 = \tilde{\mu}_n + \tilde{\lambda}_n$, and

$$2\Phi_2 = \tilde{\mu}_n - \tilde{\lambda}_n - {}^2/_3 \int (n^2\eta^4 - n^2\eta^3 + 9\eta^2 - 9\eta)\Phi_1\, d\eta \cdot - {}^4/_3 \int\left[\int \eta^4(2n^2\eta^2+9)\Phi_1\, d\eta\right] d\eta/\eta^4,$$

and the constants C_1 and C_2 are determined by the initial conditions of the problem: $\mu_n(\eta_0) = \mu^0, \lambda_n(\eta_0) = \lambda^0$.

As an example, let us consider an initially Gaussian amplitude distribution[8] for μ^0 and λ^0,

$$f^0 = \frac{1}{2\pi\Delta^{1/2}}\exp\left\{-\frac{(\mu_0-\bar{\mu})^2}{2\Delta_\mu} - \frac{(\lambda^0-\bar{\lambda})^2}{2\Delta_\lambda}\right\},$$

where $\bar{\mu}$ and $\bar{\lambda}$ are mean values, Δ_μ and Δ_λ are variances, and $\Delta = \Delta_\mu \Delta_\lambda$. From (9) we find μ^0 and λ^0 as functions of $\mu_n, \lambda_n, \tilde{\mu}_n, \tilde{\lambda}_n, \tilde{x}^\alpha$, and η, and then, using (6)–(8), we find f and F. The dynamics of the most likely fluctuations are described by the extremum equations, $\partial F/\partial \mu_n = 0$ and $\partial F/\partial \lambda_n = 0$. We can then evaluate the behavior of F near a maximum. The solution of Eq. (7) is

$$F(\mu_n, \lambda_n, \eta) = F_0 \exp\left\{\int fW\, d\tilde{\mu}_n\, d\tilde{\lambda}_n\, d^3\tilde{x}\right\},$$

where F_0 is a normalizing constant. We represent f as a power series in $\tilde{\mu}_n$ and $\tilde{\lambda}_n$, keeping terms to first order. Carrying out the integration, we find

$$\int fW\, d\tilde{\mu}_n\, d\tilde{\lambda}_n\, d^3\tilde{x} \approx \frac{\mu^*\lambda^*}{\pi\Delta^{1/2}}\left[1 + \frac{\gamma}{2}\left(\xi_1 + \frac{2-\gamma}{\gamma}\xi_2\right)\omega_1 + \frac{\alpha-\beta}{2}(\xi_1-\xi_2)\omega_2\right]\exp\left\{-\frac{\Delta_\mu}{2}\xi_1{}^2 - \frac{\Delta_\lambda}{2}\xi_2{}^2\right\} \times \int \exp(-in_\alpha\tilde{x}^\alpha) W\, d^3\tilde{x};$$

the expressions for $\alpha, \beta, \gamma, \xi_1, \xi_2, \omega_1$, and ω_2 are written out in the Appendix. Let F be normalized such that $2F_0\mu^*\lambda^* = 1$; Eq. (8) then takes the form

$$\zeta \int W\, d^3\tilde{x} = \int \exp(-in_\alpha \tilde{x}^\alpha)\, W\, d^3\tilde{x}, \qquad (10)$$

where

$$\zeta = (\alpha-\beta-2\mu^{\cdot}\lambda^{\cdot}) \times \left\{ \frac{\mu^{\cdot}\lambda^{\cdot}}{(2\pi)^{1/2}} \left[\frac{(\alpha+\beta)\,\omega_1+(\alpha-\beta)\,\omega_2}{\Delta_\mu^{1/2}} + \frac{(2-\gamma)\,\omega_1-(\alpha-\beta)\,\omega_2}{\Delta_\lambda^{1/2}} \right] \right\}^{-1}.$$

It has been assumed here that the integral on the right-hand side of Eq. (8) is evaluated near a fairly sharp maximum of the integrand. Making use of (10), we find for the most likely fluctuations

$$\mu_n = \left(\frac{1-\beta}{\alpha-\beta}\,\bar{\mu} - \frac{\beta}{\alpha-\beta}\,\bar{\lambda} \right) - \frac{1}{4}\,\zeta\,(\omega_1+\omega_2),$$
$$\lambda_n = \left(-\frac{1-\alpha}{\alpha-\beta}\,\bar{\mu} + \frac{\alpha}{\alpha-\beta}\,\bar{\lambda} \right) - \frac{1}{4}\,\zeta\,(\omega_1-\omega_2). \qquad (11)$$

The first terms on the right-hand sides in (11) correspond to linear perturbation theory.[3] The second terms are related to the collective effect already described, and if $\Delta_\mu \gg \Delta_\lambda$ (or $\Delta_\lambda \gg \Delta_\mu$), they depend only weakly on the time η.

The amplitude of the density difference δn is

$$a^2 \varkappa \rho c^2 \delta_n = \frac{2}{\eta}\,\dot{\mu}_n - \frac{n^2}{3}\,(\mu_n+\lambda_n), \qquad (12)$$

and the amplitude of the velocity perturbation δu_α is

$$3a^2 \varkappa \rho c^2 (\delta u_\alpha / a)_n = -n_\alpha (\dot{\mu}_n + \dot{\lambda}_n). \qquad (13)$$

We now evalute the spatial correlation of the density fluctuations. We extend the integration in (10) over an infinite domain, bearing in mind that $W=0$ beyond the event horizon. The integral equation thus derived is satisfied, for example, by a delta function: $W \sim \delta(\tilde{x}^\alpha - b^\alpha)$, where $b^\alpha = c^\alpha + 10^\alpha$ is a complex vector, with $n^\alpha C_\alpha = 0$ and $|c| = |d|$. We can obtain an estimate for $|d|$ from (10)—it is the spatial correlation scale length of the oscillators: $n_\alpha d^\alpha = \ln \zeta$.

If we consider the case $\Delta_\mu = \Delta_\lambda$, the expression for ζ is simplified:

$$\zeta = \frac{(2\pi\Delta_\mu)^{1/2} (\alpha-\beta-2\mu^{\cdot}\lambda^{\cdot})}{2\mu^{\cdot}\lambda^{\cdot}(\mu^{\cdot}+\lambda^{\cdot})}.$$

In that event, transforming from η to t, we find for the most likely density perturbations

$$a_0^2 \varkappa \rho_0 c^2 \delta_n = -\frac{3}{5} n^2 (\bar{\mu}+\bar{\lambda}) \left(\frac{t}{t_0}\right)^{2/3} + 3\left[\left(1 + \frac{2}{15} n^2 \eta_0^2\right) \bar{\mu} + \left(1 - \frac{2}{15} n^2 \eta_0^2\right) \bar{\lambda} \right] \left(\frac{t_0}{t}\right) + \frac{1}{4\mu^{\cdot}\lambda^{\cdot}} \left(\frac{\pi\Delta_\mu}{2}\right)^{1/2} \left(\frac{t}{t_0}\right)$$
$$\times \left[\frac{6\mu^{\cdot}}{\mu^{\cdot}+\lambda^{\cdot}} - \frac{8}{15} n^2 \left(\frac{t}{t_0}\right)^{4/3} - \frac{7}{4} n^2 \left(\frac{t}{t_0}\right)^{4/3} - 6\left(\frac{t}{t_0}\right) + \frac{9}{2}\left(\frac{t}{t_0}\right)^{2/3} \right]$$
$$- \frac{1}{4\mu^{\cdot}\lambda^{\cdot}} \left(\frac{\pi\Delta_\mu}{2}\right)^{1/2} \frac{n^2}{3} \left(\frac{t}{t_0}\right)^{5/3},$$

where $a_0 = a(\eta_0)$ and $\rho_0 = \rho(\eta_0)$. Thus, the most likely differences grow most efficiently, with $\delta_n \sim t^{8/3}$. The spatial correlation scale length for fluctuations also increases with time.

$$n_\alpha d^\alpha = \ln\left[\left(\frac{\pi\Delta_\mu}{2}\right)^{1/2} \frac{(t/t_0)^3 - 2\mu^{\cdot}\lambda^{\cdot}}{\mu^{\cdot}\lambda^{\cdot}(\mu^{\cdot}+\lambda^{\cdot})} \right].$$

APPENDIX

In the main section of the paper, we make use of the following functions:

$$\alpha = \frac{1}{2}\left(\frac{\eta}{\eta_0}\right)^3 \left[1 + \frac{2}{15} n^2\eta^2 + \left(\frac{\eta_0}{\eta}\right)^3 \left(1 - \frac{2}{15} n^2 \eta_0^2\right) \right],$$

$$\beta = \alpha - \left(\frac{\eta}{\eta_0}\right)^3,$$

$$\gamma = \alpha+\beta, \qquad \xi_1 = \frac{\alpha\mu_n + \beta\lambda_n - \bar{\mu}}{\Delta_\mu},$$

$$\xi_2 = \frac{(1-\alpha)\,\mu_n + (1-\beta)\,\lambda_n - \bar{\lambda}}{\Delta_\lambda}$$

$$\omega_1 = \mu^{\cdot} + \lambda^{\cdot},$$

$$\omega_2 = \mu^{\cdot} - \lambda^{\cdot} - (\mu^{\cdot}+\lambda^{\cdot}) \left(\frac{n^2\eta^5}{15} + \frac{n^2\eta^4}{4} + \eta^3 - \frac{9}{10}\eta^2 \right).$$

[1] L. D. Landau and E. M. Lifshitz, Statistical Physics, Part 1, Nauka, Moscow, 1976, §112 [Pergamon, 1980].
[2] I. K. Rozgacheva, Astron. Zh. **61**, 654 (1984) [Sov. Astron. **28**, 384 (1984)].
[3] E. M. Lifshitz, Zh. Eksp. Teor. Fiz. **16**, 587 (1946).
[4] A. Einstein, Collected Scientific Works, Vol. 2, Nauka, Moscow, 1966, p. 75.
[5] Yu. S. Vladimirov, Frames of Reference in Gravitation Theory, Energoizdat, Moscow, 1982, Ch. 5.
[6] A. A. Migdal, Usp. Fiz. Nauk **149**, 3 (1986) [Sov.Phys. Usp. **29**, 389 (1986)].
[7] Yu. L. Klimontovich, Kinetic Theory of Electromagnetic Processes, Nauka, Moscow, 1980, §9.
[8] A. A. Starobinskiĭ, Pis'ma Astron. Zh. **9**, 579 (1983) [Sov. Astron. Lett. **9**, 302 (1983)].

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