Magnetization precession of B phase of superfluid ^{3}He in the collisionless region

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Periodic solutions of the Leggett-Takagi equations for the superfluid B-phase of ^{3}He in the collisionless region are obtained and their stability is investigated. In contrast to hydrodynamics, a new stable mode is produced when the quasiparticle spin deviates greatly from the condensate spin. The damping of the magnetization precession is investigated in the region $\omega\tau > 1$. Interpolation equations in the intermediate region $\omega\tau \sim 1$ are also obtained.

1. INTRODUCTION

Pulsed NMR methods have been greatly improved of late and permit highly accurate measurements of the periods and relaxation times of periodic motions of magnetization. It is clear therefore that such periodic solutions play a major role in the general solution of spin-dynamics equations.

The nonlinear equations that describe the spin dynamics of the B-phase of ^{3}He have in fact been solved by now for the region $\omega\tau < 1$ corresponding to the hydrodynamic limit (here τ is the quasiparticle effective relaxation time and ω is a certain characteristic precession frequency that is equal, for different modes, either to the dipole frequency Ω or to the Larmor frequency ω_L). In view of the condition $\omega\tau < 1$, the magnetization produced by the quasiparticles deviates little from equilibrium at the given value of the condensate magnetization, so the deviation of the superfluid component from the normal decreases with the total magnetization.

The motion of magnetization in superfluid ^{3}He is characterized by several frequency scales, viz., the Larmor frequency $\omega_L = \gamma H$ (γ is the gyromagnetic ratio of the ^{3}He nuclei) and the dipole frequencies Ω. Fomin[1] was able, in view of the presence of an additional integral of the motion, to obtain all the periodic solutions of the equations of the ^{3}He B-phase spin dynamics in the hydrodynamic region, at arbitrary ratio of ω_L and Ω, and to investigate their stability and damping. In the strong-field limit $\omega_L \gg \Omega$ one of the solutions goes over into the known asymptotic law for the precession frequency that is significantly dependent on the deviation angle of the order parameter.[2,3] This dependence agrees with experiment.[4] In the opposite case the motion of the magnetization is determined mainly by the dipole forces. An exact periodic solution of the spin dynamic equations at $\omega_L = 0$, called the WP mode, has long been known.[5,6] This mode is also readily observed in experiment.[7]

The effective time τ that describes the equalization of the magnetizations of the quasiparticles and of the condensate and enters in the Leggett-Takagi equation is equal, as shown in Refs. 8–10, to the effective quasiparticle collision time τ, which is given at $T \ll T_c$ by

$$\frac{1}{\tau} \approx \frac{\pi^2}{6}\frac{\Delta}{\hbar}\frac{n_{ex}}{n}\left[\frac{\hbar}{2\pi}\nu_0^2\langle W_s\rangle\right], \qquad (1)$$

where ν_0 is the density of states on the Fermi surface, $\langle W_s\rangle$ is the averaged scattering probability amplitude, Δ is the size of the superfluid gap, and n_{ex} and n are the densities of the quasiparticles and ^{3}He atoms. The expression in the square brackets is a number of the order of unity, and the temperature dependence of τ is determined mainly by n_{ex}, which decreases exponentially with temperature:

$$n_{ex} = \nu_0(2\pi\Delta T)^{1/2}e^{-\Delta/T}.$$

Thus, when the temperature is lowered or the magnetic field is decreased, we land in the region $\omega\tau > 1$. This region can be actually reached in experiment. Thus, extrapolating (1) to the region $T \sim T_c$, we find that $\tau \sim 10^{-7}$ s on the melting curves, and for fields on the order of several hundred oersted we have $\omega\tau \sim 10^{-1}$. However, even at $T/T_c \sim 0.5$ we obtain $\omega\tau \sim 1$ and we go from the hydrodynamic to the collisionless region. This raises the question of describing motion of this kind. Strictly speaking, in the region $\omega\tau > 1$ the quasiparticles should be described by a distribution function, and the form of this function should be obtained by solving the Boltzmann equation.[11] At strong deviations from equilibrium, however, such a description becomes too complicated. A simplified interpolation theory, in which the hydrodynamic and collisionless regions can be described in a unified manner, was developed by Leggett and Takagi.[12] Their equations were developed in the spirit of the Mandel shtam-Leontovich second-viscosity theory[13] and describe the deviations of one variable from equilibrium. Such equations should, at any rate, describe more or less correctly the motion of the magnetization in the high-frequency region. For stationary periodic solutions, which are of experimental interest, this description should lead also to correct quantitative results. The degree of accuracy of the Leggett-Takagi equations is discussed in greater detail in Ref. 12.

In the present paper, in analogy with the procedure of Ref. 1, solutions of the Leggett-Takagi equations are obtained in the collisionless region, the damping and stability of the resultant equations are discussed, and interpolation equations are derived for the region $\omega\tau \sim 1$. The periodic solutions in this region are determined by the joint action of Zeeman, dipole, and Fermi-liquid forces. Up to now, theoretical investigations at arbitrary values of $\omega\tau$ were made only of continuous resonance[11,12] and of nonlinear longitudinal resonance.[11,14] It is therefore of very great interest to compare the predictions of the theory developed in the pres-

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ent paper with new experimental data (especially for NMR relaxation), for up to now the experiments were performed mainly in regions not too far from the critical temperature. The connection between the known hydrodynamic solutions of the WP-mode type and of precession in strong magnetic fields, only the one hand, and our solutions, on the other, is therefore discussed in detail. Extensions are also made to the limiting regions $\omega_L \gg \Omega$ and $\omega_L \to 0$, where the solutions have a simple physical meaning.

2. EQUATIONS OF MOTION

The equations of motion of the spin dynamics of superfluid ^{3}He were derived by Leggett and Takagi by both a macroscopic and a microscopic analysis.[12] They take the form:

$$\dot{\mathbf{S}}=\gamma[\mathbf{SH}]+\mathbf{R}_d, \tag{2}$$

$$\dot{\mathbf{S}}_p=\gamma\left[\mathbf{S}_p, \left\{\mathbf{H}-\gamma\frac{z_0}{4\chi_{n_0}}\mathbf{S}\right\}\right]+\mathbf{R}_d-\frac{1}{\tau}(\mathbf{S}_p-\lambda\mathbf{S}), \tag{3}$$

$$\dot{\mathbf{d}}(\hat{\mathbf{p}})=\left[\mathbf{d}(\hat{\mathbf{p}}), \left\{\gamma\mathbf{H}-\gamma^2\left(\frac{z_0}{4\chi_{n_0}}\mathbf{S}+\frac{1}{\chi_{p_0}}\mathbf{S}_p\right)\right\}\right]. \tag{4}$$

Here $\mathbf{S}$ and $\mathbf{S}_p$ denote the total spin and the spin of the superfluid component per unit volume of ^{3}He (the superfluid component is defined as that part $\mathbf{S}_p$ of the magnetization which is due to the change of form of the elementary-excitation spectrum, while the normal component $\mathbf{S}_q = \mathbf{S} - \mathbf{S}_p$ is the magnetization due to the change of quasiparticle occupation numbers), χ_{p0} (χ_{q0}) are the susceptibilities of the superfluid (normal) components. We present for reference the values of χ_{p0} and χ_{q0} calculated in the weak-coupling theory without allowance for the Fermi-liquid corrections (see Ref. 12):

$$\chi_{p0}=\frac{2}{3}\chi_{n0}\int_0^\infty d\varepsilon\,\frac{\Delta^2}{E_k^2}\,\mathrm{th}\,\frac{E_k}{2T}, \tag{5}$$

$$\chi_{q0}=\chi_{n0}\int_0^\infty d\varepsilon\left(\frac{1}{3}+\frac{2}{3}\frac{\varepsilon^2}{E_k^2}\right)\frac{1}{2T}\,\mathrm{sech}^2\frac{E_k}{2T},$$

$$\mathrm{sech}\,x=1/\mathrm{ch}\,x, \tag{6}$$

where $E_k^2 = \Delta^2 + \epsilon^2$ and $\chi_{n0} = \frac{1}{4}\gamma^2\hbar^2\nu_0^2$ is the susceptibility of a normal Fermi liquid. In addition, we introduce the total susceptibility $\chi_0 = \chi_{p0} + \chi_{q0}$. The quasiparticle susceptibilities tend exponentially to zero as $T \to 0$, while the susceptibility of the superfluid component tends to $2\chi_{n0}/3$.

In the equations of motion (2)–(4), the order parameter $\mathbf{d}(\hat{\mathbf{p}})$ was written in vector representation, the vector $\mathbf{d}$ being a function of the direction $\hat{\mathbf{p}}$ in momentum space. The rotation matrix $\hat{R}_{ik}$, which transforms the momentum space $\hat{\mathbf{p}}$ into $\hat{\mathbf{d}}$, is common to all $\hat{\mathbf{p}}$ and is characterized by the rotation direction $\mathbf{n}$ and by the rotation angle θ in such a way that

$$d_i(\hat{\mathbf{p}})=\hat{R}_{ik}(\mathbf{n}, \theta)\hat{p}_k, \tag{7}$$

$\mathbf{R}_d$ in (2) is the moment of the dipole-dipole forces, and is expressed in terms of the dipole energy U by the relation

$$\mathbf{R}_d=-\mathbf{n}\partial U/\partial\theta. \tag{8}$$

The dipole energy in ^{3}He–B depends only on the angle θ:

$$U={}^8/_5 g_D(\cos\theta+{}^1/_4)^2. \tag{9}$$

The terms in the curly brackets in (3) and (4) refer to the effective magnetic field, which is the sum of the external magnetic field $\mathbf{H}$ and the molecular field $\gamma(z_0/4\chi_{n0})\mathbf{S}$ (z_0 is the Fermi-liquid interaction constant).

If the last term of (3) is disregarded, Eqs. (2)–(4) become a Hamilton system with a Hamiltonian

$$E\{\mathbf{S}, \mathbf{S}_p, \mathbf{d}(\hat{\mathbf{p}})\}=U\{\mathbf{d}(\hat{\mathbf{p}})\}-\gamma\mathbf{SH}+\frac{1}{2}\frac{\gamma^2}{\chi_{p0}}\mathbf{S}_p^2$$
$$+\frac{1}{2}\frac{\gamma^2}{\chi_{q0}}(\mathbf{S}-\mathbf{S}_p)^2+\frac{1}{2}\frac{z_0}{4}\frac{\gamma^2}{\chi_{n0}}\mathbf{S}^2. \tag{10}$$

A process that is irreversible and leads to energy dissipation upon spin relaxation is the equalization of the magnetization of the quasiparticles in the condensate on account of collisions between them (it continues until $\mathbf{S}_p = \lambda\,\mathbf{S}$ is reached). This process is characterized by the effective quasiparticle collision time τ [Eq. (1)] and is the third term in the right-hand side of (3). The coefficient λ is uniquely determined by the requirement that the dissipative function W be positive for the given Hamiltonian (10):

$$W=\frac{\gamma^2\chi_0}{\tau\chi_{p0}\chi_{q0}}\{\mathbf{S}_p-\lambda\mathbf{S}\}^2 \tag{11}$$

and is given by $\lambda = \chi_{p0}/\chi_0$.

It is more convenient to change to dimensionless equations. To this end we multiply (2) by $z_0\gamma^2/4\chi_{n0}$ and (3) by γ^2/χ_{p0}; we introduce the redefined condensate spin $\mathbf{G} = \gamma^2\mathbf{S}_p/\chi_{p0}$, the redefined total spin $\mathbf{S} = \mathbf{S}' = (z_0\gamma^2/4\chi_{n0})\mathbf{S}$, and the dipole frequency $\Omega^2 = 3\gamma^2 g_D/\chi_{p0}$. After these operations, Eqs. (2)–(4) become explicitly dependent only on two parameters:

$$\mu=z_0\chi_{p0}/4\chi_{n0},\quad \mu_1=4\chi_{n0}/z_0\chi_0.$$

As a result, our initial set of equations (2)–(4), the energy (10), and the dissipative function (11) are transformed into

$$\dot{\mathbf{S}}=\gamma[\mathbf{SH}]+\mu\mathbf{R}_d, \tag{12}$$

$$\dot{\mathbf{G}}=[\mathbf{G}, \{\gamma\mathbf{H}-\mathbf{S}\}]+\mathbf{R}_d-\frac{1}{\tau}(\mathbf{G}-\mu_1\mathbf{S}), \tag{13}$$

$$\dot{\mathbf{d}}=[\mathbf{vd}], \tag{14}$$

$$\mathbf{v}=\mathbf{S}+\mathbf{G}-\gamma\mathbf{H}, \tag{15}$$

$$E=\rho_1\{\mu(1-\mu\mu_1)U-(1-\mu\mu_1)\gamma\mathbf{HS}+{}^1/_2\mu\mathbf{G}^2$$
$$+{}^1/_2(\mu_1+1-\mu\mu_1)\mathbf{S}^2-\mu\mu_1\mathbf{SG}\}, \tag{16}$$

$$W=\rho_2\{\mathbf{G}-\mu_1\mathbf{S}\}^2. \tag{17}$$

We have introduced here the notation

$$\rho_1=4\chi_{n0}\chi_0/z_0\gamma^2\chi_{q0},\quad \rho_2=\chi_0\chi_{p0}/\tau\gamma^2\chi_{q0}.$$

Retaining only the second and third terms in the right-hand side of (13) and substituting them in (14) and (15) we obtain the system (1)–(3) of Ref. 1. It was shown there that if $\gamma \to 0$, the system becomes Hamiltonian and has a set of periodic solutions. We shall investigate the opposite case $\tau \to \infty$

and show that the system (12)–(15) also has periodic solutions.

It will be convenient for this purpose to parametrize the order parameters by the Euler angles α, β, and γ defined by the usual relation

$$\hat{R}(\mathbf{n}, \theta)=\hat{R}(\alpha, \beta, \gamma)=\hat{R}_z(\alpha)\hat{R}_y(\beta)\hat{R}_z(\gamma), \qquad (18)$$

where $R_z(\alpha)$ is the matrix of rotation about the $\dot{z}$ axis through and angle α. We direct the $\dot{z}$ axis opposite to $\mathbf{H}$, so that at equilibrium $\mathbf{S}$ is directed along the $\hat{z}$ axis. The unit vectors of the moving coordinate system, which is rigidly connected with the vectors $\mathbf{d}$, by $\hat{\xi}$, $\hat{\eta}$, $\hat{\zeta}$. This coordinate system rotates about the immobile $\hat{x}, \hat{y}, \hat{z}$ with angular velocity $\mathbf{v}$ (Ref. 15):

$$\nu_z=\dot{\alpha}+\dot{\gamma}\cos\beta, \quad \nu_\zeta=\dot{\alpha}\cos\beta+\dot{\gamma}, \quad \nu_\beta=\dot{\beta}. \qquad (19)$$

We have represented here the angular velocity by the components along the axes $\hat{z}$ and $\hat{\zeta}$ and along the axis $\hat{\beta}$ which is perpendicular to the $\hat{z}\hat{\zeta}$ plane. It will be found that the system (12)–(15) of interest to us takes the simplest form precisely in terms of components along these unit vectors $\hat{z}$, $\hat{\zeta}$, and $\hat{\beta}$. First, using (19), we obtain three equations for the motion of the order parameter α, β, γ, expressed in terms of the angles and the spin components. We project next the equations of motion of the spin variables (12) and (13), obtaining thus six more equations. We need for this purpose the relations between the different parametrizations of the order parameter:

$$n_i=-\frac{1}{2\sin\theta}e_{ijk}R_{jk}(\alpha, \beta, \gamma), \qquad (20)$$

$$1+2\cos\theta=\cos\beta+\cos\Phi+\cos\beta\cos\Phi, \qquad (21)$$

where $\Phi=\alpha+\gamma$. The last relation (21) means that the dipole energy (9) depends only on the sum of the angles α and γ:

$$U(\alpha, \beta, \gamma)={}^2/_{15}\Omega^2[\cos\beta-{}^1/_2+(1+\cos\beta)\cos\Phi]^2. \qquad (22)$$

Since the axes $\hat{\zeta}$ and $\hat{\beta}$ are mobile, it is necessary to add to the resultant equations terms that take the unit-vector rotations into account. We ultimately obtain a closed system of nine nonlinear equations that can be solved in terms of the derivatives

$$\dot{\alpha}=-\omega_L+\frac{1}{\sin^2\beta}(G_z-G_\zeta\cos\beta+S_z-S_\zeta\cos\beta), \qquad (23)$$

$$\dot{\beta}=G_\beta+S_\beta, \qquad (24)$$

$$\dot{\Phi}=-\omega_L+\frac{1}{1+\cos\beta}(G_\zeta+G_z+S_\zeta+S_z), \qquad (25)$$

$$\dot{S}_z=-\mu\frac{\partial U}{\partial\Phi}, \qquad (26)$$

$$\dot{S}_\zeta=-\mu\frac{\partial U}{\partial\Phi}+\frac{1}{\sin\beta}\{S_\beta(G_z-G_\zeta\sin\beta)-G_\beta(S_z-S_\zeta\cos\beta)\}, \qquad (27)$$

$$\dot{S}_\beta=-\mu\frac{\partial U}{\partial\beta}+\frac{1}{\sin^3\beta}(S_z\cos\beta-S_\zeta)\times(S_z-S_\zeta\cos\beta+G_z-G_\zeta\cos\beta), \qquad (28)$$

$$\dot{G}_z=-\frac{\partial U}{\partial\Phi}+\frac{1}{\sin\beta}\{S_\beta(G_z\cos\beta-G_\zeta)-G_\beta(S_z\cos\beta-S_\zeta)\}-\frac{1}{\tau}(G_z-\mu_1 S_z), \qquad (29)$$

$$\dot{G}_\zeta=-\frac{\partial U}{\partial\Phi}-\frac{1}{\tau}(G_\zeta-\mu_1 S_\zeta), \qquad (30)$$

$$\dot{G}_\beta=-\frac{\partial U}{\partial\beta}+\frac{1}{\sin^3\beta}(G_\zeta\cos\beta-G_\zeta)\times(S_z-S_\zeta\cos\beta+G_z-G_\zeta\cos\beta)+\frac{1}{\sin\beta}(G_\zeta S_z-S_\zeta G_z)-\frac{1}{\tau}(G_\beta-\mu_1 S_\beta). \qquad (31)$$

The next-to-last terms in (29)–(31) describe Fermi-liquid effects, and the last ones describe the influence of the dissipation. Adding (26) to (30) multiplied by $(-\mu)$, we note that the quantity $P=S_z-\mu S_\zeta$ is conserved if dissipation is neglected. This will be used in Sec. 4 when we consider the damping of the magnetization precession.

3. PERIODIC SOLUTIONS

We neglect dissipation in this section by letting $\tau\to\infty$; the right-hand side of the system (23)–(31) does not depend explicitly on the variable α, and we seek therefore solutions in the form

$$\dot{\alpha}=\text{const},$$
$$\dot{\beta}=\dot{\Phi}=\dot{S}_z=\dot{S}_\zeta=\dot{S}_\beta=\dot{G}_z=\dot{G}_\zeta=\dot{G}_\beta=0. \qquad (32)$$

This type of periodic solutions corresponds to the type considered in Ref. 1 for the hydrodynamic region. The quantity α is a cyclic coordinate, increases linearly with time, and describes the rotation of the order parameter about the $\hat{z}$ axis. Since $\dot{\Phi}=0$, we have here also $\dot{\gamma}=-\dot{\alpha}$.

The vanishing of the right-hand sides of (26) and (30) yields

$$\partial U/\partial\Phi=-{}^8/_{15}\Omega^2(\cos\theta+{}^1/_4)\sin\Phi(1+\cos\beta)=0, \qquad (33)$$

from which we get four possibilities:

I) $\cos\theta=-{}^1/_4$, II) $\Phi=0$, III) $\Phi=\pi$, IV) $\cos\beta=-1$.

From (24) we obtain directly $G_\beta=-S_\beta$; substituting this relation in (27) and (29) we find they require either

$$S_\beta=G_\beta=0, \qquad (34)$$

or

$$G_z-G_\zeta\cos\beta=S_\zeta\cos\beta-S_z, \quad G_z\cos\beta-G_\zeta=S_\zeta-S_z\cos\beta. \qquad (35)$$

Relations (35), however, are compatible only in the special case $\cos^2\beta=1$. The condition $\cos\beta=1$ yields $\beta=0$ and $\mathbf{n}\|\hat{z}$, but it follows then from (8) that two mutually perpendicular vectors add up to zero, which is impossible. If on the other hand $\cos\beta=-1$, then also $\cos\theta=-1$, and this is the condition that the dipole energy U be an extremum, therefore $\mathbf{R}_d=0$. The solution describes rotation of the order parameter with an angular velocity parallel to the $\hat{z}$ axis and equal to $\nu_z=S_z+G_z-\gamma H$. This, however, corresponds to a maximum dipole energy, and the solution considered is expected to be unstable. This argument applies also to case IV.

Thus, Eq. (34) is valid for all the solutions I–IV, meaning that the vectors **S** and **G** do not leave the $z\hat{\zeta}$ plane. The three remaining independent equations (25), (28), and (31) contain five variables S_z, S_ζ, G_z, G_ζ, and β, and determine therefore a two-parameter family of solutions. It will be convenient to use as the parameters the quantities β and S_z. Let us consider in detail the cases obtained in the above cases.

Case I. In this case the dipole energy reaches an absolute minimum, since we have from (22) also

$$\partial U/\partial\beta=0.$$

There is no moment of the dipole forces, and the magnetization motion is determined only by the magnetic fields. Equations (25), (28), and (31) yield in this case

$$G_\zeta+G_z+S_\zeta+S_z=(1+\cos\beta)\omega_L, \quad (36)$$

$$(S_z\cos\beta-S_\zeta)(S_z-S_\zeta\cos\beta+G_z-G_\zeta\cos\beta)=0, \quad (37)$$

$$(G_z\cos\beta-G_\zeta)(S_z-S_\zeta\cos\beta+G_z-G_\zeta\cos\beta) +\sin^2\beta(G_\zeta S_z-G_zS_\zeta)=0. \quad (38)$$

This system has three nontrivial solutions:

$$\text{Ia)}\quad S_\zeta=S_z\cos\beta,\quad G_\zeta=G_z\cos\beta,\quad S_z+G_z=\omega_L. \quad (39)$$

This solution corresponds to equilibrium magnetization, when the spins of both the condensate and the quasiparticles "point" in the direction of the magnetic field along the z axis. Substitution of the solution (39) in (23) yields for the precession rate, as expected, $\dot\alpha=0$. The two independent quantities β and S_z determine completely the remaining S_ζ, G_z, and G_ζ. The angle β can take on according to (21) values from 0 to $\theta_L=\arccos(-\frac{1}{4})$. If relaxation is disregarded, S_z can assume arbitrary values. A change of β from 0 to θ_L corresponds to a change of n_z from 1 to 0. In fact, just as in hydrodynamics,[1]

$$n_z=\frac{\sin\Phi}{2\sin\theta}(1+\cos\beta)=\left[\frac{4}{5}\left(\frac{1}{4}+\cos\beta\right)\right]^{1/2}. \quad (40)$$

$$\text{Ib)}\quad S_z=S_\zeta\cos\beta,\quad G_z=G_\zeta\cos\beta,\quad S_\zeta+G_\zeta=\omega_L. \quad (41)$$

This is the Larmor precession. The spins of the quasiparticles and of the condensate are parallel, directed along the $\hat\zeta$ axis, and rotate with equal velocity $\dot\alpha=-\omega_L$. A solution is again obtained only at $0\leqslant\beta\leqslant\theta_L$, and goes over as $\omega_L\to0$ into the trivial solution $\mathbf{S}=\mathbf{G}=0$, which corresponds to equilibrium.

$$\text{Ic)}\quad S_\zeta=S_z\cos\beta,\quad G_z=G_\zeta\cos\beta,\quad S_z+G_\zeta=\omega_L. \quad (42)$$

This is a new solution and does not exist in the hydrodynamic limit. It describes the mutual rotation of the quasiparticle and condensate spins with identical frequency $\dot\alpha=\omega_L+S_z$ about the immobile total spin $\mathbf{S}\|\hat z$. The spin of the condensate is directed along the $\hat\zeta$ axis. The precession frequency is higher than the Larmor frequency (since z_0 is negative). In the spirit of the molecular-field theory, the solution can be understood as Larmor precession of each of the magnetization components in the sum of the external field and of the effective Fermi-liquid field produced by the magnetization itself. The region in which a solution exists is also restricted to the angle interval $0\leqslant\beta\leqslant\theta_L$. A mode can be excited if the spins can be "moved apart," for example by applying external nonmagnetic forces to the order parameter and by the same token causing the condensate spin to oscillate via the spin-orbit coupling.

Case II. From $\Phi=0$ and (21) we find that $\cos\theta=\cos\beta$ and

$$\partial U/\partial\beta=-{}^4/_{15}\Omega^2(4\cos\beta+1)\sin\beta. \quad (43)$$

In this case (25), (28), and (31) are transformed into

$$S_\zeta+S_z+G_\zeta+G_z=\omega_L(1+\cos\beta), \quad (44)$$

$$(S_z\cos\beta-S_\zeta)(S_z-S_\zeta\cos\beta+G_z-G_\zeta\cos\beta) +{}^4/_{15}\mu\Omega^2(4\cos\beta+1)\sin^4\beta=0, \quad (45)$$

$$(G_z\cos\beta-G_\zeta)(S_z-S_\zeta\cos\beta+G_z-G_\zeta\cos\beta) +(G_\zeta S_z-S_\zeta G_z)\sin^2\beta+{}^4/_{15}\Omega^2(4\cos\beta+1)\sin^4\beta=0. \quad (46)$$

From (44)–(46) and (23) we can obtain useful intermediate relations between the spin components and the angular velocity of the rotation:

$$S_z+G_z=\omega_L+\dot\alpha(1-\cos\beta),\qquad S_\zeta+G_\zeta=\omega_L-(\dot\alpha+\omega_L)(1-\cos\beta). \quad (47)$$

Expressing ultimately S_ζ, G_z, and G_ζ in terms of S_z and β we obtain from (44)–(46) and (23) a third-order dispersion equation for the angular velocity of the precession:

$$(\dot\alpha+\omega_L)^2\dot\alpha-S_z(\dot\alpha+\omega_L)\dot\alpha+{}^4/_{15}\Omega^2(4\cos\beta+1)(\dot\alpha+\omega_L) +\mu\cos\beta\cdot{}^4/_{15}\Omega^2(4\cos\beta+1)\dot\alpha=0. \quad (48)$$

A third-order equation can be solved in terms of radicals, but the corresponding Cardano formula is too cumbersome, and we shall not write out the solution in explicit form. It is of interest, however, to investigate the equation in the limiting cases $\Omega/\omega_L\ll1$ and $\omega_L\to0$.

Let $\Omega/\omega_L\ll1$. In the zeroth approximation in Ω/ω_L the solutions of the system (44)–(47) go over into the already investigated case *I*. In first-order approximation in Ω/ω_L the frequencies and the spin components will acquire corrections that are small to the extent that Ω/ω_L is small. We are interested only in the frequency corrections. The corrections to the spin components are important only when dissipation is considered and are taken into account in Sec. 4.

In case IIa the precession frequency is much lower than the Larmor frequency:

$$\dot\alpha=-\frac{4}{15}\frac{\Omega^2}{G_z}(4\cos\beta+1). \quad (49)$$

The relations between the spins are given by solution Ia (39). The reason the precession frequency is low is that the Zeeman moment is almost completely offset by the moment $\mathbf{R}_d$ of the dipole forces. The angle between the z axis is small to the extent that $(\Omega/\omega_L)^2\delta\beta$ is small, where $\delta\beta=\beta-\theta_L$. The Zeeman moment $\sim\Omega^2\delta\beta$ is, thus, of the same order as the dipole moment. In our case, the Larmor-precession frequency shift depends, besides on the angle β, on a second variable, viz., the spin in the direction of the $\hat{z}$ axis. The equilibrium condition [Eq. (56) below], however, fixes the value of G_z and we arrive at relation (27) of Ref. 1. The susceptibility χ_{p0} turned out here to be modified by the Fermi-liquid interaction.

In case IIb the precession frequency is somewhat higher than the Larmor frequency

$$\dot{\alpha}=-\omega_L+\frac{4}{15}\mu\frac{\Omega^2}{S_\zeta}(4\cos\beta+1). \qquad (50)$$

The spins **S** and **G** make small angles of the order of $(\Omega/\omega_L)^2\delta\beta$ with the $\hat{\zeta}$ axis, and the relations between the spins in zeroth order are given by the Ib solution (41). The correction to the Larmor frequency is determined by two quantities, β and S_ζ. In hydrodynamics, however, S_ζ is fixed by the equilibrium condition (56), and the solution (50) yields the equation first proposed by Brinkman and Smith.[2] All the Fermi-liquid corrections reduce in this case to a replacement of the susceptibility χ_{p0} in Ω by the total susceptibility $\chi_0(1+z_0\chi_0/4\chi_{n0})^{-1}$.

In case IIc, which goes over into the solution Ic, we obtain for $\beta<\theta_L$ the following equation for the precession frequency:

$$\dot{\alpha}=-\omega_L+S_z+\left\{\frac{4}{15}\frac{\Omega^2}{G_\zeta}+\frac{\mu}{15}\frac{\Omega^2}{S_z}\right\}(4\cos\beta+1). \qquad (51)$$

The vector **S** makes now an angle $\sim(\Omega/\omega_L)^2\delta\beta$ with the $\hat{z}$ axis, and **G** an angle $(\Omega/\omega_L)^2\delta\beta$ with the $\hat{\zeta}$ axis. The moment $\sim\Omega^2\delta\beta$ of the dipole forces accelerates the spin of the condensate. The latter in turn rotates the quasiparticle spin via the Fermi-liquid interaction. In view of the proximity of **S** to the $\hat{z}$ axis, the torque acting on **S** is small and is of the order of $\Omega^2\delta\beta$. It is easily seen that, as in the two preceding cases, the Fermi-liquid corrections to the precession-frequency shift [the last term in the curly brackets of (51)] reduce, when the equilibrium values of G_ζ and S_z from (56) are substituted, to a redefinition of the susceptibilities.

Let now $\omega_L=0$. A solution for this case has long been known in hydrodynamics as the WP-mode.[16] In the collisionless region at $\omega_L=0$, one of the solutions of (48) is trivial, $\dot{\alpha}=0$, and for the remaining two we get

$$\dot{\alpha}^2-S_z\dot{\alpha}+{}^4/_{15}(1+\mu\cos\beta)\Omega^2(4\cos\beta+1)=0. \qquad (52)$$

At $\beta>\theta$ there are always two real solutions of this equation, which go over for $\delta\beta>S_z/\Omega$, if Fermi-liquid corrections are disregarded, into a solution of the WP-mode type,[16] and at small angles $\delta\beta<S_z/\Omega$ they yield in one case $\dot{\alpha}=\beta^{1/2}$ and in the other $\dot{\alpha}=S_z$. The first of these corresponds to solutions IIa and IIb, and the second to IIc in strong magnetic fields $\omega_L>\Omega$. In hydrodynamics, the components of **S** along the axes $\hat{\zeta}$ and $\hat{z}$ are equal. In our case this role is assumed by the quantity $(\mathbf{S}+\mathbf{G})$. From (48) we have

$$S_\zeta+G_\zeta=(1-\cos\beta)\dot{\alpha},\quad G_z+S_z=-(1-\cos\beta)\dot{\alpha}. \qquad (53)$$

It is of interest to trace the connection between our solution and the solution in the hydrodynamic region. In the latter the solution is parametrized by one variable, in contrast to our two. We must therefore impose an equilibrium condition, i.e., a relation between the parameters S_z and β. For hydrodynamics it is known from Eqs. (23)–(25) of Ref. 1 that

$$S_z={}^2/_{15}\mu\cos\beta\Omega(4\cos\beta+1)^{1/2},$$

and this reduces (52) to the known law for the WP mode [see, e.g., Eq. (26) of Ref. 1].

All the considered solutions IIa and IIc exist in the angle region $\beta>\theta_L$. It appears that certain solutions survive also in the region $\beta<\theta_L$, but are unstable and of no physical interest.

Case III. $\cos\Phi=-1$, so that from (21) we have $\cos\theta=-1$; U is independent of the angle β and reaches a maximu: $\mathbf{R}_d=0$. The solutions of the system are then the same as in case I:

$$S_\zeta=S_z\cos\beta,\quad G_\zeta=G_z\cos\beta,\quad S_z+G_z=\omega_L,\quad \dot{\alpha}=0,$$
$$S_z=S_\zeta\cos\beta,\quad G_z=G_\zeta\cos\beta,\quad S_\zeta+G_\zeta=\omega_L,\quad \dot{\alpha}=-\omega_L,$$
$$S_\zeta=S_z\cos\beta,\quad G_z=G_\zeta\cos\beta,\quad S_z+G_\zeta=\omega_L,\quad \dot{\alpha}=-\omega_L+S_z.$$

Since U has a maximum, however, the solutions are unstable and will not be considered in detail.

Case IV. $\cos\beta=1$; the solutions are also unstable, see the discussion of relations (35).

4. DAMPING OF MAGNETIZATION PRECESSION

We recognize now that the relaxation time is finite. There are no relaxation terms in Eqs. (23)–(28), so that the cases (I)–(IV) considered satisfy these equations. The solutions of the last three equations (29)–(31), however, are altered when account is taken of the relaxation terms. Equations (29) and (31) can be preserved by redefining the quantities S_β and G_β [they are small to the extent that $(\omega\tau)^{-1}$ is small]. It was noted at the end of Sec. 2 that in the absence of dissipation the quantity $P=S_z-\mu G_\zeta$ is an integral of the motion. When dissipation is taken into account, the quantity P, and with it also other spin components S_z, S_ζ, G_z, and G_ζ are no longer conserved and the solutions are no longer strictly periodic. If, however, the dissipation is small ($\omega\tau\gg1$) the motion is almost periodic, with a frequency that varies slowly in time. This frequency is the quantity actually measured, and we shall seek the law that governs its change by relaxation. In contrast to the hydrodynamic case (see Ref. 1), ours is a two-parameter solution space. Therefore, to describe the motion for a two-parameter family we must have two equations of motion. It is convenient to use those equations that have a time derivative in the left-hand side and only terms proportional to τ^{-1} in the right-hand side. Corrections accurate to τ^{-1} need therefore not be taken into account in the periodic solutions, for this would be an exaggeration of the accuracy. One of the equations we need is for P, and the other is the energy conservation law

$$\frac{d}{dt}P=\frac{\mu}{\tau}(G_\zeta-\mu_1S_\zeta), \qquad (54)$$

$$\frac{d}{dt}E=-W. \qquad (55)$$

The equilibrium condition is that the right-hand side be zero, i.e.,

$$G_\zeta^0=\mu_1 S_\zeta^0, \quad G_z^0=\mu_1 S_z^0. \tag{56}$$

We consider the different cases.

Case Ia. In hydrodynamics, there is no relaxation at all in case I, and the magnetizations of the condensate and of the quasiparticles become equalized in the collisionless region. At equilibrium we have in this case

$$G_z^0=\mu_1 S_z^0=\mu_1\omega_L/(1+\mu_1), \tag{56a}$$

and the angle β takes on arbitrary values in the interval $0\leqslant\beta\leqslant\theta_L$. In the presence of relaxation, only one parameter S_z changes, so that only Eq. (55) is needed (it does not contain the variable β). Expanding the energy and dissipation up to terms quadratic in the deviation $\delta S_z = S_z - S_z^0$ from equilibrium, we obtain

$$E={}^1/_2\rho_1(\mu+1)(\mu_1+1)(\delta S_z)^2, \quad W=\rho_2(1+\mu_1)^2(\delta S_z)^2. \tag{57}$$

Since the energy conservation law contains the ratio $\rho_2/\rho_1=\mu$, the damping rate is expressed, as it should be according to the dynamic equations (24)–(31), only in terms of the quantities μ, μ_1, and τ. Substituting (57) in (55), we get

$$\delta S_z \propto \exp\left\{-\mu\frac{1+\mu_1}{1+\mu}\frac{t}{\tau}\right\}. \tag{58}$$

The effective relaxation time, given by (58), depends on the temperature via the susceptibilities χ_{p0} and χ_0 (2). In the region $T\ll T_c$ of greatest interest to us, however, the susceptibilities χ_{p0} and χ_0 approach exponentially constant values, and vary in the region $T\sim T_c$ within the limits of the susceptibilities themselves. To obtain numerical values of the effective relaxation times in cases when the susceptibilities do not enter in the form of the difference $(\chi_0-\chi_{p0})$ or χ_{p0}—in the denominator, we shall use their values at $T=0$; $\chi_{p0}=\chi_0=\frac{2}{3}\chi_{n0}$. We set the Fermi-liquid interaction constant equal to $z_0=-3$ and have then $\mu=\mu_1^{-1}=0.5$. For the characteristic relaxation time we obtain from (58) $\tau_{\text{eff}}\sim\tau$.

Case Ib. In this case the left-hand side of (55) contains terms that are linear in the deviation from equilibrium, given by

$$G_\zeta^0=\mu_1 S_\zeta^0=\frac{\mu_1\omega_L}{1+\mu_1}, \quad 0\leqslant\beta\leqslant\theta_L, \tag{56b}$$

and the right-hand side contains only quadratic terms. Thus, in the linear approximation we obtain from (55)

$$\frac{d}{dt}(\delta S_z-\delta S_\zeta)=0,$$

and substituting $\delta s_z=\delta S_\zeta$ in (54), we get the exponential relaxation of the ζth spin component, in a form similar to (58):

$$\delta S_\zeta \propto \exp\left\{-\mu\frac{1+\mu_1}{1+\mu}\frac{t}{\tau}\right\}. \tag{59}$$

The effective relaxation time of this mode is $\tau_{\text{eff}}\sim\tau$ [see the discussion of Eq. (58)].

Case Ic. Whereas for cases Ia and Ib the set of nondissipative solutions lies on a line through the entire two-parameter family of solutions, in case Ic the region of nondissipative solutions is a point on the (β, S_z) plane, with coordinates

$$G_\zeta^0=\mu_1 S_\zeta^0=\frac{\mu_1\omega_L}{1+\mu_1}, \quad \beta=0. \tag{56c}$$

From (54) we have directly in the linear approximation, for the case of longitudinal relaxation,

$$(1+\mu)\frac{d}{dt}\delta S_\zeta=-\frac{\mu}{\tau}(1+\mu_1)\delta S_z, \tag{60}$$

which leads in analogy with (58) to exponential relaxation in the form $\delta S\propto\exp(-t/\tau)$.

To calculate the transverse relaxation we must resort to (55). We write down only those parts of the energy and dissipation which are due to the deviation of β from zero:

$$E=\frac{1}{2}\rho_1\frac{\mu\mu_1^2}{1+\mu_1}\omega_L^2\beta^2, \quad W=\rho_2\frac{\mu_1^2}{(1+\mu_1)^2}\omega_L^2\beta^2. \tag{61}$$

It is easy to verify that the angle β also relaxes exponentially like $\beta\propto\exp(-t/\tau)$. We emphasize that all the relaxation processes considered in cases Ia–Ic alter various spin components, but are not accompanied by changes of the precession frequency.

We proceed to discuss the relaxation mechanisms when the order parameter is deflected by more than the Leggett angle. It is convenient to separate two regions—weak and strong fields.

We begin with the case of strong fields: $\Omega/\omega_L\ll1$.

Case IIa. The end result of the relaxation in this case is a state of type Ia with angle $\beta=\theta_L$. Relaxation in cases II is for two-parameter families but it turns out to be described by two greatly differing times. At first, after a time $\sim\tau$, the system relaxes according to Eq. (54), in analogy with case Ia, to a quasiequilibrium state

$$\delta G_\zeta=\mu_1\delta S_\zeta, \tag{62}$$

and then tends slowly, with (62) preserved, to $\delta\beta\to0$. We calculate the time of this process. We express for this purpose δS_z, δS_ζ, and δG_z from the system (44)–(46), in leading order in the small Ω/ω_L, in terms of the deviation from the Leggett angle $\delta\beta=\beta-\theta_L$. We use next (62) and substitute everything in (16) and (17). For E we obtain in leading order in Ω/ω_L simply the dipole energy, while for W we obtain the terms proportional to $(\Omega^2/\omega_L)^2$:

$$E=\frac{1}{2}\rho_1\mu(1-\mu\mu_1)\Omega^2\delta\beta^2,$$

$$W=-\rho_2\left[\frac{4(1+\mu_1)(1-\mu\mu_1)}{\mu_1}\frac{\Omega^2}{\omega_L}\delta\beta\right]^2. \tag{63}$$

Equation (55) gives the relaxation rate of the angle β, and with it also of the precession frequency:

$$\delta\beta\sim\exp\left\{\frac{16(1+\mu_1)^2(1-\mu\mu_1)}{\mu_1}\left(\frac{\Omega}{\omega_L}\right)^2\frac{t}{\tau}\right\}. \tag{64}$$

Using the numerical values of μ and μ_1 [see the discussion of Eq. (52) for case Ia], we easily estimate the value of the effective relaxation time $\tau_{\rm eff}$ of the considered solution. It must be recognized here that the expression for the observed dipole frequency Ω must contain the total susceptibilty χ, with account taken of the Fermi-liquid corrections, and not χ_{p0} as in (64). In addition, we retain the explicit dependence of the coefficient of τ on the condensate susceptibility χ_{p0}, since it has a singularity at $T\to T_c$. The foregoing operations yield

$$\tau_{eff}\sim\frac{1}{16}\frac{\chi_{p0}}{\chi_0-\chi_{p0}}\left(\frac{\omega_L}{\Omega}\right)^2\tau, \qquad (64a)$$

$\tau_{\rm eff}\gg\tau$—this confirms the assumption that the effective relaxation is slow, and justifies the use of the Leggett-Takagi equations. Our equation (64) for the variation of the angle β in the collisionless region is matched well to the hydrodynamic result (35) in Ref. 1—see Eq. (79) below.

Case IIb. In this case the final nondissipative state is Larmor precession with an order-parameter deflection angle $\beta=\theta_L$ (case Ib). The subdivision of the relaxation times is found to be valid in this case, too. At first, after a time $\sim\tau$, the system goes out of the equilibrium state (62), and relaxes next slowly to $\beta=\theta_L$. In analogy with case IIa, we express δS_z, δS_ζ, δG_z from the system (44)–(46), in the leading order of smallness of Ω/ω_L, in terms of the deviation $\delta\beta$. The main contribution to the expression for the energy E is made by the term linear in $\delta\beta$ and connected with the Zeeman energy, while W will contain terms $\delta S^2\propto(\Omega^2/\omega_L)^2\delta\beta$:

$$E=\left(\frac{15}{16}\right)^{1/2}\rho_1\frac{1-\mu\mu_1}{1+\mu_1}\omega_L^2\delta\beta,$$
$$W=-\rho_1(1+\mu_1)^2(1-\mu\mu_1)^2\frac{\Omega^4}{\omega_L^2}\delta\beta^2. \qquad (65)$$

It is clear right away from these expressions that the equations for the precession damping will yield a power-law variation of the precession frequency. The solution of Eq. (55) for the declination angle $\delta\beta$ reaches asymptotically

$$\delta\beta^{-1}=\left(\frac{15}{16}\right)^{1/2}\mu(1+\mu_1)^3(1-\mu\mu_1)\left(\frac{\Omega}{\omega_L}\right)^4\frac{t}{\tau}. \qquad (66)$$

The rate of change of the frequency can be easily recalculated using Eq. (50). It turns out to be even slower than in the case IIa, since it contains the additional small factor $(\Omega/\omega_L)^2$. The numerical estimates described in detail in the estimate of (64) yield an effective relaxation time

$$\tau_{eff}=\frac{1}{2}\frac{\chi_{p0}}{\chi_0-\chi_{p0}}\left(\frac{\omega_L}{\Omega}\right)^4\tau. \qquad (66a)$$

The power-law form of (66) agrees with the hydrodynamic result. This makes it easy to interpolate the result (66) to the intermediate region $\omega_L\tau\sim1$ [see Eq. (80) below]. For the present we assess, from physical considerations, the ranges of validity of Eqs. (66) and (53) obtained in various approximations.

Indeed, if the quasiparticles are dragged by the condensate via relaxation processes (the last term in the right-hand side of (13) is large compared with the first), the hydrodynamic approach is valid, but if they are dragged mainly via Fermi-liquid interaction, our results are applicable. For solution IIb we have $\delta\omega\propto\Omega^2\delta\beta/\omega_L$, the difference between the total spin and the condensate spin $\eta=\mathbf{G}-\mathbf{S}$ in hydrodynamics is $\eta_h\propto\tau\Omega^2\delta\beta$, while in the collisionless region we have $\eta_c\propto\omega_L\psi$ (ψ is the angle between $\mathbf{S}$ and $\mathbf{G}$); from (13), in turn, we have $\omega_L^2\psi\propto\Omega^2\delta\beta$. With all taken into account, we find that the condition $\eta_c<\eta_h$ under which our Eqs. (62) and (66) are valid leads to $\omega_L\tau>1$. Thus, in strong magnetic field the relaxation time τ (see the Introduction) must be compared with the Larmor-precession frequency. Precession relaxation of this type was experimentally investigated in Refs. 18 and 19. The experiments, however, were performed in the main at temperatures not too far from T_c, and the data for lower temperatures are patently insufficient. The conclusions that follow our solutions and concern the experiments directly will be discussed in greater detail in the Appendix.

Case IIc. Here $\beta=\theta_L$ is not a singular point for precession damping, and relaxation goes on to $\beta=0$, in accordance with Eqs. (60) and (61) for the case I_c.

We investigate next the motion of the magnetization when $\omega_L=0$ (WP mode). The complete equilibrium state is described by the equations

$$G_\zeta=G_z=S_z=S_\zeta=0,\quad \beta=\theta_L.$$

Just as in strong magnetic fields, the system tends initially, over a tine $\sim\tau$, to quasiequilibrium [Eq. (62)] and next relaxes in a quasiequilibrium manner to a complete equilibrium state. In our case it is convenient in this case to express the energy increment and the dissipation (16) and (17) in terms of the variable S_ζ accurate to terms of second order of smallness:

$$E=\frac{1}{2}\rho_1(\mu_1+1)(1-\mu\mu_1)S_\zeta^2,$$
$$W=\frac{48}{5}\rho_2\left[\frac{(1+\mu_1)(1-\mu\mu_1)}{\mu-{}^1/_4}S_\zeta\right]^2. \qquad (67)$$

The equations are valid for both solutions of (52). The spin S_ζ, and with it the frequency, attenuates exponentially:

$$S_\zeta\propto\exp\left\{-\frac{48}{5}\frac{\mu}{(\mu-{}^1/_4)^2}(1+\mu_1)(1-\mu\mu_1)\frac{t}{\tau}\right\}. \qquad (68)$$

For temperatures $T\ll T_c$, just as described for the case IIa, we find numerically that the effective relaxation time $\tau_{\rm eff}\propto[12(\chi_0-\chi_{p0})]^{-1}\tau$. In this region, the difference $(\chi_0-\chi_{p0})$ turns out to be small compared with χ_0 itself, therefore $\tau_{\rm eff}$ for the WP mode is also found to be much longer than τ. This justifies in turn the use of the Leggett-Takagi phenomenological equations for the WP mode in the collisionless region.

The relation (68) differs substantially from that obtained in the hydrodynamic region. The reason can be easily seen from physical considerations. In fact, in hydrodynamics we have (see, e.g., Refs. 1 and 12) for the precession frequencies the estimates $\omega \propto \delta\beta^{1/2}$ and $S \propto \delta\beta^{1/2}$, while Eq. (13) gives $\eta \propto \tau\omega S \propto \delta\beta$. This yields $E \propto S^2 \propto \delta\beta$, $W \propto \eta^2 \propto \delta\beta^2$, and from the energy-conservation law we obtain $\delta\beta \propto t^{-1}$. In our collisionless case such estimates lead to $\omega \propto \delta\beta^{1/2}$, $\delta S \propto \delta\beta^{1/2}$, $\omega S \propto S^2\psi$, i.e., $\psi \propto 1$, and hence $\eta \propto \psi S \propto \delta\beta^{1/2}$. It is easily seen that in this case $E \propto S^2 \propto \delta\beta$ and also $W \propto \eta^2 \propto \delta\beta$, and for the relaxation law we obtain the exponential $\delta\beta \propto \exp(-t/\tau)$. From similar considerations we easily estimate the ranges of validity of the hydrodynamic solution for the WP mode[16,12] and of our solution (68). Indeed, in hydrodynamics we should get in order of magnitude $\eta < S$, but (13) gives $\eta \propto \tau\Omega^2\delta\beta$, while $S \propto \Omega\beta^{1/2}$. Ultimately $\delta\beta < (\tau\Omega)^{-2}$—the hydrodynamics region. At temperatures close to T_c, the value of $\tau\Omega$ is certainly less than unity, but τ increases exponentially when the temperature is lowered (see the Introduction), and we obtain $\tau\Omega \sim 1$ already at $T/T_c \sim 0.3$. Thus, at $T/T_c < 0.3$ the angle range $\delta\beta > (\tau\Omega)^{-2}$, in which the exponential relaxation (68) should be observable, becomes experimentally accessible.

5. STABILITY OF SOLUTIONS

Only stable solutions can actually be excited in experiment. It is therefore important to investigate the stability of the obtained modes to small oscillations. We linearize for this purpose the system (24)–(31) without allowance for the dissipative increments near the obtained solutions. The result is a system of eight linear equations for eight unknowns. We seek solutions in the form $\sim \exp(i\omega t)$, so as to reduce the differential equations to an algebraic system. Since the initial system (8)–(10) is symmetric with respect to the operation $t \to -t$ (with simultaneous changes $\mathbf{H} \to -\mathbf{H}$ and $\mathbf{S} \to -\mathbf{S}$), the dispersion equation should contain even powers of the frequencies. Two frequencies in the system are identically equal to zero. One corresponds to conservation of the invariant P, and the other to the degeneracy of the system states relative to the variable α. We obtain, thus, ultimately a bicubic dispersion equations for the frequencies of the small oscillations.

In case Ia, i.e., the spins are at rest and are directed along the external field $\mathbf{H}$, while the direction of the vector $\mathbf{n}$ is arbitrary, the dispersion equation takes the form

$$\omega^6-\omega^4[G_z^2+\omega_L^2+(1+\mu)\Omega^2]+\omega^2[\omega_L^2G_z^2+(1+\mu)\Omega^2\omega_L^2+G_zn_z^2(1+\mu)\Omega^2-n_y^2\mu\Omega\omega_L(\omega_L+G_z)]-G_z^2\omega_L^2n_z^2(1+\mu)\Omega^2=0, \tag{69}$$

where n_z is defined in (40), and $n_z^2 + n_y^2 = 1$. One of the three frequencies (ω_1) is of the order of the dipole frequency Ω, and the other two are of the order of the Larmor frequency. Just as before, it is convenient to investigate (69) in various limiting cases. If $\Omega/\omega_L \ll 1$, we have in first order in terms of this small quantity

$$\omega_1^2=(1+\mu)\frac{\Omega^2}{5}(1+4\cos\beta), \quad \omega_2=\omega_L, \quad \omega_3=G_z; \tag{70}$$

here ω_1 equals the frequency of the longitudinal resonance at $\mathbf{n}\|\mathbf{H}$, ω_2 equals the frequency of the ordinary transverse resonance, and ω_3 is a new frequency connected with the Fermi-liquid interaction. In the opposite limit when $\omega_L = 0$ we have

$$\omega_1^2=(1+\mu)\Omega^2, \tag{71}$$

and the remaining frequencies vanish identically.

In case Ib and Ic we have at $\Omega/\omega_L < 1$, respectively

$$\omega_1^2=(1+\mu)\frac{\Omega^2}{5}(1+4\cos\beta), \quad \omega_2=\omega_L, \quad \omega_3=S_\zeta, \tag{72}$$

$$\omega_1^2=(1+\mu)\frac{\Omega^2}{5}(1+4\cos\beta), \quad \omega_2=S_z, \quad \omega_3=G_\zeta. \tag{73}$$

The difference between the frequencies of the small oscillations in cases Ia–Ic is due only to oscillations of the Larmor-frequency scale. For dipole frequencies, on the other hand, the result agrees with that of the hydrodynamic case.[1]

In Eqs. (70)–(73) the dipole frequency is multiplied by the coefficient $(1+\mu)^{1/2}$. This means that the renormalization of the dipole frequency is the same as carried out in expressions (50)–(52). The unrenormalized susceptibility χ_{p0} is then replaced by the complete one, with account taken of the Fermi-liquid corrections: $\chi_p = \chi_{p0}(1 + z_0\chi_{p0}/4\chi_{n0})^{-1}$.

In case II, in the strong-field limit, Eqs. (70), (72), and (73) remain in force, in first order in Ω/ω_L, for oscillations of the order of the Larmor frequencies ω_2 and ω_3, but for the dipole frequencies we obtain in all three cases

$$\omega_1^2=-{}^2/_{15}(1+\mu)\Omega^2(1+4\cos\beta)(1+\cos\beta); \tag{74}$$

thus, the frequencies ω_1 are real at $\beta > \theta_L$ and the solutions (49)–(51) are stable in this region.

If $\omega_L = 0$, we obtain for both modes (52) at angles close to θ_L,

$$\omega_1^2=-{}^4/_{15}(1+\mu)\Omega^2(1+4\cos\beta),$$
$$\omega_2=(1+\mu)\Omega+O(1+4\cos\beta). \tag{75}$$

We have effectively expanded here the oscillation frequencies in powers of the small quantity $(1+4\cos\beta)$. The origin of the factor $(1+\mu)$ is the same as considered for Eqs. (70)–(73).

It is easy to set a criterion for the validity of the method used in Sec. 4 to describe dissipation. For the adiabatic approximation to be valid it is necessary that the relaxation rate be small compared with the frequency of the oscillations themselves. The spin and orbit parameter will then not deviate greatly from the periodic solution in the process of the relaxation itself. It is necessary to satisfy mathematically the inequality

$$\frac{d}{dt}\left(\frac{1}{\omega_1}\right) \ll 1. \tag{76}$$

Clearly, it suffices to stipulate satisfaction of the criterion (76) for the lowest of the possible frequencies ω_L in (70)–(75), since (76) is then automatically satisfied also for the frequencies ω_2 and ω_3. In the case of the WP mode we obtain from (68) and (75)

$$\frac{d}{dt}\left(\frac{1}{\omega_L}\right)\infty\frac{d}{dt}\left[\frac{e^{-t/\tau}}{\Omega\beta^{1/2}}\right]\infty\frac{1}{\Omega\tau\beta^{1/2}}<1, \tag{77}$$

which coincides with the condition for the validity of our solution for the WP mode in the collisionless region [see the estimates that follow (68)]. In the case of strong fields $\Omega/\omega_L \ll 1$, on substitution of (66) and (74) for IIIb (50) in the inequality (76) the latter reduces to $\Omega/\omega_L^2\tau \ll 1$, which is certainly satisfied in the collisionless region.

In case III we find that in the limit $\Omega/\omega_L < 1$ the most dangerous dipole frequency

$$\omega_1^2=-{}^1/_5(1+\mu)\hat{\Omega}(1+\cos\beta) \tag{78}$$

is pure imaginary, so that solutions III are unstable. It can be easily shown that this conclusion remains in force also as $\omega_L \to 0$.

For case IV one should also expect instabilities to small oscillations, since the solution IV corresponds to the maximum dipole energy.

6. CONCLUSION

In sum, we can state that we have obtained all the periodic solutions with respect to the angle variable α, which are actually determined by competition between the collinear dipole and Zeeman moments and the moment of the Fermi-liquid forces. What remains open, however, is the question of the behavior of the system in the entire nine-dimensional phase space, and not only close to the periodic solutions near which the behavior of the phase trajectories is given by Eqs. (70)–(75) and (78). For physical applications, the only important solutions are I and II, whose stability is guaranteed by (70)–(78).

Let us dwell briefly on the most substantial differences between our solutions and those from hydrodynamics.[1]

First, a new nonlinear model is obtained, stemming from the Fermi-liquid interaction, and its properties are discussed. The spins of the condensate and of the quasiparticles rotate in this model in a way that ensures immobility of the total spin. In a certain sense, the analog of this mode are the spin oscillations well-known from the theory of a normal Fermi liquid, where the deviations of the distribution function from equilibrium is described by higher spherical harmonics. With account taken of higher terms of the expansion of the interaction function in spherical harmonics with a more accurate microscopic theory, other Fermi-liquid modes should also appear. To excite the mode in question, it is apparently insufficient to apply to the system only a magnetic field, since both the condensate and the quasiparticles interact equally with the field and this does not cause the spins to "move apart." This spin mode, however, should be excited if the forces applied lift the degeneracy of the order parameter of ^3He, since the condensate interacts with the dipole forces, but the quasiparticles do not.

Second, equations that differ from the hydrodynamic ones were obtained for the damping of nonlinear NMR, and were discussed in detail in Sec. 4. For precession in strong magnetic fields at deviation angles $\beta > \theta_L$, the form of the relaxation process (exponential or power-law) turns out to be the same both in hydrodynamics and in the collisionless region. This allows us to combine Fomin's results,[4,17] which are valid at $\omega_L\tau < 1$, with our (64) and (66), which hold at $\omega_L\tau > 1$, and obtain an interpolation equation that is valid also in the intermediate region $\omega_L\tau \sim 1$. For slow precession (IIa) we obtain

$$\delta\beta \infty \exp\left\{-8\frac{\chi_0-\chi_{p0}}{\chi_{p0}}\frac{\Omega^2\tau t}{A_1(\omega_L\tau)^2+1}\right\}, \tag{79}$$

where A_1 is a certain expression made up of the constant of the Fermi-liquid interaction and the susceptibilities (64), and has a smooth dependence on the temperature. However, as follows from (79), it is meaningful only if $\omega_L\tau > 1$, or, equivalently, $T \ll T_c$. To transform to Eq. (64a) it is necessary in this case to put $A_1 = 0.5$.

In the case of a precession close to the Larmor frequency [case IIb, (66)], we have an interpolation formula that is asymptotically valid for long times:

$$\delta\beta^{-1} \infty \frac{\chi_0-\chi_{p0}}{\chi_p}\frac{\Omega^4}{\omega_L^4}\frac{\tau t}{A_2(\omega_L\tau)^2+1}. \tag{80}$$

Just as in (79), A_2 is a certain expression so defined that (80) must go over into (66a) at $\omega_L\tau > 1$; at $T \ll T_c$ the numerical value of A_2 is also 0.5.

Equations (79) and (80) should be valid in the entire temperature range $0 < T < T_c$ if the appropriate time τ is substituted. As $T \to T_c$, a certain increase of the effective relaxation time is observed, due to the features of the susceptibility χ_{p0} (Ref. 18). When the temperature is slightly lowered, the relaxation rate assumes a constant value because $(\chi_0 - \chi_{p0})\tau \to \text{const}$ (Ref. 18). At still lower temperature, however, after passing through the region $\omega_L\tau \sim 1$, Eqs. (79) and (80) predict an abrupt decrease of the relaxation rate. Thus, according to the estimate made in the Introduction, lowering the temperature by a factor of two, from $T \sim 0.5T_c$ to $T \sim 0.25T_c$, causes τ to increase by an order of magnitude, and the relaxation rates (79) and (80) to decrease by two orders.

We emphasize once more that the phenomenological theory developed in the present paper should certainly be valid for times $\tau_{\text{eff}} \gg \tau$, and is therefore fully applicable to a description of precession in a strong magnetic field and of the relaxation of the WP mode, where, as we have shown, the inequality $\tau_{\text{eff}} \gg \tau$ holds.

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