Electron flow in a planar plasma layer

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The evolution of electron current in a thin plasma sheet located in a vacuum is discussed. It is shown that the nonlocal nonlinear integro-differential equation that describes this process admits of an exact analytic solution.

Studies of electron magnetic hydrodynamics (EMH) (MHD electron flows against a stationary ion background) have revealed in recent years a number of nontrivial effects associated with the evolution of the magnetic field in the plasma, namely, penetration of the field into the region forbidden by the usual skin-effect theory,[1] stabilization of necking instability of the z-pinch,[2] restricted generation of the electric field due to the thermal emf,[3] decay of the initial perturbation of $\mathbf{B}$ into individual vortices,[4] and so on. However, as a rule, all these treatments have been confined to the two-dimensional slabs (x,y) or cylindircal (r,z) situation with a single magnetic-field component (B_z or B_φ), whereas the essentially vector nature of the EMH equations suggests there should be a considerable difference between the general three-dimensional and the degenerate two-dimensional cases. It is therefore desirable to look for analytically soluable three-dimensional models.

A possible candidate for this is the flow of electrons in a thin layer (sheet) of plasma in a vacuum. This pseudo-two-dimensional situation (the magnetic field is essentially three-dimensional) is not only methodologically valuable, but may serve as a qualitative analog of a semi-infinite plasma for which the interaction between currents through the vacuum magnetic field is also significant.

Thus, let us suppose that the $z = 0$ plane contains a plasma sheet of small thickness $\delta \to 0$, but finite "surface" density $N \equiv n\delta$, conductivity $\Sigma \equiv \sigma\delta$, and current density $\mathbf{J} \equiv \mathbf{j}\delta$. The EMH equations inside the layer can be written in the standard form[1–4]

$$\mathbf{E} = \frac{1}{nec}[\mathbf{j}\times\mathbf{B}] + \frac{\mathbf{j}}{\sigma}, \qquad \frac{\partial \mathbf{B}}{\partial t} = -c \operatorname{curl} \mathbf{E}. \tag{1}$$

We are mostly interested in the z-component of the magnetic field frozen into $\mathbf{J}$ (for $\Sigma = \infty$): $B_z(x,y,0) \equiv b(x,y)$ (because the sheet is thin, i.e., $\delta \ll \delta_{\rm skin}$ or $\delta \ll c/\omega_{pe}$, the tangential components of $\mathbf{B}$ are not frozen into the current[1,4]). If the bulk characteristics n, σ, and $\mathbf{j}$ of the electron flow have no gradients along the z-axis inside the plasma layer (the tangential components would otherwise begin to transform into the normal components of $\mathbf{B}$ and vice versa), then, in the two-dimensional equation for b that follows from (1), we can replace bulk characteristics everywhere with surface characteristics:

$$\mathbf{e}_z \frac{\partial b}{\partial t} = -\operatorname{curl} \frac{[\mathbf{J}\times\mathbf{e}_z]b}{Ne} - \operatorname{curl} \frac{c\mathbf{J}}{\Sigma}. \tag{2}$$

The entire information on the latent three-dimensional nature of the problem is contained in the relationship between b and $\mathbf{J}$ (Biot-Savart law):

$$b\mathbf{e}_z = \frac{1}{c} \int \frac{[\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')]}{|\mathbf{r}-\mathbf{r}'|^3} d^2\mathbf{r}', \tag{3}$$

where the integral at $\mathbf{r}' = \mathbf{r}$ is taken in the sense of the principal value. Actually, we shall have to invert (3), which is readily done by means of Fourier transformations.

$$\mathbf{J} = -\frac{c}{4\pi^2} \int \frac{b(\mathbf{r}')[\mathbf{e}_z\times(\mathbf{r}-\mathbf{r}')]}{|\mathbf{r}-\mathbf{r}'|^3} d^2\mathbf{r}' = \frac{c}{4\pi^2} \operatorname{curl} \int \frac{b(\mathbf{r}')\mathbf{e}_z}{|\mathbf{r}-\mathbf{r}'|} d^2\mathbf{r}'. \tag{4}$$

It is therefore clear that the three-dimensional nature of the problem is now shown by the nonlocal relation between b and $\mathbf{J}$ (see Ref. 1).

1. JET FLOWS

Let $\mathbf{J} = J(x)\mathbf{e}_y$, $b = b(x)$, in which case, instead of (2), we have

$$\frac{\partial b}{\partial t} = \frac{\partial}{\partial y} \frac{1}{N} \frac{Jb}{e} - \frac{c}{\Sigma} \frac{\partial J}{\partial x}, \tag{5}$$

and b and $2\pi J/c$ are related by the Hilbert transformation [cf. (3) and (4)]:

$$b(x) = \frac{1}{\pi} \int \frac{2\pi J(x')}{c(x'-x)} dx', \qquad \frac{2\pi J(x)}{c} = -\frac{1}{\pi} \int \frac{b(x')}{x'-x} dx'. \tag{6}$$

where the integrals must be interpreted as principal values. As an example of this kind of coupling, we may mention the functions $(1 + x^2/l^2)^{-1}$ and $(x/l)(1 + x^2/l^2)^{-1}$, or $\cos(x/l)$ and $\sin(x/l)$, which we shall use below.

The analog of the usual skin problem follows from (5) and (6) when $N = \text{const}$. It is readily seen that, in this case, any initial distribution $b(x)$ with $A \equiv \int b(x)dx \neq 0$ eventually evolves to the self-similar solution with Lorentz profile

$$b = A\left[\pi\tau\left(1 + \frac{x^2}{\tau^2}\right)\right]^{-1}, \qquad \tau = \frac{c^2 t}{2\pi\Sigma}.$$

The clear singularity of this solution, which distinguishes it from the usual skin effect, i.e., the linear dependence of the scale l on time, is due to the essentially three-dimensional nature of the problem and follows even from the simple estimate $l\delta \sim c^2 t/4\pi\sigma$.

When $N = N(y)$, Eq. (5) becomes nonlinear, and the magnetic field is significantly transported by the current[1,2] (in this particular case, the nonlinearity has a peculiarly nonlocal character). In order to emphasize the difference from the paper cited above still further, which ensues from the geometry of the problem, let us begin by considering the

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linear limit of (5), when we can substitute $b = B =$ const (strong external field) and $\Sigma = \infty$ on the right-hand side, which correspond to the usual drift oscillations of the plasma. Because of its specific dispersion ($\omega \sim \text{sign}\, k$, $d\omega/dk \equiv 0$ for $k \neq 0$), the drift oscillations on the plasma plane do not propagate, which leads to the characteristic "flicker" of the initial perturbation $b_0(x)$:

$$b = b_0(x)\cos(pBt) + \frac{1}{\pi}\int \frac{b_0(x')}{x-x'}\,dx' \sin(pBt),$$
$$p = \frac{1}{2\pi e}\frac{\partial}{\partial y}\frac{1}{N}.$$

Here and below, we shall neglect the function $p(y)$, which is valid for $\partial/\partial y \ll \partial/\partial x$ and is definitely valid when $N \sim (y + y_0)^{-1}$. Nonlinear effects still ensure that the b profile is brought into motion but, in contrast to the usual planar situation,[1,2] this motion occurs without "breaking," which is clear even from the following special soliton-like solution (5) ($\Sigma = \infty$):

$$b = A[1 + (x - plAt)^2/l^2]^{-1},$$

i.e., the nonlocal nonlinearity has dispersive properties as well.

The equations describing the dynamics of the magnetic field on the plasma plane, i.e., (5) and (6), have a complete analytic solution despite their nonlinear integro-differential character. All we need to do is to consider the complex plane and look upon x as a complex quantity. According to (6), b and $2\pi J/c$ are related similarly to the way in which the real and imaginary parts of generalized susceptibility[5] are related (cf. the Kramers-Kronig relations), so that we can introduce the function $w = b + i2\pi J/c$, which is analytic in the upper half-plane of x. Equation (5) and the other equation for J, deduced from (5) by applying the Hilbert transformation, can be combined into a single equation for w:

$$i\frac{\partial w}{\partial t} = pw^2 - v\frac{\partial w}{\partial x}, \quad v = \frac{c^2}{2\pi\Sigma}, \tag{7}$$

the integration of which over the characteristics is a trivial matter. Analysis of the different solutions of (7) leads to the conclusion that there is a hierarchy of nonlinearities and diffusion, which is essentially different from the usual planar case.[1] When the problem is completely homogeneous in z, the local nonlinearity is necessarily balanced by the local dissipative term in the course of time; the steepening of the magnetic-field profile unavoidably leads to a higher rate of diffusion. In this particular geometry, this is generally not the case: there are regimes in which dissipation is always small, despite the increase in the derivatives of b, e.g., in the case of the explosive contraction of the current, described by

$$\frac{1}{w(x,t)} = \left\{\frac{x}{l} + i[1 + t(v + pA)]\right\}A^{-1},$$

where $A < -v/p$, i.e., for a sufficiently large current produced by the motion of electrons toward regions of higher density.

2. VORTEX FLOWS

Let us now suppose that $N =$ const and $\Sigma = \infty$, but that the flow of electrons over the plane; is arbitrary. In the usual planar geometry, the equation analogous to (2) then degenerates to the identity $\partial B_z/\partial t \equiv 0$ (Refs. 1 and 4), and any current distribution is stationary. In our geometry, because of a different relation between the incompressible current $\mathbf{J}$ and the frozen-in quantity $b(\mathbf{J}\cdot\nabla)b \not\equiv 0$, the stationarity condition

$$\text{curl}\,[\mathbf{J}\times\mathbf{e}_z]\,b = 0$$

is no longer trivial. Using (4), we can rewrite this in the form

$$\int \frac{b(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d^2r' = F(b),$$

where F is an arbitrary function. In particular, all circular vortices $b = b(r)$, $J = \mathbf{J}(r)\mathbf{e}_\varphi$ are stationary. In other words, in this pseudo-two-dimensional geometry, and in contrast to the cases examined in Ref. 4, interesting stationary solutions exist even when we neglect the electron inertia, which is omitted in (1) (this is valid when $l \gg a = c^2/\Omega_{pe}^2$, $\Omega_{pe}^2 \equiv 4\pi Ne^2/m$). A further characteristic feature of pseudo-two-dimensional vortices is that they are nonlocal: the magnetic field due to the current J_φ, which is bounded in r, penetrates into the vacuum and falls off at infinity only as a power law as $a \to 0$. This is a situation similar to Debye screening in a plane plasma layer.[6]

When the dynamics of b is investigated, it may be useful to identify the class of stable solutions whose analysis is particularly simply performed in terms of the integrals of motion, which, in turn, following Lamb, are conveniently written in terms of frozen-in quantities (in the present case, b).[4,7] There are four such integrals, namely, energy, momentum, angular momentum, and the frozen-in integral (cf., Ref. 4):

$$\mathscr{E} = \frac{1}{8\pi}\int \mathbf{B}^2\,d^3\mathbf{r} = \frac{1}{c^2}\int\int \frac{\mathbf{J}(\mathbf{r})\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d^2\mathbf{r}\,d^2\mathbf{r}'$$
$$= \text{const}\int\int \frac{b(\mathbf{r})\,b(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d^2\mathbf{r}\,d^2\mathbf{r}',$$
$$\mathbf{P} = -\frac{Ne}{c}\int \mathbf{A}\,d^2\mathbf{r} = \text{const}\int b[\mathbf{e}_z,\mathbf{r}]\,d^2\mathbf{r},$$
$$M_z = \text{const}\int br^2\,d^2\mathbf{r}, \quad I = \int F(b)\,d^2\mathbf{r}., \tag{8}$$

where $\mathbf{A}$ is the vector potential of $\mathbf{B}$ and F is an arbitrary function.

Analysis of (8) show that circular vortices with $d|b|/dr \leqslant 0$ are stable. They have maximum energy $\mathscr{E}$ and minimum magnitude of the angular momentum M_z for constant I; stable additional pairs of vortices with different signs of b are also possible.[4]

Vortex flows can also be generalized to the situation where the electron inertia is significant (for $l \lesssim a$), in which case $\mathbf{J}$ contains the frozen-in quantity $\omega = b + mc\mathbf{e}_z \cdot \text{curl}\,\mathbf{J}/Ne$. All that needs to be done in the integrals of motion (8) is to replace b with ω, and the charge interaction potential in the expression for $\mathscr{E}$ with $H_0(r/a) - N_0(r/a)$, where H_0 is the Struve function and N_0 the Neumann function. When $l \ll a$, the relation between the current and the quantity (curl $\mathbf{J}$) frozen into it degenerates to a local relationship, the system ceases to "feel" its final size in z, and the situation reduces to the usual two-dimensional flow of a perfect fluid.[4]

It follows that the model we have proposed provides an analytic description of the above effects, which is just as complete as the simple two-dimensional system, despite the more complicated integral relationship between **J** and b. On the one hand, this enables us to understand in specific cases the dependence of EMG effects on the geometry, and, on the other, to use this object as an example of a medium with a nonlocal nonlinearity for which our exact solution can be found.

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