Analytic results for radiative-recoil corrections in muonium

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We have calculated analytically the radiative-recoil correction to the hyperfine splitting in the ground state of muonium due to radiative corrections to the electron line. Its magnitude in units of the hyperfine splitting energy E_F is $(\alpha(Z\alpha)/\pi^2)\cdot(m/M)[6\zeta(3)+3\pi^2\ln 2+\pi^2/2+17/8]E_F$ or numerically 4.05 kHz.

1. INTRODUCTION

At the present time the hyperfine splitting in the ground state of muonium has been measured with a relative error $3.6\cdot 10^{-8}$,[1] and further improvement of the experimental results is planned.[2] The theoretical calculations of this quantity, which have a history of many years and which have been closely related to the principal steps in the development of quantum mechanics and quantum electrodynamics, still have not reached such accuracy. At the same time, since muonium is a purely electrodynamical system, there are no fundamental difficulties in the path of increasing the accuracy of the theory of muonium.

The main contribution to the hyperfine splitting in a hydrogen-like ion, which corresponds to the nonrelativistic interaction of the two magnetic moments, was calculated by Fermi[3] and is[1)]

$$\nu_F=\frac{16}{3}(Z\alpha)^2R_\infty c\frac{\mu}{\mu_B}\left(1+\frac{m}{M}\right)^{-3}. \quad (1)$$

Here $R_\infty = mc\alpha^2/2h$ is the Rydberg constant, c is the velocity of light, m is the electron mass, h is Planck's constant, μ_B is the Bohr magneton, Z is the charge of the nucleus in units of the electron charge ($Z=1$ for the proton and the muon), M is the mass of the nucleus and μ is the magnetic moment of the nucleus.

It is convenient to classify the corrections to the Fermi splitting energy (1) according to their dependence on the mass ratio of the electron and the nucleus. Corrections of the first type do not depend at all on the mass ratio and are calculated in the external-field approximation. Examples of such contributions are the Breit correction[4] of order $(Z\alpha)^2$ which is due to the relativistic behavior of the electron, and the radiative correction of first order (contribution $\alpha(Z\alpha)$ to the hyperfine splitting) in the external field. Calculation of this contribution[5,6] was one of the first impressive achievements of the renormalization technique in quantum electrodynamics. At the present time corrections of order $\alpha(Z\alpha)^2$ which do not depend on the mass ratio have also been calculated (see for example the review of Ref. 7).

Corrections of the second type are linear in the mass ratio m/M, i.e., they explicitly take into account recoil effects. The relativistic two-particle equations are used for calculation of these corrections. The early studies made use of the Bethe-Salpeter (BS) equation,[8] by means of which the leading correction for recoil of order $(Z\alpha)(m/M)$ was obtained.[9,10] Investigation of other corrections has turned out to be considerably more laborious and has led to development of various modifications of the BS equation which are more convenient for applications (for more detail consult Section 2). A specially developed two-particle formalism has recently been used to calculate the contribution of order $(Z\alpha)^2(m/M)$.[11,7] Radiative corrections to recoil of order $\alpha(Z\alpha)(m/M)$ were not known until recently. In this order there are contributions proportional to the square and first power of the logarithm of the mass ratio, and also a term which does not depend on the mass ratio. All of these terms, which are related to vacuum polarization, were obtained in Ref. 12. In regard to radiative corrections to the electron line, only the coefficient of the logarithm of the mass ratio has previously been calculated analytically.[12] The present work is devoted to analytical calculation of the term which does not depend on the mass ratio. (The results of our work have been published in the form of letters.[13,14])

The article is organized as follows. In the second section we describe briefly the two-particle formalism version used by us. The third section is devoted to selection of diagrams which lead to the contribution of order $\alpha(Z\alpha)E_F$ to the hyperfine splitting. Here we also discuss questions related to the choice of gauge and the problem of infrared and ultraviolet divergences. In this section it is found in what mode it is necessary to calculate the diagrams which contribute to the hyperfine splitting. In the fourth section for the case of diagrams with a self-energy correction we have illustrated the basic procedures with which we have been able to obtain an answer in closed form. Here we also give the results of calculation of the remaining diagrams which give a contribution of order $\alpha(Z\alpha)E_F$. The last section contains the principal result of the present work—the radiative correction to recoil of order $\alpha(Z\alpha)(m/M)E_F$. In this section we also discuss prospects for further improvement of the value of the hyperfine splitting in the ground state of muonium and for improvement of the value of the fine-structure constant.

2. THE EFFECTIVE DIRAC EQUATION

The BS equation has a number of shortcomings which complicate its use in exact calculations: the existence of a nonphysical variable of the relative energy (or relative time) in the BS wave function, which hinders its physical interpretation; irreducibility of the ladder approximation to the Dirac equation if one of the particles is very heavy; absence of an exactly soluble zero approximation, and so forth. While the first shortcoming reflects to a certain degree the multiparticle nature of bound states in quantum field theory, the remaining result primarily from the nonphysical nature of the diagram classification used in the BS equation by means of the concept of two-particle irreducibility. Actual-

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ly, the contributions of ladder graphs and graphs with crossing photons have a tendency to cancel, while the means of dealing with these graphs in terms of the BS equation are completely different. It is therefore natural to reconstruct the BS equation in such a way that these graphs enter more equally and to attempt to avoid simultaneously all of the enumerated shortcomings. We shall follow the procedure of Gross,[15] who proposed to select the free two-particle propagator in a form corresponding to propagation of the heavy particle on the mass shell. This choice of the bare propagator corresponds to the physics of the problem, since it is clear that in a weakly bound system the heavy particle is mainly near the mass shell. Specification of the free propagator determines all the structural blocks of the two-particle equation. The corresponding formulas have been obtained in Refs. 16 and 17, and we shall not give them here. We note only that a three-dimensional equation is obtained for the wave function, which is a four-component spinor in the electron index and two-component spinor in the muon index. This equation is close in form to the ordinary single-particle Dirac equation, the wave function in it depends only on the three-dimensional momentum, and we shall call it the effective Dirac equation (EDE). The kernel of the EDE is expressed in terms of the kernel of the BS equation by the formula

$$K=K_{BS}+K_{BS}(S_0-S)K_{BS}+K_{BS}(S_0-S)K_{BS}(S_0-S)K_{BS}+\dots, \tag{2}$$

where K and $K_{\rm BS}$ are respectively the kernels of the EDE and the BS equation, and the difference of the free two-particle propagator and the Gross propagator is

$$S_0-S=\frac{i}{\hat{p}-M}\frac{i}{\hat{E}-\hat{p}-m}-2\pi i\delta^{(+)}(p^2-M^2)\frac{\hat{p}+M}{\hat{E}-\hat{p}-m}. \tag{3}$$

In this difference the factors which contain the electron mass m act on the electron indices, while those in the muon mass M act on the muon indices. The EDE and the series for its kernel (2) are shown graphically in Figs. 1(a) and 1(b), where the "small rectangle" denotes the difference propagator (3).

We choose the zeroth kernel approximation K_0, which includes Coulomb exchange, in a form which permits exact solution.[17] The obtained energy-level values correctly describe the fine structure (with inclusion of the reduced mass), and the wave functions can be approximated with

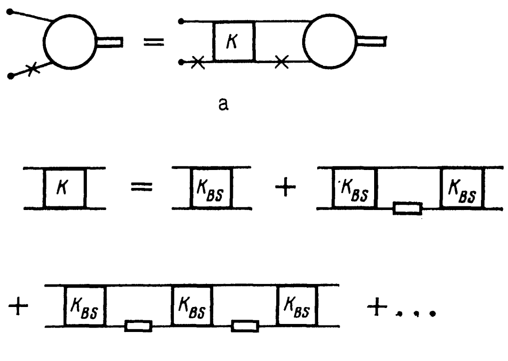

FIG. 1. Effective Dirac equation for bound states (a) and the series for its kernel (b) in terms of the Bethe-Salpeter equation.

accuracy sufficient for our purposes by the product of the Coulomb functions for the reduced-mass Schrödinger equation and the free electron spinor. In the limit of infinite mass of the heavy particle the EDE, in contrast to the BS equation, goes over already in the zeroth approximation to the Dirac equation in an external field. As a consequence of the dependence of the perturbation $\delta K=K-K_0$ on energy, the perturbation-theory formulas contain in addition to the usual terms the derivatives of the perturbation with respect to energy (in what follows derivatives are indicated by primes):

$$E_n=E_n{}^0+(n|i\delta K(E_n{}^0)|n)\times[1+(n|i\delta K'(E_n{}^0)|n)]+(n|i\delta K(E_n{}^0)\times G_{n0}(E_n{}^0)i\delta K(E_n{}^0)|n)[1+(n|i\delta K'(E_n{}^0)|n)]+\dots. \tag{4}$$

Here $G_{n0}(E)$ is the Green function of the EDE with the bare kernel K_0, in which we have carried out the subtraction

$$G_{n0}(E)=G_0(E)-|n)(n|/(E-E_n{}^0).$$

3. DIAGRAMS GIVING CONTRIBUTIONS OF ORDER $\alpha(Z\alpha)E_F$

The main contribution (1) to the hyperfine splitting is from the part of the one-photon exchange entering into the perturbation δK, which depends explicitly on the muon spin. Of course, in calculation of this contribution we obtain as the factor μ_B in (1) the Dirac value of the muon magnetic moment. To obtain the total value of the magnetic moment it is necessary to take into account the vertex correction to the muon line. As we shall see, the contribution of order $\alpha(Z\alpha)(m/M)E_F$ is due to integration over the large virtual momenta, when the vertex correction to the muon line already does not reduce to an anomalous moment. Therefore all corrections below are calculated in units of ν_F (see Eq. (1)), where μ is taken to be the Dirac value of the magnetic moment. The contribution of radiative corrections to the muon line must be calculated individually.

In this study we shall investigate contributions of order $\alpha(Z\alpha)E_F$ associated with radiative corrections to the electron line. In analysis of these corrections it is convenient to use the Fried-Yennie gauge (FY) gauge[18] for the radiative (attached at both ends to the electron line) photon. This gauge is distinguished by the fact that in it infrared divergences are softened; for example, there is no Abrikosov logarithm[19] in the infrared asymptote of the electron propagator. In the FY gauge it is not necessary to introduce the infrared mass of the photon in the usual subtraction procedure on the mass shell. The exchange photons (fastened at one end to the electron line and at the other end to the muon line) are taken in the Coulomb gauge, since it is well known that any other gauge for the exchange photons leads to fictitious contributions to the energy of bound states. The possibility of using different gauges simultaneously for the radiative and exchange photons is due to the Abelian nature of quantum electrodynamics. In non-Abelian theory vector particles carry charge, and therefore all of their propagators must be chosen in the same gauge. In the FY gauge the subdiagram with the radiative correction has a softer behavior at low momenta than does the corresponding skeleton block. This

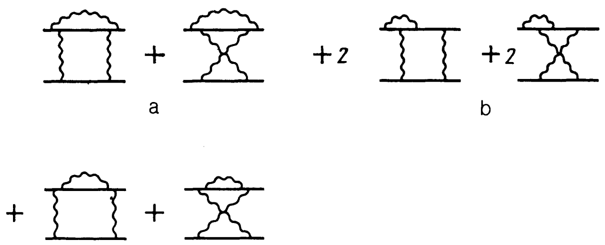

FIG. 2. Complete gauge-invariant set of diagrams contributing to radiative corrections to the recoil: (a)—diagrams with two exchange photons, (b)—diagrams with a renormalized vertex operator, (c)—diagrams with a renormalized self-energy operator.

greatly facilitates the analysis of the possible contributions and is the main reason for use of the FY gauge.

Even in the FY gauge the analysis of all diagrams with radiative corrections to the electron line entering into the kernel of the EDE (2) and into the perturbation-theory formula (4) remains a very awkward problem. We shall give here only the results of this treatment. All diagrams with radiative corrections are conveniently classified depending on how many exchange photons the radiative photon spans. It can be shown that diagrams in which the radiative photon spans more than two exchange photons do not lead at all in the FY gauge to contributions of order $\alpha(Z\alpha)E_F$. (In any other gauge these diagrams have even a still larger contribution αE_F.) The contribution of all diagrams spanning two exchange photons reduces to the diagrams of Fig. 2(a). The matrix element of these diagrams must be calculated between the large components of the electron and muon spinors, neglecting the momenta of the wave functions inside the diagrams. That is, in calculation of the loop integrals it is assumed that the external momenta are on the mass shell, and that their spatial components are equal to zero. The contribution of individual diagrams to the energy reduces to the product of the matrix element calculated in this way by the square, at zero, of the Coulomb wave function of the Schrödinger equation with reduced mass. The contributions of the diagrams encountered in the following must also be calculated under these, as we shall call them, standard conditions (SC).

Analysis of diagrams with a vertex correction is complicated by ultraviolet divergences. Accurate investigation with use of gauge-invariant ultraviolet regularization (for example, by introduction of a heavy photon) shows that as a consequence of the Ward identity the divergent parts of the vertex and self-energy corrections cancel. This cancellation occurs not in the kernel of the EDE (2), but only in the expression for the energy (4). Next we shall deal only with the finite part of the vertex, and then with the self-energy corrections. The contribution of the anomalous magnetic moment we shall also consider separately. This is due to the fact that the anomalous magnetic moment behaves in a different way than the remaining form factors, which in the FY gauge fall off rapidly on approach to zero of the momentum transfer and of the virtualities of the electron ends. It is possible to show by explicit calculation that the anomalous moment does not lead to corrections of order $\alpha(Z\alpha)E_F$. We shall assume further that the anomalous moment has been subtracted from the vertex. A detailed analysis shows that the contribution of all diagrams with a vertex correction reduces to the sum of the diagrams of Fig. 2(b), which must be calculated under the standard conditions. Note that in constructing the kernel of the EDE (2) and the perturbation theory for the energy levels (4) a nontrivial cancellation of the subtraction diagrams with the vertex correction has occurred.

The investigation of diagrams with a self-energy correction is distinguished by a number of features, as a result of the fact that the simplest kernel of the BS equation with this correction contains the inverse muon propagator. No other kernel contains such an object, which plays a specific role in perturbation theory. We note also that terms with derivatives of the kernel in Eq. (4) act just for diagrams with a divergent part of the self-energy correction and provide cancellation of ultraviolet divergences, which was mentioned in discussion of vertex corrections. The subtractive diagrams with a self-energy insert cancel, as in the case of diagrams with a vertex correction. The entire contribution of diagrams with a self-energy correction reduces therefore to the sum of the diagrams of Fig. 2(c), which must be calculated under the standard conditions.

Therefore in calculation of the contribution of radiative corrections to the electron line it is sufficient to calculate the sum of the diagrams of Fig. 2 under the standard conditions. These diagrams form a complete gauge-invariant set, and the standard conditions mean that all outer ends are on the mass shell. Therefore we can by means of a Feynman trick go over to a covariant gauge in the exchange photons. It is possible also to go over to a Feynman gauge for the radiative photons, since in the diagrams of Fig. 2 there are a renormalized mass operator and a renormalized vertex. This transformation permits us to use the well known expression for a renormalized vertex with one virtual electron end.[20]

As we have already mentioned, the sum of the diagrams of Fig. 2 does not contain infrared divergences. The individual diagrams in the FY gauge also do not contain them. However, in the Feynman gauge the individual diagrams diverge in the infrared and we regularize them, introducing a mass of the radiative photon. The cancellation of these divergences in the sum of the diagrams can be understood also without reference to the FY gauge. Indeed, the diagrams of Fig. 2 describe radiative corrections to forward scattering of an electron by a muon at threshold. Ordinarily in the scattering cross section the infrared divergences due to emission of virtual photons cancel the divergences from emission of real bremsstrahlung photons. In forward scattering there is no bremsstrahlung, and therefore there are no infrared divergences associated with the radiative corrections. In regard to the divergences at threshold, which exist already in the sum of the two-photon graphs without radiative corrections, the addition of these corrections to the electron line removes the divergence, since the corrections are weak functions of the loop momentum.

We note that in any gauge except the FY gauge there are many other diagrams in addition to those shown in Fig. 2 which lead to corrections of order $\alpha(Z\alpha)$ to the hyperfine splitting. The contributions of these diagrams, as the analysis given above shows, cancel, and for calculation of the total contribution of radiative corrections to the electron line it is sufficient to calculate the matrix elements of the diagrams of Fig. 2 in an arbitrary gauge under the standard conditions.

4. ANALYTICAL CALCULATION OF THE DIAGRAMS

The simplest diagrams from the calculational point of view are those of Fig. 2(c), which contain a self-energy correction. We shall describe below the main steps of the calculations, devoting particular attention to those procedures which have enabled us to obtain an answer in closed form.

Substitution of the known expression[20] for the self-energy operator into the matrix element of the diagrams of Fig. 2(c) under the standard conditions leads to the following contribution to the hyperfine splitting:

$$\delta E_\Sigma=\frac{\alpha(Z\alpha)}{\pi^2}\frac{m}{M}E_F\frac{3i}{8\pi^2\mu^2}$$

$$\times\int_0^1 dx\int_0^x dy\int\frac{d^4k}{k^4}\left(\frac{1}{k^2+\mu^{-1}k_0+i0}+\frac{1}{k^2-\mu^{-1}k_0+i0}\right)$$

$$\times\frac{1}{-k^2+2k_0+a_1{}^2-i0}\left[h_1k_0-h_2\left(k_0{}^2-\frac{2}{3}\mathbf{k}^2\right)\right]. \quad (5)$$

Here x and y are Feynman parameters from the expression for the self-energy correction,[20] $\mu=m/2M$ is a small parameter, and the momentum integration is over the exchange-photon momentum made dimensionless by the electron mass. The weighting functions of the Feynman parameters are given by the relations

$$h_1(x,y)=\frac{1+x}{y},\qquad h_2(x,y)=\frac{1-x}{y}\left[1-\frac{2(1+x)}{x^2+\lambda^2}y\right],$$

$$a_1{}^2(x,y)=\frac{x^2+\lambda^2}{(1-x)y}, \quad (6)$$

where λ is the infrared mass of the radiative photon in units of the electron mass. Everywhere below where not specifically stated otherwise we shall omit the dimensional factor $[\alpha(Z\alpha)/\pi^2](m/M)E_F$ in the expressions for the energy and shall calculate the dimensionless integrals which remain.

The main contribution of order $1/\mu$ to the integral (5) is from the residue at the muon pole, which corresponds to motion of the muon on the mass shell. This same residue is related to the leading infrared divergence of the integral, which is proportional to $\lambda^{-1/2}$; the remaining contributions diverge only logarithmically. These two circumstances make it convenient to calculate separately the contribution of the mass shell and of the remaining difference, not only for the contributions of the diagrams of Fig. 2(c), but also for the diagrams of Figs. 2(a) and 2(b). It turns out that the structure of the infrared divergences for all pole contributions is the same, and these divergences cancel individually within the sum of the pole and nonpole contributions of the diagrams of Fig. 2. For the pole contribution to the energy shift (5), leaving in the integrand only the leading term of the nonrelativistic expansion with respect to the muon mass, we have

$$\delta E_\Sigma(m.s.)=\frac{3\pi}{2\mu(1+2\mu)^{1/2}}$$

$$\times\int_0^1 dx\int_0^x dy\frac{1}{a_1(x,y)}\left[\mu h_1(x,y)+\frac{2}{3}h_2(x,y)\right]. \quad (7)$$

The result of the integration in (7) is conveniently represented in the form

$$\delta E_\Sigma(m.s.)=\frac{1}{2\mu}\left(-2I_\lambda+\frac{11\pi^2}{6}\right)+\left(I_\lambda+\frac{23\pi^2}{24}\right), \quad (8)$$

where we have introduced the standard infrared-divergent integral

$$I_\lambda=\frac{4\pi}{3}\int_0^1 dx\left(\frac{x}{x^2+\lambda^2}\right)^{3/2}(1-x)^{1/2}\sim\frac{1}{\lambda^{1/2}}. \quad (9)$$

The integral I_λ does not need to be calculated, since it arises also in the pole contributions of the diagrams of Figs. 2(a) and 2(b) and cancels in the sum of these diagrams.

Returning to the calculation of the total contribution of the self-energy correction (5) we see at once that the usual combining of the photon, electron, and muon denominators by means of additional Feynman parameters u and v and subsequent momentum integration does not lead to success. In fact, in this procedure we arrive at a fourfold integral over Feynman parameters with a typical denominator $(1-x)yuv+\mu f(x,y,u,v)$. In the last term the function f is of order unity, but expansion of the denominator in the small parameter μ is invalid since its first term vanishes in the region of integration. On the other hand, in an exact calculation of the fourfold integral, technical difficulties arise which cannot be overcome. Therefore we do not introduce additional Feynman parameters into the expression (5), but make a Wick rotation (this is valid since the diagram is calculated at threshold) and symmetrize the integrand in k_0:

$$\delta E_\Sigma=-\frac{3}{\pi\mu^2}\int_0^1 dx\int_0^x dy\,h_1\int_0^\infty dk^2\int_0^\pi d\theta$$

$$\times\frac{\sin^2\theta\cos^2\theta}{k^2+\mu^{-2}\cos^2\theta}\frac{1}{(k^2+a_1{}^2)^2+4k^2\cos^2\theta}$$

$$+\frac{1}{2\pi\mu^2}\int_0^1 dx\int_0^x dy\,h_2\int_0^\infty dk^2(k^2+a_1{}^2)\int_0^\pi d\theta$$

$$\times\frac{\sin^2\theta(2+\cos^2\theta)}{k^2+\mu^{-2}\cos^2\theta}\frac{1}{(k^2+a_1{}^2)^2+4k^2\cos^2\theta}=\delta E_{\Sigma1}+\delta E_{\Sigma2}. \quad (10)$$

Direct calculation of this integral is hindered as before by the impossibility of using the smallness of the parameter μ. We shall use the obvious identity

$$\frac{\cos^2\theta}{(k^2+\mu^{-2}\cos^2\theta)[(k^2+a_1{}^2)^2+4k^2\cos^2\theta]}=\frac{\mu^2}{4k^4\mu^2-(k^2+a_1{}^2)^2}$$

$$\times\left\{\frac{k^2}{k^2+\mu^{-2}\cos^2\theta}-\frac{(k^2+a_1{}^2)^2}{(k^2+a_1{}^2)^2+4k^2\cos^2\theta}\right\}. \quad (11)$$

The denominator of the factor in front of the curly brackets on the right-hand side of the identity contains two terms, the first of them being much smaller than the second, since $\mu^2k^4/(k^2+a_1^2)\leqslant\mu^2\leqslant1$. Since we shall follow here only the contributions of order $1/\mu$ and 1 to the energy (10), this term can be omitted, making use of the presence in the problem of a small parameter. Now the contribution to the energy, which contains the weighting function h_1, after angular integration will take the form

$$\delta E_{\Sigma1}=3\int_0^1 dx\int_0^x dy\,h_1(x,y)\int_0^\infty dk^2$$

$$\times\left\{-\frac{1}{(k^2+a_1^2)^2}\left[\mu k(1+\mu^2k^2)^{1/2}-\mu^2k^2\right]\right.$$

$$\left.+\frac{1}{4k^2}\left[\frac{1}{k^2+a_1^2}\left((k^2+a_1^2)^2+4k^2\right)^{1/2}-1\right]\right\}. \qquad (12)$$

In the first term of the expression (12) there is a small parameter μ (we shall call these integrals μ-integrals), while the second term does not depend on μ (we shall call these integrals c-integrals). Calculation of a μ-integral in closed form is impossible. We shall break down the region of integration into parts by means of a parameter σ such that

$$1\ll\sigma\ll\mu^{-1}. \qquad (13)$$

In the region of small momenta $0<k<\sigma$ we shall use the smallness of μk in comparison with unity and shall expand the integrand in series, after which the integration is easy to perform. In the region of large momenta $k\geqslant\sigma$ we shall use for simplification of the calculations the fact that $k\geqslant 1$. At momenta $k\sim\sigma$ the two approximations are applicable simultaneously and therefore in the sum of the contributions of the two regions the auxiliary parameter σ cancels. After making the calculations described, we obtain

$$\delta E_{\Sigma 1}{}^{\mu}=0. \qquad (14)$$

For calculation of the c-integral from Eq. (12) we shall first carry out the momentum integration

$$\delta E_{\Sigma 1}{}^{c}=3\int_0^1 dx\int_0^x dy\, h_1(x,y)\left\{\frac{1}{a_1}\operatorname{arctg}\frac{1}{a_1}-\frac{1}{2}\ln\frac{1+a_1^2}{a_1^2}\right\}. \qquad (15)$$

As we have already mentioned above, it is more convenient to calculate the difference of the total and pole contributions to the energy shifts. The pole terms, for example (7), contain contributions of order $1/\mu$ and 1, which we shall denote respectively by $\delta E^{(0)}(m.s.)$ and $\delta E^{(1)}(m.s.)$, so that $\delta E(m.s.)=\delta E^{(0)}(m.s.)+\delta E^{(1)}(m.s.)$. Note that a contribution to the total energy shift of order $1/\mu$ is given only by the μ-integrals, while the c-integrals give only a contribution of order unity, since they do not contain the parameter μ at all. It is evident from (7) that subtraction from the c-integral (15) of the pole contribution $\delta E^{(1)}_{\Sigma 1}$ signifies the substitution in the integrand

$$\operatorname{arctg}\frac{1}{a_1}\to\operatorname{arctg}\frac{1}{a_1}-\frac{\pi}{2}=-\operatorname{arctg}a_1. \qquad (16)$$

It turns out that the rule (16) for subtraction of the contribution of the mass shell has a general nature and is valid for calculation of all remaining c-integrals (associated with the diagrams of Figs. 2(a) and (b)), differing only in the explicit expression for the weighting function $a(x,y)$. As a result of the calculations we obtain

$$\delta E^{c}_{\Sigma 1}-\delta E^{(1)}_{\Sigma 1}(m.s.)=-21\pi^2/16. \qquad (17)$$

The calculation of the remaining contribution $\delta E_{\Sigma 2}$, Eq. (10), of the self-energy correction is carried out like that described above. The only difference is that now the contribution of the μ-integral is not equated to zero, and we subtract from it the contribution of the mass shell $\delta E^{(0)}(m.s.)$. The subtraction leads to the following substitution in the integrand for the μ-integral:

$$\frac{1}{\mu k}(1+\mu^2k^2)^{1/2}\to\frac{1}{\mu k}\left[(1+\mu^2k^2)^{1/2}-1\right]. \qquad (18)$$

This rule for subtraction of the pole contribution, like the similar rule for the c-integrals (16), is universal and applies to all μ-integrals. We shall calculate the contribution $\delta E_{\Sigma 2}^{\mu}-\delta E^{(0)}_{\Sigma 2}(m.s.)$ by breaking down the region of integration by means of the parameter σ (see Eq. (13)). At small momenta we shall first integrate over momentum, and then over the Feynman parameter; at large momenta the reverse order of the integrations is appropriate. In the course of the calculations, characteristic integrals arise which involve the Euler logarithm and frequently are not given in standard handbooks.

As a result we obtain for the sum of the μ- and c-integrals

$$\delta E_{\Sigma 2}-\delta E_{\Sigma 2}(m.s.)=-\frac{3}{4}\ln^2\frac{M}{m}-\frac{9}{2}\ln\frac{M}{m}-3\ln\frac{M}{m}\ln\lambda+\frac{13\pi^2}{16}-9. \qquad (19)$$

The total contribution of diagrams with a self-energy correction of Fig. 2(c) is equal to the sum of the contributions (8), (17), and (19):

$$\delta E_{\Sigma}=\frac{1}{2\mu}\left(-2I_\lambda+\frac{11\pi^2}{6}\right)+\left(I_\lambda+\frac{23\pi^2}{24}\right)+\left(-\frac{3}{4}\ln^2\frac{M}{m}-3\ln\frac{M}{m}\ln\lambda-\frac{9}{2}\ln\frac{M}{m}-\frac{\pi^2}{2}-9\right). \qquad (20)$$

The leading logarithms in this expression are due to the asymptotic behavior of the self-energy operator and cancel in the complete gauge-invariant set of diagrams of Fig. 2. This cancellation can easily be proved by means of Ward identities, and in any event such logarithms do not appear at all in the individual diagrams in the Landau gauge.

The contribution to the hyperfine splitting of diagrams with a spanning photon of Fig. 2(a) is given by an integral similar to (5) for the self-energy corrections:

$$\delta E_{B}=-\frac{i}{16\pi^2\mu^2}\int_0^1 dx\int_0^x dy\,(x-y)\times\int\frac{d^4k}{k^4}\left(\frac{1}{k^2+\mu^{-1}k_0+i0}+\frac{1}{k^2-\mu^{-1}k_0+i0}\right)$$

$$\times\left\{2(3k_0^2-2\mathbf{k}^2)\left[\frac{1-2y}{\Delta}+\frac{1}{\Delta^2}\left[-2+2x(1-y)-x^2\right]\right]-6bk_0\left[\frac{1-2y}{\Delta}+\frac{1}{\Delta^2}\left[2bk_0y(1-y)+2x(1-y)+x^2\right]\right]\right\}. \qquad (21)$$

Here

$$\Delta=[-k^2+2bk_0+a^2-i0]y(1-y),$$
$$a^2=(x^2+\lambda^2)/y(1-y),\quad b=(1-x)/(1-y). \qquad (22)$$

The integral (21) is calculated by means of the same procedures as the self-energy contribution (5), but the calculations are still more cumbersome. As the result of lack of space we do not have the possibility of giving intermediate results and shall at once write out the answer:

$$\delta E_{\Xi}=\frac{1}{2\mu}\left(-2I_\lambda+4\pi^2\ln 2-\frac{13\pi^2}{6}\right)+\left(I_\lambda+2\pi^2\ln 2-\frac{25\pi^2}{24}\right)$$
$$+\left(-\frac{3}{4}\ln^2\frac{M}{m}-3\ln\frac{M}{m}\ln\lambda-\frac{15}{4}\ln\frac{M}{m}+3\zeta(3)\right.$$
$$\left.-2\pi^2\ln 2+\frac{7\pi^2}{4}-\frac{107}{8}\right). \qquad (23)$$

Here $\zeta(3)$ is the Riemann zeta function; it arises in calculation of integrals with a trilogarithm, similar to the Euler dilogarithm mentioned above. As in the preceding case, the leading logarithms in (23) are easily obtained from the corresponding asymptotic behavior.

Calculation of the contribution of the diagrams of Fig. 2(b) with a vertex correction is extremely cumbersome. We used the well known expression[20] for a renormalized vertex function with one virtual end, which contains six independent form factors. We present the result of the calculations:

$$\delta E_{\Lambda}=\frac{1}{\mu}\left(2I_\lambda-\frac{3\pi^2}{2}\ln 2-\frac{35\pi^2}{24}\right)+\left(-2I_\lambda+5\pi^2\ln 2-\frac{\pi^2}{6}\right)$$
$$+\left(\frac{3}{2}\ln^2\frac{M}{m}+6\ln\frac{M}{m}\ln\lambda+12\ln\frac{M}{m}+3\zeta(3)\right.$$
$$\left.-2\pi^2\ln 2-\frac{\pi^2}{2}-\frac{49}{2}\right). \qquad (24)$$

It is easy to see that the leading logarithms in this formula are due to the asymptotic behavior of the vertex function

5. DISCUSSION OF RESULTS

Collecting the obtained contributions to the hyperfine splitting and restoring the omitted factor $[\alpha(Z\alpha)/\pi^2](m/M)E_F$, we obtain

$$\delta E_{\Sigma}+\delta E_{\Xi}+\delta E_{\Lambda}=\alpha(Z\alpha)E_F\left[\ln 2-\frac{13}{4}\right]$$
$$+\frac{\alpha(Z\alpha)}{\pi^2}\frac{m}{M}E_F\left[\frac{15}{4}\ln\frac{M}{m}+6\zeta(3)+3\pi^2\ln 2+\frac{\pi^2}{2}+\frac{17}{8}\right]. \qquad (25)$$

The first term in the expression (25) is the radiative correction of first order in the external field and after taking into account the anomalous moment of the muon it reproduces the known result.[5,6] The coefficient of the logarithm in the second term has also been obtained previously.[12] The remaining terms, which we shall write out individually, have been calculated analytically for the first time in the present work and are its principal result:

$$\delta\nu_{rr}=\frac{\alpha(Z\alpha)}{\pi^2}\frac{m}{M}\nu_F\left[6\zeta(3)+3\pi^2\ln 2+\frac{\pi^2}{2}+\frac{17}{8}\right]. \qquad (26)$$

Note that the scale of the correction (26) is characterized by a coefficient π^2 in a number of terms. This also is to be expected, since usually in quantum electrodynamics the constant correction to the leading term of the asymptotic behavior has a scale π^2 if this leading term is the square of a large logarithm (recall the canceled squares of logarithms in Eq. (25)). The radiative correction to the recoil (26) has recently been obtained numerically[21] and turns out to be $\delta\nu_{rr} = 3.83(7)$ kHz. Our result (26) is $\delta\nu_{rr} = 4.05$ kHz, which differs from the result given above[21] by three standard deviations of the numerical integration. If we recognize that the integrands have singularities, the error in the numerical integration may very likely be underestimated. The calculations in Ref. 21 and in the present work were carried out in different gauges, and different methods of removing the infrared and ultraviolet divergences were used. We consider that the difference of only three standard deviations between the numbers obtained should be regarded as an indication of the compatibility of these results.

We shall give the complete theoretical formula for the magnitude of the hyperfine splitting with inclusion of the correction (26) (see for example the review Ref. 7):

$$\delta\nu=\nu_F(1+a_\mu)\left\{1+\frac{3}{2}(Z\alpha)^2+a_e+\alpha(Z\alpha)\left(\ln 2-\frac{5}{2}\right)\right.$$
$$-\frac{8\alpha(Z\alpha)^2}{3\pi}\ln(Z\alpha)\left[\ln(Z\alpha)-\ln 4+\frac{281}{480}\right]$$
$$+\frac{\alpha(Z\alpha)^2}{\pi}(15{,}38(29))+(1+a_\mu)^{-1}\left[-\frac{3(Z\alpha)}{\pi}\frac{mM}{M^2-m^2}\ln\frac{M}{m}\right.$$
$$\left.\left.+(Z\alpha)^2\frac{mM}{(M+m)^2}\left(-2\ln(Z\alpha)-8\ln 2+\frac{65}{18}\right)+\delta_\mu\right]\right\}, \qquad (27)$$

where a_μ is the muon anomalous magnetic moment, and the radiative correction to the recoil δ_μ is given by the relation

$$\delta_\mu=\frac{\alpha(Z\alpha)}{\pi^2}\frac{m}{M}\left[\left(-2\ln^2\frac{M}{m}-\frac{8}{3}\ln\frac{M}{m}-\frac{29}{9}-\frac{\pi^2}{3}+1.9(3)\right)\right.$$
$$\left.+\left(\frac{15}{4}\ln\frac{M}{m}+6\zeta(3)+3\pi^2\ln 2+\frac{\pi^2}{2}+\frac{17}{8}\right)\right]$$
$$-(Z\alpha)(Z^2\alpha)\frac{m}{M}\cdot 1{,}037(9)-\frac{4}{3}\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m}{M}\ln^3\frac{M}{m}. \qquad (28)$$

Let us make clear the origin of those terms in Eq. (28) which are not contained in the review of Ref. 7. The last term in the square brackets is the result of the present work (26), and the next to the last term in (28) is the result of the numerical integration of Ref. 21 and corresponds to the contribution of radiative corrections to the muon line. At the present time we are performing an analytical calculation of this contribution and hope to report the result in the very near future. Finally, the last term in (28), which was obtained in Ref. 22, is the leading three-loop contribution. In this contribution the cube of the large logarithm contains the extra factor α; we note, however, that there are also other uncalculated contributions of order $\alpha^2(Z\alpha)(m/M)E_F$.

Calculation of the magnitude of the hyperfine splitting according to Eq. (27) gives

$$\delta\nu_{\rm theor}=4463304.77(98)(133)(60)\ \text{kHz}, \qquad (29)$$

where we have used the known value of the fine-structure constant obtained on the basis of the Josephson effect,[23]

$$\alpha_J^{-1}=137.035963(15) \quad (\delta=0.11\cdot 10^{-6}) \qquad (30)$$

and the experimental value[1] for the ratio of the muon and electron masses

$$M/m=206.768262(62) \quad (\delta=0.3\cdot 10^{-6}). \qquad (31)$$

The letter δ above denotes the magnitude of the relative error. In the parentheses in Eq. (29) we have given the errors of the last significant digits. The still uncalculated terms of order $\alpha^2(Z\alpha)(m/M)E_F$ have been estimated as 0.60 kHz ($\delta = 0.13\cdot 10^{-6}$). The error 1.33 kHz ($\delta = 0.3\cdot 10^{-6}$) originates from the error in measurement of the mass ratio (31). Finally, the error of 0.98 kHz ($\delta = 0.22\cdot 10^{-6}$) is due to the uncertainty in determination of the fine-structure constant. The correction (26) obtained in the present work is 4.05 kHz ($\delta = 0.91\cdot 10^{-6}$) and makes a substantial contribution to the value (29). The experimental result for the hyperfine splitting[1]

$$\delta\nu_{\rm exp} = 4\,463\,302{,}88(16)\ \text{kHz} \qquad (32)$$

has a substantially smaller value than the theoretical number. Note, however, that it is planned[2] to make a substantial increase in the accuracy of measurement of the mass ratio (31), which contributes the greatest uncertainty to the theoretical value (29). The agreement of the experimental value (32) and the theoretical value (29) for the magnitude of the hyperfine splitting must be considered quite satisfactory.

From the data on hyperfine splitting in muonium (32) it is possible by means of Eqs. (27) and (28) to find the value of the fine-structure constant:

$$\alpha_{\rm Mu}^{-1} = 137.035992(22) \qquad (\delta = 0.16\cdot 10^{-6}). \qquad (33)$$

We give also the two other most accurate values of the fine-structure constant:

$$\alpha_e^{-1} = 137.035994(5) \qquad (\delta = 0.04\cdot 10^{-6}), \qquad (34)$$

$$\alpha_H^{-1} = 137.035968(23) \qquad (\delta = 0.17\cdot 10^{-6}), \qquad (35)$$

which were obtained respectively from measurement of the electron anomalous magnetic moment[24] and experiments on the quantum Hall effect.[25] Without going into detailed discussions of the experimental and theoretical problems involved in the values (34) and (35), we mention only that there are real possibilities for improvement of their accuracy. The quantum-electrodynamical values of the fine-structure constant (33) and (34) agree beautifully with each other, and the rigid-body values (30) and (35) agree equally satisfactorily. Only further investigations will show whether the discrepancy noted above between the two groups of data must be taken seriously. Investigations of this type are being carried out in various directions; in regard to the theory of the hyperfine splitting in muonium, the problems closest to solution are analytic calculations of radiative corrections to recoil associated with the muonium line, and purely radiative corrections of order $\alpha^2(Z\alpha)E_F$.

[1)]Experimentally one measures the frequency, which is related to the energy as $E_F = h\nu_F$. In what follows we shall speak of the energy levels, having in mind this relation.

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