Diffraction of nonlinear spatially incoherent wave

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The diffraction of an intense spatially incoherent electromagnetic wave by a slit placed in front of a nonlinear medium is considered within the framework of the nonlinear parabolic equation. The average intensity of the diffracted radiation in the Fraunhofer zone is calculated by the inverse-problem method for two limiting cases, weak and strong incoherence of the incident wave, both for a focusing medium and for a defocusing one.

1. INTRODUCTION

A high-power electromagnetic pulse propagating through a medium usually alters the optical density of the latter, since the dependence of the refractive index on the field amplitude becomes substantial. The ensuing inhomogeneity of the region leads either to self-contraction of the beam or to its broadening. A theoretical explanation and a qualitative interpretation of such effects were first considered in Refs. 1 and 2 (see also Refs. 3, 4, and 5).

In the simplest case, the effects of self-action of an electromagnetic wave are described by a cubic nonlinearity. The corresponding nonlinear increment to the induction vector in the case of a linearly or circularly polarized wave is usually written in the form[5]

$$\vec{\mathscr{D}}^{(nl)} = \varepsilon_2 |\vec{\mathscr{E}}|^2 \vec{\mathscr{E}}. \tag{1.1}$$

If $\varepsilon_2 > 0$, the phase velocity of plane monochromatic waves decreases with increasing amplitude (focusing medium). In a defocusing medium ($\varepsilon_2 < 0$) the phase velocity increases with increase of amplitude.

We consider the influence of the nonlinearity of the medium on a monochromatic wave propagating along the z axis:

$$\vec{\mathscr{E}} = {}^1/_2 \mathbf{e}\{E(z, \mathbf{r}_\perp)\exp[i(\omega t - kz)] + \text{c.c.}\}, \tag{1.2}$$

where $E(z,\mathbf{r}_\perp)$ is a slowly varying function of the coordinates (the so-called stationary focusing), and $\mathbf{r}_\perp$ is a vector perpendicular to the propagation direction. Substituting (1.1) in Maxwell's equations with account taken of (1.2), we arrive in the principal approximation at the equation[1–5]

$$2ik\partial E/\partial z + \Delta_\perp E = -k^2(\varepsilon_2/\varepsilon_0)|E|^2 E. \tag{1.3}$$

The solutions of Eq. (1.3) that describes the focusing are the subject of an appreciable number of studies (see, e.g., Refs. 1, 2, and 6–10). If the field E depends only on two variables (z and x), that stationary-focusing problem described by (1.3) admits of an exact solution, for in this case (1.3) is the known nonlinear Schrödinger equation (NSE), which is exactly solvable by the method of the inverse scattering problem.[8]

Within the framework of Eq. (1.3) it is possible to investigate exactly not only effects peculiar to essentially nonlinear media, but also classical (linear) effects, such as Fraunhofer diffraction. The use of the powerful formalism of the inverse-scattering problem, proposed for the NSE by Zakharov and Shabat,[8] has enabled Manakov[9] to obtain an exact expression for the direction distribution of the diffracted-radiation intensity for wave diffraction by a slotted screen placed in front of a nonlinear medium.

At present, however there are no accurate results whatever for the propagation of a high-power incoherent pulse in a nonlinear medium, although problems of this kind were posed quite long ago (see, e.g., Refs. 3 ad 10).

Mathematically, the problem of propagation of an incoherent radiation pulse from a random source in a nonlinear medium reduces to the Cauchy problem for a nonlinear partial differential equation with random initial conditions.

We consider in this paper, as a specific physical version of the problem, Fraunhofer diffraction of a monochromatic spatially incoherent wave by a slit, with allowance for the nonlinearity of the medium behind the screen. In Sec. 2 we consider nonlinear diffraction, in the Fraunhofer zone, of a wave whose amplitude contains on top of a regular component also a small random component that describes the spatial incoherence of the field in the plane of the screen. Using a perturbation theory based on the inverse-problem method, we obtain the correction to the average intensity of the diffracted radiation, and determine the diffraction-pattern changes due to the presence of the weak coherence of the incident wave, for two cases—focusing and defocusing media. The third section of the paper is devoted to diffraction of an essentially incoherent wave. We assume for simplicity that the incident wave has no regular component at all, and investigate in this case the Zakharov-Shabat spectral problem, connected with Eq. (1.3) in the inverse-problem method, with a random potential on a segment of length L. The probability density for the scattering coefficient, which determines the distribution of the average intensity of the diffracted radiation, is determined for a number of limiting cases.

2. DIFFRACTION OF A PARTIALLY INCOHERENT BEAM

We introduce a new variable $\tau = z/2k$ and denote $k^2\varepsilon_2/\varepsilon_0$ by $\varkappa$. Equation (1.3) takes then in the two-dimensional case (1.3) the form

$$i\partial E/\partial\tau + \partial^2 E/\partial x^2 + \varkappa|E|^2 E = 0. \tag{2.1}$$

We choose the initial conditions for (2.1) such that they describe the diffraction of an electromagnetic field by a slit. Let the screen be located in th $z = 0$ plane, and let the slit be the strip $0 < x > L$. The field in the plane of the screen is assumed given:

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$$E(x,0)=\begin{cases}0, & x<0, \quad x>L,\\ E(x), & 0<x<L.\end{cases} \tag{2.2}$$

According to the inverse-scattering-problem method,[8,11] to deduce $E(x,z)$ from the initial value $E(x,0)$ we must solve an auxiliary scattering problem. The NSE is associated with the Zakharov-Shabat spectral problem

$$\begin{aligned}\partial\psi_1/\partial x&=i\lambda\psi_1+ir^*(x)\psi_2\,\mathrm{sgn}\,\varkappa,\\ \partial\psi_2/\partial x&=-i\lambda\psi_2+ir(x)\psi_1\end{aligned} \tag{2.3}$$

for the column function $\psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$, where λ is a spectral parameter, and the function $r(x)$ is simply related with the initial condition: $r(x)=(|\varkappa|/2)^{1/2}E(x,0)$. The solution of the spectral problem (2.3), as is usual in scattering problems, is represented by the so-called scattering data for a continuous (real λ) and a discrete (complex λ) spectrum. The principal role in the determination of the scattering data is assumed by the normalized solutions of the system (2.3), viz., the Jost functions, which are introduced in terms of the asymptotics

$$\begin{aligned}\psi(x,\lambda)&\to\begin{pmatrix}e^{i\lambda x}\\ 0\end{pmatrix}, \quad x\to+\infty,\\ \varphi(x,\lambda)&\to\begin{pmatrix}0\\ e^{-i\lambda x}\end{pmatrix}, \quad x\to-\infty.\end{aligned} \tag{2.4}$$

The behavior of the Jost functions at $-\infty$ and $+\infty$ is characterized respectivity by two complex functions of λ—the scattering data $a(\lambda)$ and $b(\lambda)$:

$$\begin{aligned}a(\lambda)&=\psi_1(x,\lambda)\varphi_2(x,\lambda)-\psi_2(x,\lambda)\varphi_1(x,\lambda),\\ b(\lambda)&=\psi_1^*(x,\lambda)\varphi_1(x,\lambda)+\psi_2^*(x,\lambda)\varphi_2(x,\lambda).\end{aligned}$$

We consider hereafter only a situation in which the coefficient is not equal to zero for any λ, i.e., there is no discrete spectrum.[11] The determination of the solution $r(x,\tau)$ from the initial condition is the subject of the inverse-scattering problem, on which we shall not dwell in detail, referring the reader to the original paper[8] and to Ref. 11 (pp. 247–251). We shall use hereafter only one result of the inverse-problem method. This result enables us to express the asymptote of the function $|r(x,\tau)|^2$ at large values of τ in terms of the scattering coefficient $a(\lambda)$ (Refs. 9 and 11):

$$|r(x,\tau)|^2=\frac{1}{4\pi\tau}\ln\left|a\left(-\frac{x}{4\tau}\right)\right|^{-2\,\mathrm{sgn}\,\varkappa}. \tag{2.5}$$

As indicated in Ref. 11, one can determine from (2.5) the radiation-intensity distribution in nonlinear Fraunhofer diffraction by a slit. Since Eq. (1.3) itself is valid only for diffraction by small angles θ, the intensity of the radiation diffracted in the angle interval from θ to $\theta+d\theta$ can be written in the form

$$dI(\theta)/d\theta=(k/\pi\varkappa)\,(\ln|a(\lambda)|^{-2})_{\lambda=-k\theta/2}. \tag{2.6}$$

For Fraunhofer diffraction in a focusing medium ($\varkappa>0$), when the amplitude of the incident electromagnetic wave is constant, $E(x)=E_0=\mathrm{const}$ (the Manakov problem[9]), the solution of the scattering problem yields

$$|a(\lambda)|^2=(\lambda^2+|r_0|^2\cos^2\chi L)(\lambda^2+|r_0|^2)^{-1}, \tag{2.7}$$

where

$$\chi=\chi(\lambda)=(\lambda^2+|r_0|^2)^{1/2}, \tag{2.8}$$

$$r_0=(|\varkappa|/2)^{1/2}E_0. \tag{2.9}$$

Therefore

$$\frac{dI(\theta)}{d\theta}=\frac{k}{\pi\varkappa}\ln\left(\frac{k^2\theta^2/4+|r_0|^2}{k^2\theta^2/4+|r_0|^2\cos^2[L(k^2\theta^2/4+|r_0|^2)^{1/2}]}\right). \tag{2.10}$$

This equation is valid if the initial conditions do not lead to waveguide channels (NES solitons), i.e., so long as the integrated intensity $I_{\rm inc}=I_0=|E_0|^2L$ is less than[9]

$$I_{\rm cr}=\frac{\pi^2}{2L}\frac{\varepsilon_0}{\varepsilon_2k^2}. \tag{2.11}$$

Focusing medium

We proceed now to investigate, within the framework of the nonlinear parabolic equation (1.3), the diffraction of a random electromagnetic field. It is natural to consider first the case of a partially incoherent beam, when the complex envelope of the electric field in the screen plane ($z=0$) has besides the stationary component E_0 also a small component $\varepsilon(x)$ that is a random function with paired correlators

$$\langle\varepsilon(x)\varepsilon(x')\rangle=\sigma_1^2B_{l_1}(x-x'), \tag{2.12}$$

$$\langle\varepsilon(x)\varepsilon^*(x')\rangle=\sigma_2^2D_{l_2}(x-x'), \tag{2.13}$$

where the subscripts l_1 and l_2 were introduced to denote the correlation radii, and the angle brackets stand for averaging over all the realizations of the random function $\varepsilon(x)$. We note that the functions $B_{l_1}(\xi)$ and $D_{l_2}(\xi)$ must satisfy the obvious relations

$$B_{l_1}(\xi)=B_{l_1}(-\xi), \quad D_{l_2}(\xi)=D_{l_2}^*(-\xi).$$

The weak-incoherence condition means that perturbation theory can be used in the calculation of the properties of the diffraction pattern.

The starting point in the calculation of the radiation intensity in the Fraunhofer zone is the general equation (2.6). Recognizing that the Jost coefficient is a functional of the amplitude of the electromagnetic field incident on the slit, i.e.,

$$a=a\{r_0+\eta,\ r_0^*+\eta^*\},$$

where

$$\eta\equiv\eta(x)=(\varkappa/2)^{1/2}\varepsilon(x), \tag{2.14}$$

the function $f=\ln|a(\lambda)|^{-2}$ can be expanded in a functional Taylor series in powers of $\eta(x)$ and $\eta^*(x)$. After averaging over the realizations of the random function [$\eta(x)$ and $\eta^*(x)$], we obtain the average intensity of the diffracted radiation in the form

$$\langle dI(\theta)/d\theta\rangle=(dI(\theta)/d\theta)_0+(2k/\pi\varkappa^2)G(\theta). \tag{2.15}$$

Here $(dI/d\theta)_0$ is the coherent-diffracted-field intensity, which takes the form (2.10) at $\varkappa>0$, and the correction accounts for the incoherence of the radiation and is determined by the random-function correlators:

$$G(\theta)=G(\lambda)|_{\lambda=-k\theta/2},$$

$$G(\lambda)=-\mathrm{Re}\int_0^L dx\int_0^L dx'\left\{\sigma_1^2\left[\frac{1}{a^2(\lambda)}\psi_1^2(x)\varphi_1^2(x')\right.\right.$$

$$+\frac{1}{a^{*2}(\lambda)}\psi_2^{*2}(x)\varphi_2^{*2}(x')\Big]B_{l_1}(x-x')$$

$$-\sigma_2^2\Big[\frac{1}{a^2(\lambda)}\psi_1^2(x)\varphi_2^2(x')$$

$$+\frac{1}{a^{*2}(\lambda)}\psi_2^{*2}(x)\varphi_1^{*2}(x')\Big]D_{l_2}(x-x')\Big\} \quad (2.16)$$

We use now the explicit forms of the Jost functions, obtained directly by solving the scattering problem (2.3), and make the change of variables

$$x-x'=\xi,\quad x+x'=\zeta.$$

The integral with respect to ζ can then be easily calculated, and ultimately the general expression for the correction to the average intensity of the diffracted radiation is

$$G(\lambda)=\mathrm{Re}\int_0^L d\xi\{\sigma_1^2A_0(\xi)B_{l_1}(\xi)+\sigma_2^2A_1(\xi)D_{l_2}(\xi)\}. \quad (2.17)$$

Here

$$A_0(\xi)=\frac{1}{a^2(\lambda)}e^{2i\lambda L}Q(L-\xi)+\frac{1}{a^{*2}(\lambda)}e^{-2i\lambda L}Q(\xi-L), \quad (2.18)$$

$$A_1(\xi)=\frac{1}{a^2(\lambda)}e^{2i\lambda L}P(\xi-L)+\frac{1}{a^{*2}(\lambda)}e^{-2i\lambda L}N(\xi-L), \quad (2.19)$$

where

$$Q(z)=\frac{r_0^{*2}}{4\chi^4}\Big\{\frac{|r_0|^2-2\lambda^2}{2\chi}\sin 2\chi z+4i\lambda\sin^2\chi z$$

$$-2z[|r_0|^2-{}^1/_2(\chi^2+\lambda^2)]\cos 2\chi z+i\chi\lambda\sin 2\chi z\Big\}, \quad (2.20)$$

$$P(z)=\frac{1}{4\chi^4}\Big\{-2z[|r_0|^4+{}^1/_2(\chi^2+\lambda^2)^2\cos 2\chi z$$

$$+2i\chi\lambda(\chi^2+\lambda^2)\sin 2\chi z$$

$$+2\chi^2\lambda^2\cos 2\chi z]+\frac{1}{2\chi}[4\chi^2\lambda^2-(\chi^2+\lambda^2)(2\lambda^2+5|r_0|^2)]$$

$$\times\sin 2\chi z-8i\lambda|r_0|^2\sin^2\chi z\Big\}, \quad (2.21)$$

$$N(z)=\frac{|r_0|^4}{4\chi^4}\Big\{\frac{3}{2\chi}\sin 2\chi z-z[1+2\cos^2\chi z]\Big\}. \quad (2.22)$$

It is easy to verify that the correction $G(\theta)$ to the average intensity of the diffracted radiation alters the intensity at the point of the principal diffraction maximum[9]

$$(dI(\theta)/d\theta)_{max}=(k/\pi\varkappa)\ln\{\cos^{-2}[(I_0\varkappa L/2)^{1/2}]\}$$

by an amount

$$\frac{2k}{\pi\varkappa^2}\sigma_2^2(|r_0|\cos^2|r_0|L)^{-1}\int_0^L d\xi\,F[|r_0|(\xi-L)]D_{l_2}(\xi), \quad (2.23)$$

where

$$F(x)=\sin 2x-{}^1/_2x(1+2\cos^2x).$$

For example, the value of (2.23) for the correlation function

$$D_{l_2}(\xi)=(1/2l_2)\exp(-|\xi|/l_2) \quad (2.24)$$

is easily calculated in explicit form and becomes equal to

$$\frac{2k\sigma_2^2}{\pi l_2\varkappa^{5/2}E_0}\cos^{-2}\Big[\Big(\frac{\varkappa}{2}\Big)^{1/2}E_0L\Big]$$

$$\times\Big\{\frac{4(l_2-1)}{4+\varkappa^2l_2^2E_0^2}\sin\Big[(2\varkappa)^{1/2}E_0L+\frac{\pi}{4}\Big]$$

$$+\Big(\frac{\varkappa}{2}\Big)^{1/2}E_0\Big(L-\frac{l_2}{2}\Big)$$

$$-\exp\Big(-\frac{2L}{l_2}\Big)\Big(2^{3/2}+\varkappa^{1/2}E_0l_2-\frac{1}{4}\varkappa^{3/2}l_2^3E_0^3\Big)$$

$$\times(4+\varkappa^2l_2^2E_0^2)^{-1}\Big\}.$$

In addition, the locations of the diffraction maxima and minima are changed respectively by amounts

$$\Delta\theta_{max}^{(n)}=-\frac{2k}{\pi\varkappa^2}G'(\theta_{max}^{(n)})\Big/\Big(\frac{dI}{d\theta}\Big)''\Big|_{\theta=\theta_{max}^{(n)}},$$

$$\Delta\theta_{min}^{(n)}=-\frac{2k}{\pi\varkappa^2}G'(\theta_{min}^{(n)})\Big/\Big(\frac{dI}{d\theta}\Big)''\Big|_{\theta=\theta_{min}^{(n)}},$$

where $\theta_{max}^{(n)}$ and $\theta_{min}^{(n)}$ are the positions, determined from (2.10), of the maxima and minima of the radiation.

Since the instant of waveguide channel (NSE soliton) formation is determined by the onset of a singularity in the first term of (2.15), the criterion for the formation of the waveguide channel in terms of the dc component of the incident-wave amplitude remains the same as before, $I_0=|E_0|^2L<I_{cr}$, where I_{cr} is defined in (2.11). The average intensity of the incident radiation, however, is now

$$\langle I_{inc}\rangle=\int_0^L\langle|E(x,0)|^2\rangle dx=|E_0|^2L+\sigma_2^2D_{l_2}(0)L. \quad (2.25)$$

At average incident-wave intensity, the condition for formation of a waveguide channel is therefore

$$\langle I_{inc}\rangle<I_{cr}+\sigma_2^2D_{l_2}(0)L. \quad (2.26)$$

The validity of the employed perturbation theory is governed by the extent to which the second term in (2.26) is small compared with the first. Our equations (2.16)–(2.22), as well as the correction (2.23) to the principal maximum, are therefore valid at

$$\sigma_2^2D_{l_2}(0)\ll|E_0|^2.$$

The presence of weak incoherence of the diffracted beam leaves thus the overall picture of the nonlinear Fraunhofer diffraction unchanged. Allowance for this incoherence alters the form of the diffraction dependence of $\langle dI(\theta)/d\theta\rangle$, as well as the condition for the formation of the waveguide channels. The calculations demonstrate, however, the general character of the change in the diffraction pattern when the incident-field incoherence is allowed for, viz., the directions of the shifts of the diffraction minima and maxima, the broadening of the diffraction peaks, and the strengthening of the conditions for waveguide-channel formation.

Defocusing medium

The medium is defocusing if the nonlinearity parameter $\varkappa$ in (1.3) is negative. Accordingly, at $\varkappa<0$, the squared modulus of the Jost coefficient $a(\lambda)$, which determines as before the intensity of the diffracted radiation, is

$$|a(\lambda)|^2=[\lambda^2-|r_0|^2\cos^2\tilde{\chi}(\lambda)L]/(\lambda^2-|r_0|^2), \quad (2.27)$$

where

$$\tilde{\chi}(\lambda)=(\lambda^2-|r_0|^2)^{1/2}. \quad (2.28)$$

Taking into account all the changes necessitated by the negative sign of the nonlinearity in (1.3), we can formulate the final result as follows: The average intensity of the diffracted radiation is determined by Eq. (2.15), where $(dI(\theta)/d\theta)_0$ for a defocusing medium must be taken to mean Eq. (1.6) with the reflection coefficient defined by (2.27), while the correction $G(\theta)$ needed to account for the weak incoherence of the diffracted wave is described by Eqs. (2.16)–(2.22), in which we must substitute $\sigma_1^2 \to -\sigma_1^2$, $\chi(\lambda)\to\tilde{\chi}(\lambda)$, $\tilde{\chi}(\lambda)$ is defined in (2.28).

3. NONLINEAR DIFFRACTION OF THE RANDOM FIELD

The perturbation theory used above is restricted to sufficiently small values of the random-field variance. For correlation functions such as (2.24) (when the variance is inversely proportional to the field-correlation radius) this imposes a lower bound on the field correlation radius in the screen plane. To investigate the evolution of the large-variance beams we must solve exactly a stochastic nonlinear problem. Since the propagation of a quasimonochromatic wave in a nonlinear medium is described by an exactly integrable NSE, the inverse-scattering problem yields, generally speaking, the solution for an arbitrary realization of the field in the screen plane. What is important in this case is that the intensity distribution in the screen plane (which serves as the initial condition) is limited by the slit dimensions, i.e., we can use for it the inverse-scattering-problem method in the classical formulation. The random initial conditions can be governed by the incoherent character of the source, for example by partial incoherence of the laser pulse, or by the randomness of the screen (as the beam passes through a narrow layer of a randomly inhomogeneous medium). For propagation of an incoherent pulse, interest attaches to its statistical properties. We confine ourselves to calculation of the mean beam intensity in the Fraunhofer zone.

We turn to the inverse-scattering-problem method. The solution of the NSE is determined by scattering properties of the corresponding Zakharov-Shabat system (2.3), where the potential $r(x)$ should be taken to be the initial condition of the problem (the intensity distribution in the screen plane). In the present section we assume for simplicity that the incident wave has no regular component at all, and consider the diffraction of an essentially incoherent field, when $r(x)$ is a random function.

The nonlinear-diffraction problem can be solved by the following procedure. The distribution of the beam in the screen region is used to determine the statistics of the scattering properties of the Zakharov-Shabat system. The inverse-problem method is used to reconstruct from them the statistics of the NSE solutions, after which the radiation characteristics in the Fraunhofer zone are calculated.

We seek the solution of (2.3) in the form

$$\begin{aligned} &A_L\varphi(x-L,\lambda)+B_L\psi(x-L,\lambda), \quad x>L,\\ &A(x)\varphi(x,\lambda)+B(x)\psi(x,\lambda), \quad 0<x<L,\\ &A_0\varphi(x,\lambda), \quad x<0, \end{aligned}$$

where φ and ψ are defined in (2.4). It is easily verified that the functions $A(x)$ and $B(x)$ satisfy respectively the equations

$$\begin{aligned} dB/dx&=ir^*(x)e^{-2i\lambda x}A(x)\ \mathrm{sgn}\ \varkappa,\\ dA/dx&=ir(x)e^{2i\lambda x}B(x) \end{aligned} \quad (3.1)$$

with boundary conditions

$$A_L=A(L)e^{-i\lambda L},\quad B_L=B(L)e^{i\lambda L},\quad x=L;$$
$$A(0)=A_0,\quad B(0)=0,\quad x=0.$$

We introduce an auxiliary reflection coefficient

$$R(x)=B(x)A^{-1}(x)e^{2i\lambda x}. \quad (3.2)$$

According to the "immersion" method, it is easy to derive for the reflection coefficient $R(L)$ (L is regarded here as the argument) the equation

$$dR(L)/dL=2i\lambda R(L)+ir^*(L)\,\mathrm{sgn}\,\varkappa-ir(L)R^2(L) \quad (3.3)$$

with initial condition $R(L=0)=0$.

Equation (3.3) is the fundamental for the study of the diffraction of essentially incoherent radiation in a nonlinear medium.

Focusing medium

Consider the problem with real initial conditions. This case corresponds to propagation of a linearly polarized wave. We assume the field in the plane of the screen to the Gaussian with statistical characteristics

$$\langle r(L)\rangle=0,\quad \langle r(L)r(L+\xi)\rangle=2\sigma^2\delta(\xi). \quad (3.4)$$

To find the statistics of the scattering data, we consider Eq. (3.3). It is convenient to write the reflection coefficient in the form

$$R(L)=\mathscr{R}(L)e^{i\varphi(L)}, \quad (3.5)$$

inasmuch as by virtue of (2.6) the average beam intensity in the Fraunhofer zone is determined by the modulus of the reflection coefficient, i.e., by the function $\mathscr{R}(L)$. We are consequently interested in the one-parameter probability density. Substituting

$$\Psi=\mathrm{arctg}\,\mathscr{R},\quad \vartheta=\varphi-2\lambda L-\pi/2 \quad (3.6)$$

we obtain from (3.3) and (3.5) the system of equations

$$d\Psi/dL=r(L)\cos(2\lambda L+\vartheta), \quad (3.7a)$$
$$d\vartheta/dL=-2r(L)\sin(2\lambda L+\vartheta)\,\mathrm{ctg}\,2\Psi. \quad (3.7b)$$

The system (3.7) is analyzed by known methods.[12] We introduce the probability density

$$P(x_1,x_2;L)=\langle\delta(\Psi-x_1)\delta(\vartheta-x_2)\rangle, \quad (3.8)$$

where, as before, the angle brackets denote averaging over all the realizations of the random function $r(L)$. Using the standard procedure, we obtain an equation for $P(x_1,x_2;L)$:

$$\frac{\partial P}{\partial L}=\sigma^2\frac{\partial}{\partial x_i}\left[\mathscr{D}_i\frac{\partial}{\partial x_k}(\mathscr{D}_kP)\right],\quad i,k=1,2, \quad (3.9)$$

where

$$\mathscr{D}_1=\cos(2\lambda L+x_2),\quad \mathscr{D}_2=-2\sin(2\lambda L+x_2)\,\mathrm{ctg}\,2x_1.$$

It is impossible to find an exact solution of (3.9). We analyze the solution by using two facts mentioned above: first, since we are interested in nonlinear Fraunhofer diffraction, we put $\lambda L \ll 1 < \theta \ll 1$); second, the use of the variables introduced in (3.5) allows us to confine ourselves to one-parameter probability density for x_1. Integration with respect to x_2 reduces (3.9) to the form

$$\frac{\partial P_1}{\partial L} - \sigma^2 \frac{\partial^2 P_1}{\partial x_1^2}$$

$$= \sigma^2 \left\{ \frac{1}{2} \frac{\partial^2}{\partial x_1^2} + \operatorname{ctg} 2x_1 \frac{\partial}{\partial x_1} - 2 \sin^{-2} 2x_1 \right\} (\tilde{P} - P_1). \quad (3.10)$$

We have introduced here the notation

$$P_1 = \int_0^\pi dx_2\, P(x_1, x_2; L),$$

$$\tilde{P} = \int_0^\pi dx_2 \cos[z(2\lambda L + x_2)] P(x_1, x_2; L),$$

where P_1 is the probability density of interest to us. We solve the resultant equation by iteration. To this end, we consider first the probability density $P_1(0)$ for the intensity on the axis, which describes the radiation diffracted through zero angle ($\lambda = 0$). It is easily seen that a solution of (3.3) with allowance for (3.5) is

$$\mathcal{R} = \operatorname{tg} \left| \int_0^L r(x)\, dx \right|, \quad \varphi = \frac{\pi}{2} \operatorname{sgn} \int_0^L r(x)\, dx. \quad (3.11)$$

The probability density for $\mathcal{R}$ is determined from (3.9). At $P = P_1$ we have, with allowance for the initial condition $(\partial P / \partial \mathcal{R})_0 = 0$ and the boundary condition $P(L = 0) = \delta(x_1)$,

$$P_1^{(0)}(x_1; L) = \frac{2}{\pi} \sum_{-\infty}^{\infty} \exp[-4n^2\sigma^2 L] \cos 2nx_1. \quad (3.12)$$

A general expression for the average intensity is

$$\left\langle \frac{dI(\theta)}{d\theta} \right\rangle = -\frac{k}{\pi \varkappa} \int_0^{\pi/2} dx_1 \ln[\cos^2 x_1] P_1^{(0)}(x_1; L). \quad (3.13)$$

Expression (3.12) therefore determines completely the intensity of the radiation diffracted through zero angle:

$$\left\langle \frac{dI(\theta)}{d\theta} \right\rangle_{\theta=0} = \frac{2k}{\pi \varkappa} \left[\ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \exp(-4n^2\sigma^2 L) \right]. \quad (3.14)$$

It is curious that a similar analysis of random fields with arbitrary correlation radii leads to a result obtained from (3.14) by the simple substitution

$$L \to \int_0^L d\xi \int_0^\xi B(\xi - \xi')\, d\xi',$$

where $B(\xi - \xi') = \langle r(\xi) r(\xi') \rangle$, and take consequently into account the dependence of the average intensity on the beam-inhomogeneity scale.

It is easy to obtain from (3.14) in the linear limit ($I_0 L \varkappa \to 0$) the well known value of the intensity of diffracted radiation propagating in a linear medium:

$$\langle dI(\theta)/d\theta \rangle_{\theta=0} \to k I_0 L / 2\pi,$$

where $I_0 = 4\sigma^2/\varkappa$ is the incident-wave intensity determined from the condition $\langle E(x) E(x') \rangle = I_0 \delta(x - x')$.

In the limiting case of strong incoherence ($I_0 L \varkappa \gg 1$) the sum in (3.14) is exponentially small compared with the first term in the square brackets. We get therefore

$$\langle dI(\theta)/d\theta \rangle_{\theta=0} \to (2k/\pi\varkappa) \ln 2.$$

We proceed now to study small-angle diffraction ($\theta \neq 0$). At $\lambda L \ll 1$ we must substitute in the right-hand side of (3.10) the unperturbed value of the probability density. As a result we obtain a solution with an initial condition $P_{1|L=0}$:

$$P_1^{(1)}(x_1; L) = -\frac{16\lambda^2}{\pi \sigma^4} \sum_{n=1}^{\infty} 2n \cos(2nx_1) \exp[-(2n\sigma)^2 L]$$

$$\times \left\{ -\frac{L^3 \sigma^6}{3}(n-1) + \sum_{k=1}^{n-1} \exp[-16(k-n)\sigma^2 kL] \left(-\frac{L^2\sigma^4}{16k(k-n)} - \frac{2\sigma^2 L}{[16k(k-n)]^2} - \frac{2}{[16k(k-n)]^3} \right) + \frac{2}{[16k(k-n)]^3} \right\}. \quad (3.15)$$

With (3.15) taken into account, an expression can be obtained for the intensity of the small-angle diffracted radiation. This expression is too unwieldy to present here.

Defocusing medium

We assume, as before, that $r(x)$ is a Gaussian random function with statistical properties (3.4). It is convenient then to introduce the functions Ψ and ϑ defined as

$$\vartheta = \varphi - 2\lambda L - \pi/2, \quad (3.16)$$

$$\Psi = \operatorname{arcth} \mathcal{R} = {}^1/_2 \ln[(1+\mathcal{R})/(1-\mathcal{R})], \quad (3.17)$$

$\mathcal{R}$ and φ are respectively te amplitude and phase of the reflection coefficient defined by Eq. (3.5).

Taking (3.16) and (3.17) into account, we get from (3.3)

$$d\Psi/dL = -r(L) \cos(2\lambda L + \vartheta), \quad (3.18)$$

$$d\vartheta/dL = 2r(L) \sin(2\lambda L + \vartheta) \operatorname{cth} 2\Psi. \quad (3.19)$$

We introduce the probability density defined by (3.8).

We obtain as before for the one-parameter probability density $P(x_1, L)$ an expression that describes the intensity of the diffracted radiation:

$$\frac{\partial P_1}{\partial L} - \sigma^2 \frac{\partial^2 P_1}{\partial x_1^2}$$

$$= \sigma^2 \left[\frac{1}{2} \frac{\partial^2}{\partial x_1^2} + \operatorname{cth} 2x_1 \frac{\partial}{\partial x_1} - 2 \operatorname{ch}^{-2} 2x_1 \right] (P_1 - \tilde{P}). \quad (3.20)$$

The notation here is the same as in the preceding subsection.

We find first the zero-angle-diffracted radiation. Equation (3.3) is easily solved explicitly:

$$\mathcal{R} = \operatorname{th} \left| \int_0^L r(x)\, dx \right|, \quad \varphi = -\frac{\pi}{2} \operatorname{sgn} \int_0^L r(x)\, dx. \quad (3.21)$$

For $P_1^{(0)}(x_1; L)$ we have correspondingly

$$P_1^{(0)}(x_1;L)=\frac{1}{2\sigma(\pi L)^{1/2}(1-x_1^2)}\times\exp\left\{-\frac{\ln^2[(1+x_1)/(1-x_1)]}{16\sigma^2 L}\right\}, \quad (3.22)$$

and the average intensity is given by

$$\left\langle\frac{dI(\theta)}{d\theta}\right\rangle_{\theta=0}=\frac{k}{\pi|\varkappa|}\frac{1}{\sigma(\pi L)^{1/2}}\int_0^\infty dy\ln(1+\text{th}^2 y)\times\exp\left(-\frac{y^2}{4\sigma^2 L}\right). \quad (3.23)$$

In the limiting case we have from (3.23)

$$\left\langle\frac{dI(\theta)}{d\theta}\right\rangle_{\theta=0}\approx\begin{cases}k\ln 2/\pi|\varkappa|, & \sigma^2 L\gg 1,\\ 2k\sigma^2 L/\pi|\varkappa|, & \sigma^2 L\ll 1.\end{cases}$$

Note that the argument of the logarithm in the integrand of (3.23) does not vanish at any value of y, meaning that no waveguide channels can be produced in a defocusing medium. The analogous equation for a focusing medium contained an integrable singularity.

The probability density describing the non-zero-angle diffracted radiation is

$$P_1^{(1)}(x_1;L)=\frac{\lambda^2}{\pi^{1/2}}\int_0^L dx\frac{x^2\sin^2(2\lambda x)}{(L-x)^{3/2}}\times\int_0^\infty d\xi\left\{(\xi-x_1)\exp\left[-\frac{(\xi-x_1)^2}{4\sigma^2(L-x)}\right]+(\xi+x_1)\exp\left[-\frac{(\xi+x_1)^2}{4\sigma^2(L-x)}\right]\right\}\left[\frac{1}{2}\frac{\partial}{\partial\xi}+\text{cth}\,2\xi\right]P_1^{(0)}(x,\xi).$$

Validity of the approximations employed

We now discuss briefly the validity of the employed approximation with a δ-correlated random field $r(x)$. The validity question is raised for two reasons. First, the δ-correlated field model can strictly speaking not be used to study nonlinear Fraunhofer diffraction, since the condition for the validity of the inverse-scattering-problem method used to obtain the main results is absolute integrability of the function $r(x)$, which is certainly not met for a δ-correlated field. Second, the nonlinear parabolic equation that describes the self-action of a nonlinear wave is applicable for quasi-planar packets. This means that the scale of variation of $E(x)$ in a plane perpendicular to the wave-propagation direction must satisfy the condition $l\ll k^{-1}$. The equations obtained above must therefore be regarded as first-order approximations in the small parameter l/L, where l is the scale of the beam inhomogeneity (the correlation radius) and L is the characteristic length over which the system parameters vary (the slit width). The interval of the modulus of the beam intensity over the entire screen area must therefore be bounded. For the described approach to be valid it is necessary to meet definite compatibility conditions. These are easily obtained from the exact form of the equation for the probability density of a beam with arbitrary inhomogeneity scale, by writing down the conditions under which the scale of variation of the beam correlation function in the screen plane is a minimum in all the integrals contained in this equation. A straightforward but rather cumbersome calculation leads to the condition

$$l\ll\lambda^{-1},\quad L,\ \sigma^{-2}, \quad (3.24)$$

where $\lambda\sim k\theta$ and σ is the intensity of the fluctuations [see (2.4)]. The inequalities (3.24) is thus the condition for the validity of the δ-correlated-field approximation. These inequalities lead finally to the condition for the validity of the proposed theory

$$k^{-1}\ll l\ll\lambda^{-1},\ L,\ \sigma^{-2}.$$

Allowance for the finite correlation radii yields a much more complicated dependence of the diffracted-radiation intensity on the slit width and on the diffraction angle.

4. CONCLUSION

We have considered nonlinear Fraunhofer diffraction of a spatially incoherent wave. Obviously, the questions investigated are but part of the problem of random-field propagation in substantially nonlinear media even in the case of the simplest exactly integrable mode. It remains, for example, to develop a rigorous correlation theory for a nonlinear random field, to calculate the statistical characteristics of the diffracted radiation, to determine the stochastic properties (not only the asymptotics) of the field along the entire propagation direction, and others. Another urgent task, in our opinion, is further development of methods for statistically describing the solutions of nonlinear partial differential equations with random initial conditions, and finding results, similar to those given above, for other nonlinear equations, particularly those for which the inverse-scattering-problem method is applicable. We have in mind here the study of the dynamics of pulses, randomly specified at the initial instant, in essentially nonlinear systems.

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