Latent symmetry of quantum spin systems with uniaxial anisotropy

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Generators of the auxiliary Hamiltonian symmetry group, which are strictly of quantum origin and vanish in the classical limit, are found for a spin magnetic system with uniaxial anisotropy of the easy-plane type. It is shown that a latent symmetry of isolated spins exists for selected values of the longitudinal magnetic field, and in a definite field region for a macroscopic system of interacting spins. A connection is established between the obtained quantum integrals of motion and the degenerate energy levels of the system, and the possibility of suitably classifying the stationary state is indicated.

The total-spin component S_z along a chosen axis of a uniaxial magnet is known to be conserved in a longitudinal external magnetic field. This property is possessed by both classical and quantum systems. In an easy-plane classical magnet, in fields weaker than critical, when the component of each spin along the z axis is smaller in the equilibrium configuration than the maximum, the system is degenerate in the azimuth angle. This degeneracy vanishes if the field reaches the critical value at which all the spins at equilibrium are directed along the z axis. The fact that the azimuth angle is meaningless for quantum spins raises the question whether an additional quantum-system symmetry exists if the field is weaker than critical.

We show in this paper that for a magnetically dilute system with uniaxial anisotropy, at a finite field-value number that depends on the spin, the symmetry is higher than that corresponding to rotations about the z axis. For such fields, part of the energy levels is doubly degenerate in the spin component S_z. Given the value of the spin, the number of such levels is determined by the value of the field. We have found, for a corresponding symmetry group, generators that are of pure quantum origin (they vanish as $S\to\infty$). This additional symmetry is absent in fields stronger than critical.

If the system consists of N interacting spins and has easy-plane anisotropy, most magnetic-field values at which the energy levels are degenerate are determined both by the individual spins and by their number N. With increase of the number of spins, the permissible values of the field come closer together and in the limit as N they fill a finite interval whose boundaries are determined by the system parameters. If the field is stronger than critical, the ground ferromagnetic state is separated by a gap, and the corresponding additional integrals of motion vanish. Just as in the case of one spin, these integrals are entirely of quantum origin and are absent in the classical limit.

1. We consider a magnetically dilute crystal in which each magnetic ion is in a crystal field described by the anisotropy energy. Allowing for the Zeeman energy, the spin Hamiltonian of an individual magnetic ion takes in an external field H applied along the chosen axis the form

$$\mathscr{H}={}^1/_2\alpha S_z^2-2\mu H S_z, \tag{1}$$

where S_z is the operator of the spin z-component, $\alpha>0$ is the magnetic anisotropy constant, and μ is the magneton. The pressence of a symmetry axis is manifested by the conservation of S_z. We shall show, however, that at some definite field values the system (1) has a higher symmetry that leads to the existence of additional integrals of motion.

The determination of all the conserved quantities of a spin system can be reduced to solution of a differential equation by using the representation of the spin coherent states.[1,2] In the case of system (1), this equation, which expresses the commutativity of a certain quantity with the Hamiltonian, is of the form

$$\left[\frac{\alpha}{2}\left(S-z\frac{\partial}{\partial z}\right)^2-2\mu H\left(S-z\frac{\partial}{\partial z}\right)-\frac{\alpha}{2}\left(S-z^*\frac{\partial}{\partial z^*}\right)^2+2\mu H\left(S-z^*\frac{\partial}{\partial z^*}\right)\right]F=0, \tag{2}$$

where

$$F(z,z^*)=(1+|z|^2)^{2S}\langle z|I|z\rangle,$$

$|z\rangle$ is the vector of the spin coherent state numbered by the complex parameter z, while S is the value of the spin at the site. The change of variable $z=|z|\exp(i\varphi)$ recasts the equation for F in the form

$$[{}^1/_2\alpha(2S-|z|\partial/\partial|z|)-2\mu H]\,\partial F/\partial\varphi=0. \tag{3}$$

The solutions of (3) are

$$F_1=F_1(|z|),\quad F_2=f(z/z^*)|z|^r,\quad r=2(S-2\mu H/\alpha). \tag{b}$$

The quantities z and z^* are connected with the spin projections averaged over the coherent state by the relations

$$\langle z|S^+|z\rangle=\langle z|S^-|z\rangle^*=2Sz/(1+|z|^2),$$
$$\langle z|S^z|z\rangle=S(1-|z|^2)/(1+|z|^2). \tag{4}$$

These relations allow us, given a function of z and z^*, to reconstruct the corresponding operator. Since any operator that depends on spin components reduces for any finite spin to a polynomial, its mean value will be a polynomial in the mean values of the components, of degree not higher than $2S$. The function F will then be a polynomial in z and z^* of degree not higher than $2S$ in each argument.

The first of the solutions of (3), as seen from (4), corresponds to a function (polynomial) of the operator S_z for any field, i.e., this solution yields nothing new. The second solution leads, apart from a factor, to the function

$$F_2=z^{l+r/2}(z^*)^{-l+r/2}, \tag{5}$$

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where the following conditions must be met:

$$l+r/2\leqslant 2S,\quad -l+r/2\geqslant 0,\quad \pm l+r/2 - \text{integer}. \qquad (6)$$

It follows hence that r must be an integer, and $2l$ must have the same parity as r. In addition it is seen that at $r>4S$ and $r<0$ these conditions are not met and there is no solution F_2. The last inequalities define the field region in which degeneracy exists:

$$|2\mu H/\alpha|\leqslant S.$$

The inequalities (6) determine the number of integrals of motion, which are numbered 1 for a given value of r (of the field H).

In accord with (2) and (4), we seek the operator $\tilde{I}_l^-$ that corresponds to the function (5) in the form

$$\tilde{I}_l^- = \sum_{m=0}^{2S-2l} a_m S_z^m (S^-)^{2l}, \qquad (7)$$

where the coefficients a_m are the solutions of a system of $2S-2l+1$ equations

$$\sum_{m=0}^{2S-2l} a_m (S-p)^m = \delta_{p,l+r/2},\quad p=2l,\ 2l+1,\dots,2S \qquad (8)$$

and are expressed in terms of Vandermonde determinants. The sought polynomial can be obtained also directly if it is recognized that, according to Eqs. (8), the eigenvalues $S-p$ of the operator S_z are the roots of the polynomial at $p=2l,\ 2l+1,\dots,2S$, with exception of $p=l+r/2$. We therefore have

$$\sum_{m=0}^{2S-2l} a_m S_z^m = \prod_{\substack{p=2l\\ p\neq l+r/2}}^{2S} \frac{S_z-(S-p)}{S-l-r/2-(S-p)}. \qquad (9)$$

The operator (7), as follows from (3), commutes with the spin Hamiltonian (1). This means that the system has an additional symmetry. It can be verified that, given nonzero r and l, the normalized operators

$$I_l^- = \left[\frac{(2S-l-r/2)!(-l+r/2)!}{(2S+l-r/2)!(l+r/2)!}\right]^{1/2} \tilde{I}_l^-,\quad I_l^+=(I_l^-)^+,$$

$$I_l^z = \frac{1}{2}[I_l^+, I_l^-]$$

satisfy the commutation relations

$$[I_l^+, I_l^-]=2I_l^z,\quad [I_l^z, I_l^+]=I_l^+,\quad [I_l^z, I_l^-]=-I_l^-,$$

which coincide with the commutation rules for the angular-momentum components. Thus, the operators I_l^-, I_l^+, I_l^z are generators of the group $SO(3)$, the invariance group of the Hamiltonian. Generators with different values of l commute, and their number is equal to the number of degenerate energy levels.

The stationary states of the spin Hamiltonian (1) can be classified in accordance with the projection of the spin on the z axis. The operators I_l^-, I_l^+, I_l^z act in the following manner on the corresponding sationary states of the Hamiltonian:

$$I_l^-|S-n\rangle=\delta_{n,-l+r/2}|S-n-2l\rangle,$$

$$I_l^+|S-n\rangle=\delta_{n,l+r/2}|S-n+2l\rangle,$$

$$I_l^z|S-n\rangle={}^1/_2(\delta_{n,-l+r/2}-\delta_{n,l+r/2})|S-n\rangle, \qquad (10)$$

i. e., the nonzero matrix elements of these operators are

$$\langle S-l-r/2|I_l^-|S+l-r/2\rangle,\quad \langle S+l-r/2|I_l^+|S-l-r/2\rangle,$$
$$\langle S-l-r/2|I_l^z|S-l-r/2\rangle,\quad \langle S+l-r/2|I_l^z|S+l-r/2\rangle.$$

Consequently, the transformations of the Hamiltonian invariance groups $I_l^\pm$ change states with $l\pm r/2$ excitations into states with $l\mp r/2$ excitations. The latter means, in particular, that the system has one and the same energy in the states $|S-l-r/2\rangle$ and $|S+l-r/2\rangle$.

Each of the Hermitian integrals of motion

$$I_l^x=(I_l^++I_l^-)/2,\quad I_l^y=(I_l^+-I_l^-)/2i$$

has a common complete set of eigenvectors with the Hamiltonians. It is easy to verify by starting from (10) that only the vectors $|S-l-r/2\rangle \pm |S+l-r/2\rangle$ and $|S-l-r/2\rangle \pm i|S+l-r/2\rangle$ correspond to nonzero eigenvalues, $\pm 1/2$, of the operators I_l^x and I_l^y.

A situation similar to that considered above obtains for an anharmonic oscillator with a Hamiltonian[1)]

$$\mathcal{H}=\beta a^+a-(a^+a)^2,\quad [a, a^+]=1.$$

For arbitrary β, the only conserved quantity is a^+a. If $\beta=n$ ($n=1,2,\dots.$), there exist additional integrals of motion that are of purely quantum origin:

$$I_l^-=(a^+)^{-l+n/2}|0\rangle\langle 0|a^{l+n/2},\quad I_l^+=(I_l^-)^+$$

($2l$ has the same parity as $2l\leqslant n$), where $|0\rangle\langle 0|$ is a projector on the "vacuum" and $a|0\rangle=0$. The additional symmetry is due here to the degeneracy of the energy levels for the states $|-l+n/2\rangle$ and $|l+n/2\rangle$, so that the number of such pairs increases with increase of n.

2. We consider now a system of interacting spins constituting a uniaxial ferrodielectric described by the Heisenberg model. In the presence of an external field H applied along the chosen axis, the spin-Hamiltonian of the system takes, with allowance for the one-ion anisotropy, the form

$$\mathcal{H}=-J\sum_{\mathbf{n}\boldsymbol{\delta}}(S_\mathbf{n}^xS_{\mathbf{n}+\boldsymbol{\delta}}^x+S_\mathbf{n}^yS_{\mathbf{n}+\boldsymbol{\delta}}^y+\gamma S_\mathbf{n}^zS_{\mathbf{n}+\boldsymbol{\delta}}^z)$$
$$+\frac{\alpha}{2}\sum_\mathbf{n}(S_\mathbf{n}^z)^2-2\mu H\sum_n S_\mathbf{n}^z, \qquad (11)$$

where $J>0$ is the exchange integral, α the anisotropy constant, $0\leqslant\gamma\leqslant 1$, $\mathbf{n}$ the vector number of the site, and $\boldsymbol{\delta}$ a vector joining nearest neighbors.

The system (11), just as (1), has axial symmetry that manifests itself in conservation of the z-component of the total spin $S_z=\Sigma_\mathbf{n}S_\mathbf{n}^z$. This system, however, also has a higher symmetry, and furthermore not at individual points but in the entire range of values of the external field. The boundaries of this region are determined by the system parameters. As in the case of a single spin, the determination of all the integrals of motion can be reduced to solution of a differential equation in the coherent-states representation. For the spin system (11) this equation, which expresses the commutativity of a certain quantity I with the Hamiltonian, is of the form

$$(K-K^*)F(z,\ z^*)=0, \tag{12}$$

where

$$K=\frac{J}{2}\sum_{\mathbf{n}\boldsymbol{\delta}}(z_\mathbf{n}^2+z_{\mathbf{n}+\boldsymbol{\delta}}^2-2\gamma z_\mathbf{n}z_{\mathbf{n}+\boldsymbol{\delta}})\frac{\partial^2}{\partial z_\mathbf{n}\partial z_{\mathbf{n}+\boldsymbol{\delta}}}+\frac{\alpha}{2}\sum_\mathbf{n} z_\mathbf{n}^2\frac{\partial^2}{\partial z_\mathbf{n}^2}-JS\sum_{\mathbf{n}\boldsymbol{\delta}}\left\{z_\mathbf{n}\frac{\partial}{\partial z_{\mathbf{n}+\boldsymbol{\delta}}}+z_{\mathbf{n}+\boldsymbol{\delta}}\frac{\partial}{\partial z_\mathbf{n}}+\left[\frac{\alpha(2S-1)}{2JSq}-2\gamma-\frac{2\mu H}{JSq}\right]z_\mathbf{n}\frac{\partial}{\partial z_\mathbf{n}}\right\}, \tag{13}$$

q is the number of nearewt neighbors, K^* is obtained from K by replacing all the z_n by $z_n{}^*$, while the function F is proportional to the integral of motion averaged over the coherent state

$$F=\prod_\mathbf{n}(1+|z_\mathbf{n}|^2)^{2S}\langle z|I|z\rangle,\quad |z\rangle\equiv\prod_\mathbf{m}|z_\mathbf{m}\rangle. \tag{14}$$

Equation (12) is a second-order differential equation for a function of many variables $z_\mathbf{n}$ and $z_\mathbf{n}^*$. It is easily seen that the operator K is homogeneous: it is invariant to a similarity transformation for the variables $z_\mathbf{n}$ and $z_\mathbf{n}^*$. This is consequence of that axial symmetry of the Hamiltonian (11) and permits the solutions of (12) to be regarded as homogeneous polynomials.

The simplest solutions are functions linear in $z_\mathbf{n}$ and $z_\mathbf{n}^*$:

$$F=\sum_\mathbf{m} A_\mathbf{m}z_\mathbf{m},\quad F^*=\sum_\mathbf{m} A_\mathbf{m}^*z_\mathbf{m}^*. \tag{15}$$

Substituting (15) and (12) we get

$$\sum_{\mathbf{n}\boldsymbol{\delta}}-JSz_\mathbf{n}\left\{A_{\mathbf{n}+\boldsymbol{\delta}}+A_{\mathbf{n}-\boldsymbol{\delta}}-\left[2\gamma-\frac{\alpha(2S-1)}{2JSq}+\frac{2\mu H}{JSq}\right]A_\mathbf{n}\right\}-\text{c.c.}=0.$$

Hence

$$\sum_{\boldsymbol{\delta}}\left\{A_{\mathbf{n}+\boldsymbol{\delta}}+A_{\mathbf{n}-\boldsymbol{\delta}}-\left[2\gamma+\frac{2\mu H-\alpha(S-{}^1/_2)}{JSq}\right]A_\mathbf{n}\right\}=0.$$

A normalized solution of this finite-difference equation is

$$A_\mathbf{n}=(2S/N^{1/2})\exp\left[i\sum_{\boldsymbol{\delta}}(\mathbf{n}\boldsymbol{\delta})\varphi\right], \tag{16}$$

where

$$\cos\varphi=\gamma+[2\mu H-\alpha(S-{}^1/_2)]/2JSq, \tag{17}$$

and N is the number of spins. This solution exists at field values meeting the condition

$$|\gamma+[2\mu H-\alpha(S-{}^1/_2)]/2JSq|\leqslant 1. \tag{18}$$

This inequality coincides with the condition for the absence of a gap in the spectrum of states with one flipped spin above the ferromagntic state. For fields outside the region (16), the solution (16) is unbounded in the case of an infinite change, and in the presence of boundaries it will not satisfy the boundary conditions.

In the one-dimensional case the solution (16) becomes

$$A_n=(2S/N)^{1/2}e^{in\varphi}. \tag{16'}$$

For a closed finite chain ($N+1\equiv 1$), the periodicity requirement singles out the admissible set of values of the phase φ:

$$\varphi_l=2\pi l/N\quad (l=0,\ 1,\ldots,\ N-1). \tag{19}$$

In appropriate fields, the functions (15) satisfy Eq. (12), meaning that they determine an integral of the motion. In the multidimensional case the periodicity requirements in the different dimensions can be compatible only for a crystal of cubic form.

If the condition (19) is not met, the nonconservation of I is due only to edge effects. It can therefore be assumed for a one-dimensional chain with $N\gg 1$ that the solution obtained yields an integral in the entire interval of subcritical fields (18).

The operators I^- and I^+ corresponding to the functions (15) have according to (4) and (14) the form

$$I^-=\sum_m A_mQ_{2S-1}(S_m^z)S_m^-\prod_{j\neq m}P_{2S}(S_j^z),\quad I^+=(I^-)^+.$$

Here

$$Q_{2S-1}(S_m^z)=\sum_{n=0}^{2S-1}a_n(S_m^z)^n,\quad P_{2S}(S_m^z)=\sum_{n=0}^{2S}b_n(S_m^z)^n, \tag{20}$$

and the coefficients a_n ($n=0, 1,\ldots, 2S-1$) and b_n ($n=0, 1,\ldots, 2S$) are respectively solutions of the following systems of linear equations:

$$\sum_{n=0}^{2S-1}a_n(S-p)^n=\delta_{p,1}/2S,\quad p=1,2,\ldots,2S, \tag{21}$$

$$\sum_{n=0}^{2S}b_n(S-p)^n=\delta_{p,0},\quad p=0,1,\ldots,2S. \tag{22}$$

Just as for the system (8), the polynomials $Q_{2S-1}(S_m^z)$ and $P_{2S}(S_m^z)$ can be written in the form

$$Q_{2S-1}(S_m^z)=\frac{1}{(2S)!}\prod_{n=2}^{2S}[S_m^z-(S-n)],$$

$$P_{2S}(S_m^z)=\frac{1}{(2S)!}\prod_{n=1}^{2S}[S_m^z-(S-n)].$$

The obtained integrals of motion are generators of the group of the additional symmetry of the Hamiltonian (11), a symmetry due to the degeneracy of the energy of the "ferromagnetic" state: at field values given by (17) and (19), the same energy is possessed by the one-magnon state. The operators I^+,I^- and $I^z=\frac{1}{2}[I^+,I^-]$ satisfy, just as in the one-spin case considered above, the commutation relations for the angular momentum. The corresponding symmetry group is therefore $SO(3)$ with generators I^+,I^-, and I^z. It is easy to verify that the eigenvalues of each of the components I^x, I^y, and I^z are equal to zero in all stationary cases, except the "ferromagnetic" and one-magnon state having the same energy. The operators I^+ and I^- transform these states into each other, and the eigenvalues of each of the projections are equal to $\pm 1/2$. The properties of the operator $I^\pm$ follow from the fact that the polynomials $P_{2S}(S_j^z)$ and $Q_{2S-1}(S_j^z)$ are respectively projectors on a state with projection S and on a two-dimensional subspace of states with spin components S and $S-1$ along the z axis in the site j.

The integrals of motion, which are polynomials of higher degrees in z in the coherent-states representation, are generators of a symmetry group that corresponds to equality of the energy level of the "ferromagnetic" state to one of the levels of the r-magnon state (r is the power of the polynomial). Finally, polynomials with specified powers of z and z^* in each term define generators that correspond to overlap of the bands of the r-magnon and l-magnon states (r is the power of z and l the power of z^*. In the general case of the Hamiltonian (11), the determination of the "higher" integrals calls for the solution of a system of differential equations, which reduces in the simplest case $S=1/2$ to the equation of the Bethe Ansatz. In the particular case of an XY chain ($\gamma=0$) one can obtain for $S=1/2$ explicit expressions for all the integrals of motion, as well as relations that determine the regions of their existence. By way of example, we present the expression for the generators of the symmetry group generated by equality of the energies of the ferrormagnetic and two-magnon states:

$$I_{02}^-=\frac{1}{N}\sum_{m,n}A_{mn}S_m^-S_n^-\prod_{j\neq m,n}S_j^+S_j^-, \tag{23}$$

where

$$A_{mn}=\{\exp[i(k_1m+k_2n)]-\exp[i(k_1n+k_2m)]\}\operatorname{sgn}(n-m),$$

and the wave numbers k_1 and k_2 are connected by the equation

$$J(\cos k_1+\cos k_2)-4\mu H=0, \tag{24}$$

which means equality of the energies of the ferromagnetic and two-magnon states. In the case of a finite closed chain, both wave numbers satisfy the condition $\exp(ikN)=-1$, so that (24) determines the field in which (23) is an integral of the motion. Clearly, just as in the case of one-magnon states, the integral (23) vanishes at $2\mu H>1$.

An integral of motion of another type corresponds to overlap of the magnon bands. For example, if $r=2$ and $1=1$ we have

$$I_{12}^-=\frac{1}{N^{3/2}}\sum_{m,n,p}A_{m,n;p}S_m^-S_n^-S_p^+\prod_{j\neq m,n,p}S_j^+S_j^-, \tag{25}$$

where

$$A_{m,n;p}=\{\exp[i(k_1m+k_2n)]-\exp[i(k_1n+k_2m)]\}\times\exp(ik_3p)\operatorname{sgn}(n-m),$$

and the wave numbers k_1, k_2, and k_3 are connected by the condition that the energies of the one-magnon and two-magnon states be equal:

$$J(\cos k_1+\cos k_2-\cos k_3)-2\mu H=0.$$

If $r=l$, the field drops out of the condition that the energies be equal, so that there are no constraints whatever on the field strength. Since there is no band overlap at $2\mu H>J$, the existence of an integral in this field region is not connected with an additional symmetry of the quantum system. This means that integrals of this type are of classical origin and at $S\gg 1$ their classical Poisson bracket with the Hamiltonian is zero.

Thus, a quantum spin system with Hamiltonian (11) has in the field region (18) a set of integrals of motion I_{lr}^-, I_{lr}^+, I_{lr}^z, that correspond to overlap of r-magnon and l-magnon bands ($r\neq l$) and vanish in the classical limit. These integrals of motion can be used to classify stationary states with overlap of the magnetic-excitation bands. If on knows the set of quantum numbers that number the states in a subspace with specified value of the projection S_z of the total spin, the eigenvalues ($\pm 1/2$) of the projection I_{lr}^x or I_{lr}^y define a linear combination of vectors with l and r excitations that have equal energies.

[1)] Note that this Hamiltonian is obtained from (1) is the Holstein-Primakoff approximation[4] $S_z=S-a^+a$ is used for the operator S_z.

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