# Path expansion technique in the nonlinear electrodynamics of a nonequilibrium plasma

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A new technique is proposed for studying nonlinear effects in nonequilibrium plasmas. The technique is based on an expansion in perturbations of the particle paths. In this approach, both the nonlinear interaction of the waves and oscillations of plasma particles (on the one hand) and the effect of the average force exerted by the field on the particles (on the other) are taken into account. Some specific examples of nonlinear process in an unstable beam plasma are considered. In particular, the example of multimode excitation of waves, in the case in which turbulence arises in the plasma, is examined. The particle distribution function in a slightly nonlinear plasma is constructed, and its structure studied.

The effects of nonlinear wave interactions[1] and of the average force, quadratic in the wave amplitudes, on the charged particles[2] have been studied in some detail in nonlinear plasma electrodynamics.[3] In a nonequilibrium plasma these effects may be important if nonlinear saturation of instabilities which occur in the plasma develops. In the present paper we adopt the case of a beam plasma for a systematic analysis of the interaction of waves and of the average force. We offer a nonlinear theory for the radiative instability of an electron beam in a slow-wave system. The theory is based on a technique of expanding the characteristic system of the Vlasov equation in perturbations of the particle paths. This technique makes it possible to construct a chain of nonlinear equations for the wave amplitudes which incorporate the slowing of the beam caused by the average force. The presence of a small parameter, determining the resonant interaction of a plasma wave of the beam and the radiation, makes it possible to truncate this chain of equations for the amplitudes and to find several analytic solutions. We will show that the average force and the nonresonant interaction of harmonics of the plasma wave of the beam disrupt the synchronization of this wave with the radiation. In a resonant interaction of the harmonics of the plasma wave, the wave energy becomes redistributed over the spectrum. We also examine the structure of the distribution function of the beam electrons and the dynamics of the radiation over a broad spectral range. The only limitation which is of importance ot the discussion below is the assumption that the density modulation of the beam is small; in our case, this assumption is equivalent to the assumption that the electron velocity field is single-valued. The latter assumption means that we can eliminate from consideration resonant wave-particle interactions. Such interactions, however, will have no effect at all on the dynamics of the system in which we are interested here, since the beam electron velocities differ substantially from the wave phase velocities.

## 1. FIELD EQUATIONS WHICH EXPLICITLY CONTAIN EQUATIONS FOR THE PATHS OF THE BEAM PARTICLES

An important property of a beam-plasma system is its instability in the field of a slow electromagnetic wave ("slow" here means that the phase velocity is below the velocity of light). In a monoenergetic electron beam, an instability sets in at essentially arbitrarily small amplitude of an initial electromagnetic wave. Such a wave initiates a stimulated emission of beam electrons and itself increases in amplitude over space or time. Since the instability occurs in a comparatively narrow interval of wave numbers, the wave which is radiated may be regarded as approximately monochromatic.

An electromagnetic wave is excited as a result of its resonant interaction with plasma oscillations of the beam. Furthermore, the interaction of the fundamental mode of plasma oscillations with their higher harmonics is important. All of these processes can be described as a nonlinear wave interaction. There is, however, yet another process, which results from the average force exerted on the plasma electrons by the radiation field. This effect shifts the symmetry point of the electron distribution function and strongly influences the nonlinear dynamics of the instability. To take these processes into account in a systematic way, we will derive nonlinear equations which explicitly contain equations for the electron paths.

We denote by $n_b$ the unperturbed density of the beam electrons, by $u$ their unperturbed velocity, directed along the $z$ axis, and by $2\pi/k$ the wavelength of perturbations along this axis. We assume that the system is immersed in a longitudinal magnetic field strong enough that we can treat the electron motion as one-dimensional:

$$m\frac{d^2z}{dt^2}=F(t,z). \tag{1.1}$$

Here $z(t,z_0)$ is the path of an electron which begins (at

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$t = 0$) at the point $z_0$, while $F(t,z)$ is the force, which includes both the effect of the field of the high-frequency space charge of the beam (i.e., the field of the plasma waves) and the field of the electromagnetic wave which is excited.

Knowing $z(t,z_0)$, we can find the perturbation of the electron velocity:

$$\tilde{n}=n_b\left\{\int\delta[z-z(t,z_0)]dz_0-1\right\}. \quad (1.2)$$

It is this perturbation which determines the high-frequency space charge. Expanding (1.2) in a Fourier series,

$$\frac{\tilde{n}}{n_b}=\frac{1}{2}\sum_{n=1}^{\infty}(\rho_n e^{iny}+\text{c.c.}),\quad \rho_n=\frac{1}{\pi}\int_0^{2\pi}e^{-iny(\tau,y_0)}dy_0, \quad (1.3)$$

and determining the space-charge field of the electrons from the Poisson equation, we can finally put Eq. (1.1) in the form

$$\frac{d^2y}{d\tau^2}=-\frac{i}{2}\sum_{n=1}^{\infty}\frac{\alpha_n}{n}(\rho_n e^{iny}-\text{c.c.})+\frac{\nu}{2}(\varepsilon e^{iy-i\eta_0\tau}+\text{c.c.}). \quad (1.4)$$

Here we have switched to the dimensionless variables $y = k\times(z-ut)$, $y_0 = kz_0$, $\tau = \Omega_b t$, where $\Omega_b$ is the frequency of the natural oscillations of the electrons in the frame of reference of the beam, and we have taken the geometry of the problem into account.[1] This geometry is also incorporated in the coefficients $\alpha_b$ for $n\ 2\geqslant(\alpha_1 = 1)$. The second term on the right side of (1.4) reflects the force exerted by the electromagnetic wave. Here $\eta_0 = (\omega - ku)/\Omega_b$, where $\omega$ is the frequency of the electromagnetic wave, $\varepsilon$ is its dimensionless amplitude, and the dimensionless quantity $\nu$ depends on the mechanism which couples the electromagnetic wave with the beam.[2]

Equation (1.4) should be supplemented with the equation for the excitation of an electromagnetic wave by the beam. We write this equation in the form

$$d\varepsilon/d\tau=-\nu\rho_1 e^{i\eta_0\tau}. \quad (1.5)$$

Equations (1.4) and (1.5) constitute the equations which we are seeking for the nonlinear electrodynamics of a nonequilibrium beam plasma. These are the equations to which we will apply the technique of an expansion in electron paths. These equations are completely equivalent to the Vlasov-Maxwell equations with a self-consistent field.

Linearizing (1.4) and (1.5), we can easily show that if the beam is coupled only weakly with the electromagnetic wave, and the relation $\nu\ll 1$ holds, the growth rate is a maximum at $\eta_0 = -1$ or, equivalently, at

$$\omega-ku=-\Omega_b. \quad (1.6)$$

The condition corresponds to a synchronization of the electromagnetic wave with a beam wave of negative energy.[6] The (dimensional) growth rate is given by the expression $(\omega\to\omega+i\delta)$

$$\delta=2^{-1/2}\nu\Omega_b, \quad (1.7)$$

which determines the physical meaning of the parameter $\nu$. We will be analyzing only this case of weak coupling ($\nu\ll 1$) below, since in this case the electron velocity field is single-valued, and our path expansion technique is valid.

## 2. TECHNIQUE OF EXPANSION IN PARTICLE PATHS

We write the path of an electron as

$$y=y_0+w(\tau)+\tilde{x}(y_0,\tau), \quad (2.1)$$

where $w(\tau)$ is the displacement associated with the translational motion and due to the average force; $\tilde{x}(y_0,\tau)$ is a periodic function of $y_0$, with a period of $2\pi$, which is a consequence of the waves and their interaction. The technique which we will be presenting below is essentially one of expanding the nonlinearities of Eqs. (1.3)–(1.5) in powers of the perturbation $\tilde{x}$, which is assumed to be small. We write $\tilde{x}(y_0,\tau)$ as the Fourier series

$$\tilde{x}(y_0,\tau)=\frac{1}{2}\sum_{n=1}^{\infty}[a_n(\tau)e^{iny_0}+\text{c.c.}], \quad (2.2)$$

substitute (2.1) and (2.2) into (1.4) and (1.5), and expand the exponential functions in these equations in power series in $\tilde{x}$. Equating the coefficients of $e^{iny_0}$ for $n = 0, 1, 2,\ldots$, we find an infinite hierarchy of equations for $a_n$, $\varepsilon$, and $w$:

$$\frac{dw}{d\tau}=-\tfrac{1}{4}(|\varepsilon|^2-|\varepsilon_0|^2),$$

$$\frac{d\varepsilon}{d\tau}=-\nu e^{i\eta_0\tau-iw}\sum_{k=1}^{\infty}\frac{(-i)^k}{k!}a_{1k}, \quad (2.3)$$

$$\frac{d^2a_n}{d\tau^2}=\frac{1}{n}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}n^k}{k!}a_{nk}$$

$$-\frac{i}{2}\sum_{p=1}^{\infty}\left\{\frac{1}{p}\left(\sum_{k=1}^{\infty}\frac{(-1)^kp^n}{k!}a_{pk}\right)\right.$$

$$\left.\times\sum_{m=1}^{\infty}\frac{i^mp^m}{m!}a_{n-p,m}-\frac{1}{p}\left(\sum_{k=1}^{\infty}\frac{i^kp^k}{k!}a_{pk}^*\right)\sum_{m=1}^{\infty}\frac{(-1)^mp^m}{m!}a_{n+p,m}\right\}$$

$$+\frac{\nu}{2}\varepsilon e^{-i\eta_0\tau+iw}\left(a_{n-1,0}+\sum_{k=1}^{\infty}\frac{i^k}{k!}a_{n-1,k}\right)$$

$$+\frac{\nu}{2}\varepsilon^*e^{i\eta_0\tau-iw}\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}a_{n+1,k}.$$

Here $\varepsilon_0$ is the initial wave amplitude (at $\tau = 0$), and the coefficients $a_{nk}$ are expressed in terms of $a_n$ $(n,k = 0,1,\ldots)$:

$$a_{nk}=\frac{1}{\pi}\int_0^{2\pi}\tilde{x}^ke^{-iny_0}dy_0. \quad (2.4)$$

We can also write an expression for the amplitudes of the harmonics of the charge density:

$$\rho_n=e^{-inw}\sum_{k=1}^{\infty}\frac{(-i)^kn^k}{k!}a_{nk}. \quad (2.5)$$

Equations (2.3), which constitute an infinite chain of equations, are equivalent to system (1.4), (1.5) and are therefore just as complicated. We can simplify the equations considerably by assuming the following hierarchy of values of the small quantities in (2.4): $a_1$ and $\varepsilon$ are small quantities of first order, $a_2$ is a small quantity of second order, $a_3$ is a small quantity of third order, and so forth. Under this assumption, we can prove several important assertions.

1. The coefficients $a_{nk}$ $(n,k \geqslant 1)$ are small quantities at least max $(n,k)$.

2. The amplitudes $\rho_n$ and the equations for $\alpha_n$ contain only terms of the following orders: $n$, $n = 2$, $n = 4,\ldots$, $n = 2k$, where $(n,k) \geqslant 1$.

3. The amplitudes $\rho_n$ and the equations for $a_n$ do not contain $a_n$ in a linear way in an order $(m > n)$ higher than $n$.

The last two of these assertions justify our assumption above regarding the order of the small quantities $\varepsilon$ and $a_n$, and they give us a recipe for cutting off the infinite chain of equations (2.3). Specifically, to find a closed system of equations for $\varepsilon, a_1,\ldots, a_{p-1}$ we set $a_n \equiv 0$ at $n \geqslant p$. We substitute the values of $a_{mk}$ calculated from (2.2) and (2.4) in this case, with $(m,k) \leqslant p$, into system (2.3), written for $n \leqslant p-1$. We retain terms of up to order $p$ inclusively. System (2.3) for $\varepsilon, a_1,\ldots,a_{p-1}$ now contains all terms which are small quantities of order up to $p$, while $\rho_n$ is expressed in terms of these quantities by means of (2.5) with $n \leqslant p-1$. To find $\rho_p$ we must use system (2.3) to higher order and incorporate the amplitude $a_p$. The amplitude $a_p$ itslef does not contribute a small quantity of order $p$ to the system for $\varepsilon, a_1,\ldots, a_{p-1}$. Following this recipe, we write the system (2.3) to fourth order in the quantities $\varepsilon$, $w$, $a_1$, $a_2$, and $a_3$:

$$
\begin{gathered}
d\varepsilon/d\tau=\nu(ia_1+{}^1/_2a_1^*a_2-{}^1/_8i|a_1|^2a_1)e^{i\eta_0\tau-iw},\\
dw/d\tau=-{}^1/_4(|\varepsilon|^2-|\varepsilon_0|^2),\\
d^2a_1/d\tau^2=-a_1+i(1-\alpha_2)a_1^*a_2-{}^1/_2(1-\alpha_2)|a_1|^2a_1\\
+\nu[\varepsilon(1-{}^1/_4|a_1|^2)e^{-i\eta_0\tau+iw}+(ia_2+{}^1/_4a_1^2)\varepsilon^*e^{i\eta_0\tau-iw}],\\
\frac{d^2a_2}{d\tau^2}=-\alpha_2a_2+\frac{i}{2}(\alpha_2-1)a_1^2+i\left(\alpha_2-\frac{3}{2}\alpha_3+\frac{1}{2}\right)a_3a_1^*\\
+\left(2\alpha_2+\frac{1}{4}\right.\\
\left.-\frac{9}{4}\alpha_3\right)|a_1|^2a_2-\frac{i}{2}\left(\frac{4}{3}\alpha_2-\frac{9}{8}\alpha_3-\frac{5}{24}\right)|a_1|^2a_1^2\\
+\frac{\nu}{2}\left[\varepsilon\left(ia_1-\frac{1}{2}a_2a_1^*-\frac{i}{8}|a_1|^2a_1\right)e^{-i\eta_0\tau+iw}\right.\\
\left.+\varepsilon^*\left(-ia_3-\frac{1}{2}a_1a_2+\frac{i}{24}a_1^3\right)e^{i\eta_0\tau-iw}\right],\\
d^2a_3/d\tau^2=-\alpha_3a_3+i\left(\frac{3}{2}\alpha_3-\alpha_2-\frac{1}{2}\right)a_1a_2\\
+\left(\frac{3}{8}\alpha_3+\frac{1}{8}-\frac{1}{2}\alpha_2\right)a_1^3+\frac{\nu}{2}\varepsilon\left(ia_2-\frac{1}{4}a_1^2\right)e^{-i\eta_0\tau+iw}.
\end{gathered}
\tag{2.6}
$$

From (2.5) we find

$$
\begin{gathered}
\rho_1=-i[(1-{}^1/_8|a_1|^2)a_1-{}^1/_2ia_1^*a_2]e^{-iw},\\
\rho_2=-2i[(1-|a_1|^2)a_2-{}^1/_2ia_1^2]e^{-2iw},\\
\rho_3=-3i(a_3-{}^3/_2ia_1a_2-{}^3/_8a_1^3)e^{-3iw}.
\end{gathered}
\tag{2.7}
$$

The system (2.6), like (2.7), contains all the small terms of fourth order and thereby describes all possible wave interaction processes in a nonequilibrium beam plasma.

It can be seen from expressions (2.7) that the amplitude $\rho_n$ contains a term $a_1^n$. It can be shown that this is true for arbitrary $n$. It follows that for $|a_1| > 1$ series (1.3) contains all the terms and cannot be truncated. The inequality[3)]

$$
|a_1|<1 \tag{2.8}
$$

is therefore a necessary condition for the applicability of system (2.3). The truncated system of equations (2.6) is applicable under the strong version of inequality (2.8), i.e., for $|a_1| \ll 1$. Numerical calculations have shown that with $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $\nu < 0.16$ the solution of system (2.6) is of such a nature that the relation $|a_1| < 1$ holds. For larger values of $\nu$, on the other hand, this method is not applicable. For other values of $\alpha_n$, the applicability condition reduces to a similar restriction on the parameter $\nu$ (in the case $\alpha_1 = 1$, $\alpha_2 = 4$, $\alpha_3 = 9$, e.g., the restriction would be $\nu < 0.27$).

In this particle path expansion technique, the only thing we are assuming to be small is therefore the oscillatory motion of the electrons which is a consequence of the waves and their interaction. The average motion, described by $w(\tau)$, in contrast, is not assumed to be small, as can be seen directly from the system (2.6), (2.7). The quantity $w(\tau)$ also describes the average effect of the field of the electromagnetic wave on the plasma, while $a_n$ represents the amplitudes of the waves interacting in the plasma.

To finally solve the problem to fourth order, we may need the amplitude $\rho_4$, which can be shown to be given by the expression

$$
\rho_4=-4i(a_4-2ia_1a_3-ia_2^2-2a_1^2a_2-{}^1/_3ia_1^4)e^{-i4w}, \tag{2.9}
$$

while $a_4$ satisfies the equation

$$
\begin{gathered}
d^2a_4/d\tau^2=-\alpha_4a_4+{}^1/_2\nu\varepsilon(ia_3-{}^1/_2a_1a_2-{}^1/_{24}ia_1^3)e^{-i\eta_0\tau+iw}\\
+i(2\alpha_4-{}^1/_2\alpha_1-{}^3/_2\alpha_3)a_1a_3+i(\alpha_4-\alpha_2)a_2^2\\
+(2\alpha_4-{}^1/_4\alpha_1-{}^9/_4\alpha_3)a_1^2a_2\\
-{}^1/_2i({}^2/_3\alpha_4-{}^1/_{24}\alpha_1+{}^1/_2\alpha_2-{}^9/_8\alpha_3)a_1^4.
\end{gathered}
\tag{2.10}
$$

Solving the latter equation is totally unnecessary for solving the basic system (2.6). The quantities $\varepsilon$, $a_1$, $a_2$ and $a_3$, on the other hand, unambiguously determine $a_4$ according to Eq. (2.10), as a forced vibration (the case $\alpha_4 = 16$ is exceptional, in which there is no dispersion of the plasma waves, and small terms of order higher than fourth must be taken into account).

Equations (2.6) and (2.10) have yet another curious property: In the case $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$, all the expressions in parentheses containing these coefficients vanish, and the equations simplify substantially.

### 3. EXAMPLES OF NONLINEAR WAVES

3.1. We begin our analysis of system (2.6), (2.7), which can be solved analytically in a quite general case, with a consideration of the simplest solutions. For example, if there is no coupling of the electromagnetic wave with plasma waves, i.e., if $\nu = 0$, and if the oscillations are strictly one-dimensional, i.e., $\alpha_n \equiv 1$, so that we have $\Omega_b = \omega_b$, then it follows from (2.6) that we have $\varepsilon = \varepsilon_0$, $w = 0$, and for all $n$ we can write

$$d^2a_n/d\tau^2+a_n=0. \tag{3.1}$$

This result means that for an arbitrary amplitude of the plasma waves, regardless of their wavelength, their frequency will be[7] $\omega^2 = \omega_b^2$. The wave itself contains multiple spatial harmonics in this case. Even if we have $a_n = 0$ at $n > 2$ and $a_1 \neq 0$, we can find the structure of the harmonics of the space-charge waves from (2.7):

$$\rho_1=-i(1-{}^1/_8|a_1|^2)a_1, \quad \rho_2=-a_1^2, \quad \rho_3={}^9/_8 ia_1^3. \tag{3.2}$$

In the case $\nu = 0$, the average force exhibited by the radiation vanishes, and the analogous process just reduces to a nonlinear interaction.

We now turn on the interaction between the electromagnetic wave and the plasma wave, i.e., we set $\nu \neq 0$. Again assuming $\alpha_n = 1$, and restricting the analysis to small terms up to third order, we find a solution of system (2.6) in the following form (in this case, $a_2$ is a forced vibration which is a consequence of the interaction with the electromagnetic wave):

$$a_1=a(\tau)e^{i\tau}, \quad a_2=b(\tau)e^{2i\tau}. \tag{3.3}$$

We are assuming that $a(\tau)$ and $b(\tau)$ are slowly varying functions of the time, i.e.,

$$\left|\frac{1}{a}\frac{da}{d\tau}\right| \ll 1, \quad \left|\frac{1}{b}\frac{db}{d\tau}\right| \ll 1. \tag{3.4}$$

Making the substitution $\varepsilon \to \varepsilon e^{iw}$, and eliminating the functions $b(\tau)$ and $w(\tau)$ from system (2.6), we find the following system of equations for the case of the maximum instability growth rate (1.6), i.e., in the case $\eta_0 = -1$:

$$\frac{d\varepsilon}{d\tau}+\frac{i}{4}\left(|\varepsilon|^2-|\varepsilon_0|^2+\frac{\nu^2}{3}|a|^2\right)\varepsilon=i\nu\left(1-\frac{1}{8}|a|^2\right)a,$$
$$\frac{da}{d\tau}-\frac{i}{24}\nu^2|\varepsilon|^2a=-\frac{i}{2}\nu\left(1-\frac{1}{4}|a|^2\right)\varepsilon+\frac{i}{16}\nu a^2\varepsilon^*. \tag{3.5}$$

This system of equations has the first integral

$$|\varepsilon|^2-|\varepsilon_0|^2=2|a|^2. \tag{3.6}$$

The left sides of Eqs. (3.5) contain nonlinear frequency shifts: $i(|\varepsilon|^2 - |\varepsilon_0|^2)\varepsilon/4$, which is the shift due to the average force (the slowing of the beam on the average), and $\nu^2|a|^2\varepsilon/12$ and $\nu^2|\varepsilon|^2a/24$, which are shifts due to the interaction of the waves. The two latter shifts are insignificant because of the small value of $\nu$.

We further assume that the field $\varepsilon$ is turned on adiabatically as $\tau \to -\infty$, i.e., $\varepsilon_0 = 0$. Assuming

$$a\to ae^{i\varphi}, \quad \varepsilon=2^{1/2}ae^{i\psi}, \quad \Phi=\psi-\varphi \tag{3.7}$$

in this case, we find from (3.5) the following equations and phase integral:

$$\frac{da}{d\tau}=\frac{\nu}{2^{1/2}}\left(1-\frac{1}{8}|a|^2\right)a\sin\Phi,$$
$$\frac{d\Phi}{d\tau}=-\frac{1}{2}\left(1+\frac{\nu^2}{3}\right)a^2+2^{1/2}\nu\left(1-\frac{a^2}{4}\right)\cos\Phi, \tag{3.8}$$
$$\cos\Phi=\frac{2^{1/2}}{\nu}\left(1+\frac{\nu^2}{3}\right)\frac{a^2/8}{1-a^2/8}.$$

We then find one equation for $x \equiv a^2/8$:

$$\frac{dx}{d\tau}=2^{1/2}\nu x(1-x)\left[1-\frac{2}{\nu^2}\left(\frac{x}{1-x}\right)^2\right]^{1/2}. \tag{3.9}$$

Determining the maximum root of the right side of Eq. (3.9), we find the maximum value of the amplitude $a$:

$$a_{max}=2\left(\frac{2\nu}{2^{1/2}+\nu}\right)^{1/2}\approx 2^{5/4}\nu^{1/2}. \tag{3.10}$$

If we now impose the condition for the applicability of the path expansion, (2.3), and the (stronger) condition for the applicability of system (2.6), $a < 1$, we reach the conclusion that weak coupling, $\nu < 1$, is necessary. Under this condition, Eq. (3.9) has the soliton solution

$$x=\nu/2^{1/2}\,\mathrm{ch}\,(2^{1/2}\nu\tau), \tag{3.11}$$

which agrees with the requirement of weak coupling, $\nu < 1$, when the inequalities (3.4) are taken into account. According to Eqs. (3.11) and (3.6), saturation of the wave amplitudes results from a nonlinear frequency shift due to the average force. Solution (3.11) was derived in Ref. 8 by a less rigorous approach.

3.2. We turn now to an analysis of system (2.6) in the case of multidimensional plasma waves, with $\alpha_n \neq 1$. If there is no radiation ($\nu = 0$), Eq. (2.6) reduces to the following equations to third order:

$$d^2a_1/d\tau^2+a_1=i(1-\alpha_2)a_1^*a_2+{}^1/_2(1-\alpha_2)|a_1|^2a_1,$$
$$d^2a_2/d\tau^2+\alpha_2a_2=-{}^1/_2 i(1-\alpha_2)a_1^2. \tag{3.12}$$

Here, in contrast with (3.1), the second harmonic $a_2$ is coupled with $a_1$, with the result that there is a nonlinear correction to the frequency $\Omega_b$. Since $a_2$ is a forced vibration with respect to $a_1$, we seek a solution of (3.12) in the form $a_1 = Ae^{i\delta\tau}$, $a_2 \sim e^{2i\delta\tau}$, where $A$ = const. As a result, we find from (3.12) the nonlinear spectrum (for $|A| < 1$)

$$\delta^2=1+\frac{3}{2}\frac{1-\alpha_2}{\alpha_2-4}|A|^2, \tag{3.13}$$

which determines the dependence of the frequency of the plasma waves ($\omega^2 = \delta^2\Omega_b{}^2$) on their amplitude.

We now consider the interaction between a multidimensional beam wave ($\alpha_2 \neq 1$) and an electromagnetic wave. Repeating almost literally the derivation of (3.10) and (3.11), we find the solutions

$$a_{max}=2^{7/4}\nu^{1/2}[|\alpha_2-4|/(\alpha_2+5)]^{1/2}, \tag{3.14}$$

$$x=2^{1/2}\nu|\alpha_2-4|/(\alpha_2+5)\,\mathrm{ch}\,(2^{1/2}\nu\tau). \tag{3.15}$$

While (3.10) and (3.11) incorporate only a nonlinear frequency shift, due to a change in the beam velocity caused by the average force, Eqs. (3.14) and (3.15) also include a nonlinear correction[4)] to the spectrum (3.13). In the case $\alpha_2 = 1$, Eqs. (3.14) and (3.15) become (3.10) and (3.11).

3.3. In the case $\alpha_2 = 4$, solutions (3.13)–(3.15) do not apply, and the case $\alpha_n = n^2$ will in general require special treatment, since it pertains to a linear dispersion law for a plasma beam wave,[5] $\Omega_b \sim k$, and it requires that the resonant interaction of plasma wave harmonics be treated. We will examine this interaction (with $\nu = 0$) in the simplest case, with $\alpha_2 = 4$ and $\alpha_3 \neq 9$. In this case we can restrict the analysis exclusively to those equations which have a cubic nonlinearity, i.e., system (3.12). Substituting (3.3) into (3.12) under the condition $\alpha_2 = 4$, we find the following equations for the slow amplitudes of the harmonics:

$$da/d\tau - {}^3/_4 i|a|^2 a = -{}^3/_2 a^* b, \quad db/d\tau = {}^3/_8 a^2. \tag{3.16}$$

These equations have the first integral

$${}^1/_4|a|^2 + |b|^2 = {}^1/_4|a_0|^2, \tag{3.17}$$

where $a_0$ is the amplitude of the first harmonic at $\tau = 0$ ($b = 0$ at $\tau = 0$). Now introducing the real amplitudes and phases

$$a \to a e^{i\varphi}, \quad b \to b e^{i\psi}, \quad \Phi = \psi - 2\varphi, \tag{3.18}$$

we find from (3.16) the following equations and phase integral

$$da/d\tau = -{}^3/_2 ab\cos\Phi, \quad db/d\tau = {}^3/_8 a^2 \cos\Phi,$$
$$\sin\Phi = (a^4 - a_0^4)/4a^2 b = -(a_0^2 - a^2)^{1/2}(a_0^2 + a^2)/2a^2. \tag{3.19}$$

Using (3.17), we can now easily find one equation for the quantity $s = a^2$ ($s_0 = a_0^2$) from (3.19):

$$ds/d\tau = -{}^3/_4\{(s_0 - s)(4s^2 + s^3 - s_0^2 s + s_0 s^2 - s_0^3)\}^{1/2}. \tag{3.20}$$

The latter equation is easily integrated under the condition $s_0 \ll 1$, which is the same as the condition for the applicability of the entire method. The final solution is

$$s = s_0\,\mathrm{cn}^2(z, p) + {}^1/_2 s_0^{3/2}\,\mathrm{sn}^2(z, p), \tag{3.21}$$

where $z = {}^3/_8 s_0^{1/2}\tau$, $p = 1 - {}^1/_2 s_0^{1/2}$. We see that the first harmonic of the plasma wave decays. Its amplitude decreases, and at

$$\tau = \tau_p = ({}^8/_3 s_0^{-1/2})\ln(4s_0^{-1/4}) \tag{3.22}$$

it reaches a minimum value

$$s = s_{min} = s_0^{3/2}/2. \tag{3.23}$$

Since we have $s_{\min}/s_0 = s_0^{1/2}/2 \ll 1$, the decay of the first harmonic is extremely important.

If $\alpha_2 = 4$ and $\alpha_3 = 9$, then energy is also pumped to a resonant third harmonic. This process can be analyzed with the help of Eqs. (2.6) (with $\nu = 0$). If in addition the condition $\alpha_4 = 16$ holds, then energy is also pumped to the fourth harmonic. Equations (2.6) and (2.10) are no longer sufficient for studying this process, since small terms of fifth order must now be taken into account. We will not take up these rather complicated problems here.

3.4. There is another interesting case: $\nu \neq 0$, $\alpha_2 = 4$, $\alpha_3 \neq 9$. This is the interaction of the electromagnetic wave with the first two plasma wave harmonics. Using the substitution (3.3) and the replacement $\varepsilon \to \varepsilon e^{-iw}$, we can reduce Eqs. (2.6) (with $a_3 = 0$) for this case to

$$d\varepsilon/d\tau + {}^1/_4 i|\varepsilon|^2\varepsilon = i\nu(1 - {}^1/_8|a|^2)a + {}^1/_2\nu a^* b,$$
$$da/d\tau - {}^3/_4 i|a|^2 a$$
$$= -{}^3/_2 a^* b - {}^1/_2 i\nu[\varepsilon(1 - {}^1/_4|a|^2) - {}^1/_2(ib + {}^1/_4 a^2)\varepsilon^*], \tag{3.24}$$
$$db/d\tau = {}^3/_8 a^2 + {}^1/_8\nu\varepsilon a.$$

The latter system of equations has only one integral,

$$|\varepsilon|^2 = 2(|a|^2 + 4|b|^2), \tag{3.25}$$

so that it cannot be solved analytically. We note the most obvious feature of the solution. The time scale of the interaction of waves $\varepsilon$ and $a$ is of order $\nu^{-1}$, as can be seen from (3.11), while the typical amplitude of $a$ is of order $\nu^{1/2}$. On the other hand, the interaction time of waves $a$ and $b$ [see (3.22)] is $a^{-1} \sim \nu^{-1/2}$ in order of magnitude, much smaller than $\nu^{-1}$ (in the case $\nu \ll 1$). Consequently, energy is repeatedly pumped from $a$ to $b$ and back before the amplitude of the electromagnetic wave, $\varepsilon$, reaches saturation. Figure 1 shows a numerical solution of system (3.24), which clearly illustrates the above remarks. In principle, we might expect some interesting features from the solutions of Eqs. (3.24) at certain values of the parameters ($\nu$ and $\varepsilon_0$). We intend to take up this question in future studies.

#### 4. DISTRIBUTION FUNCTION AND MULTIMODE INSTABILITY

The distribution function of a system of particles with an initially uniform velocity can be expressed in terms of their paths $y(y_0, \tau)$ by the formula (we are normalizing the distribution function)

$$f = \int \delta[y - y(y_0, \tau)]\delta[\dot{y} - \dot{y}(y_0, \tau)]dy_0. \tag{4.1}$$

Function (4.1) is not very informative, so instead of analyzing it we will analyze the coordinate-independent function $\langle f \rangle$, where the average is over the wavelengths of the plasma waves:

$$\langle f \rangle = \frac{1}{2\pi}\int_0^{2\pi} f(y, \dot{y})\,dy = \frac{1}{2\pi}\int \delta[\dot{y} - \dot{y}(y_0, \tau)]dy_0. \tag{4.2}$$

The latter expression can be rewritten as

$$\langle f \rangle = \frac{1}{2\pi}\sum_j \left| \frac{\partial \dot{y}}{\partial y_0}(y_{0j}, \tau) \right|^{-1}, \tag{4.3}$$

where $y_{0j}$ is the $j$th root of the equation

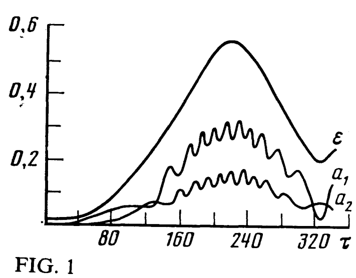


FIG. 1

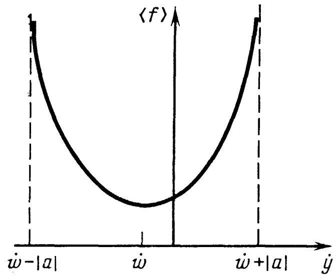


FIG. 2

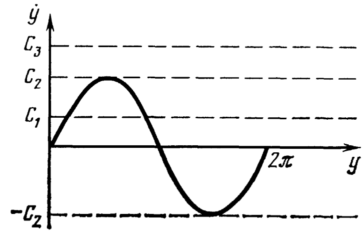


FIG. 3

$$\dot{y}=\dot{y}(y_0,\ \tau). \tag{4.4}$$

Substituting the expansion (2.1), (2.2) into (4.4), and finding the roots, we can in principle construct the distribution function (4.3). In General, however, Eq. (4.4) cannot be solved. An exceptional case is that of a one-dimensional plasma ($\alpha_n = 1$), and it is this case which we will examine in order to determine the structure of the distribution function.

In a one-dimensional plasma, the only nonzero coefficient in the expansion (2.2) is $a_1$ [see (3.1); this is true even in the case $\nu\neq 0$, $a_2\sim \nu a_1{}^2 \ll a_1$]. Accordingly, taking the first equation in (3.3) into account, we can reduce Eq. (4.4) to

$$\dot{y}=\dot{w}-a\sin(y_0+\tau+\varphi), \tag{4.5}$$

where $\varphi$ is a phase, and $y = y_0 + w + a\cos(y_0 + \tau + \varphi)$. Determining $y_0$ from (4.5) [we need to allow for the fact that there are either zero or two such solutions on the interval $(0, 2\pi)$], and substituting into (4.3), we find, after some simple manipulations,

$$\langle f\rangle=\frac{1}{\pi}\left\{\begin{array}{c}[|a|^2-(\dot{y}+{}^1/_4|\varepsilon|^2)^2]^{-1/2}\\ 0\end{array}\right. . \tag{4.6}$$

The upper row in (4.6) refers to the case in which the expression in the radical is positive; in the opposite case we would have $\langle f\rangle = 0$. In deriving (4.6) we used $\dot{w} = -\frac{1}{4}|\varepsilon|^2$.

Figure 2 shows function (4.6). Figure 3 shows the phase plane of the electrons (we are not considering $\dot{w}$), which illustrates the structure of the solution (4.6) (in the case $\dot{y} = C_1 < |a|$, the electron density integrated over the wavelength is finite, in the case $\dot{y} = C_2 = \pm|a|$ it is infinite, and in the case $\dot{y} = C_3$ it vanishes).

Using the formula

$$\int_0^1\frac{x^m\,dx}{(1-x^2)^{1/2}}=\frac{\pi^{1/2}}{2}\frac{\Gamma[0.5(m+1)]}{\Gamma(0.5m+1)} \tag{4.7}$$

we can calculate the moments of the distribution function:

$$J_0=\int_{-\infty}^{\infty}\langle f\rangle d\dot{y}=1,\qquad J_1=\int_{-\infty}^{\infty}\dot{y}\langle f\rangle d\dot{y}=-{}^1/_4|\varepsilon|^2J_0,$$
$$J_2=\int_{-\infty}^{\infty}(\dot{y}-J_1)^2\langle f\rangle d\dot{y}={}^1/_2|a|^2. \tag{4.8}$$

The latter relations are quite obvious; the moment $J_2$ can be identified with the beam temperature.

Up to this point we have been considering a single-mode excitation of an electromagnetic wave by a beam (with the possible excitation of multiple harmonics). We now consider the multimode case, which occurs if the initial electromagnetic wave is not monochromatic and can be represented by a large set of modes which are approximately the same. We write a multimode analog of Eqs. (1.4) and (1.5):

$$\frac{d\varepsilon_s}{d\tau}=-\nu\rho_s e^{i\eta_{0s}\tau},\qquad \rho_s=\frac{2}{kL}\int_0^{kL}e^{-isy}\,dy_0,$$
$$\frac{d^2y}{d\tau^2}=-\frac{1}{2}i\sum_s\frac{1}{s}(\rho_s e^{isy}-\text{c.c.})+\frac{\nu}{2}\sum_s(\varepsilon_s e^{isy-i\eta_{0s}\tau}+\text{c.c.}), \tag{4.9}$$

where $L$ is the spatial period of the initial perturbation, and $\eta_{0s} = (\omega_s - sku)/\Omega_b$. At this point we set $\omega_s = skc_0$, where $c_0 < u$ is the phase velocity of the electromagnetic wave. Here we have $\eta_{0s} = -s$.

By analogy with (2.1) and (2.2) we write the electron trajectory as

$$y=y_0+w+\frac{1}{2}\sum_s(a_s e^{isy_0}+\text{c.c.}). \tag{4.10}$$

In substituting (4.10) into (4.9), we need to recall that the wave $a_s$ may interact with its own harmonics, $a_{2s}$, $a_{3s}$, etc. In the one-dimensional case, however, as we have already shown, there is no such interaction so that the analysis below is simplified substantially.

Omitting the fairly simple calculations, we write the following system of equations for the amplitudes $\varepsilon_s$ and $a_s$:

$$\frac{d\varepsilon_s}{d\tau}=\nu a_s e^{-isw},\qquad \frac{da_s}{d\tau}+i\frac{s-1}{s}a_s=\frac{1}{2}\nu\varepsilon_s e^{isw}, \tag{4.11}$$
$$\frac{dw}{d\tau}=-\frac{1}{4}\sum_s(|\varepsilon_s|^2-|\varepsilon_{0s}|^2)=-\frac{1}{2}\sum_s|a_s|^2.$$

In deriving (4.11) we used the replacement $a_s\to s^{-1}a_s e^{is\tau}$. Also using the replacement $a_s\to a_s e^{isw}$, we can reduce (4.11) to the final form:

$$\frac{d\varepsilon_s}{d\tau}=\nu a_s,\qquad \frac{da_s}{d\tau}+i\Delta_s a_s=\frac{1}{2}\nu\varepsilon_s, \tag{4.12}$$

where

$$\Delta_s=s\left[\frac{s-1}{s}-\frac{1}{2}\sum_s|a_s|^2\right] \tag{4.13}$$

is a nonlinear frequency shift. By analogy with $J_2$ in (4.8), the quantity

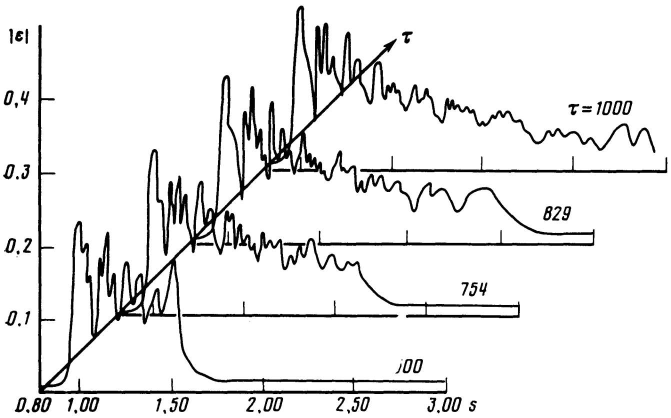


FIG. 4

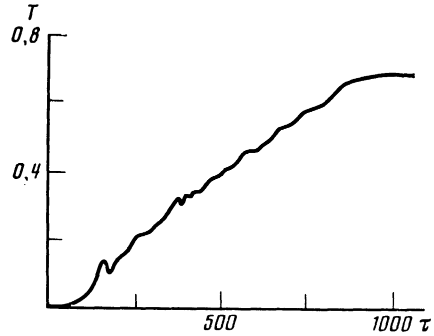


FIG. 5

$$T=\frac{1}{2}\sum_s |a_s|^2 \qquad (4.14)$$

may be interpreted as the plasma temperature. Equations (4.12) and (4.13) are the equations which we have been seeking for a multimode instability.

We can draw the following picture of the dynamics of the excitation of the spectrum: At a given instant, the mode which is excited is that for which the condition $\Delta_s = 0$ holds. As time elapses, the temperature $T$ increases, as does the indexs of the mode which is excited. In the limits $\tau \to \infty$ and $s \to \infty$, we have $T \to 1$ (since $\Delta_s = 0$ and $s \to \infty$). It follows from the last equation in (4.11) that in the limit $\tau \to \infty$ we have

$$dw/d\tau=-T=-1. \qquad (4.15)$$

Using the definition of $\tau$ and $w$, and switching to dimensional variables, we can reduce (4.15) to the form

$$\langle v\rangle=u-\Omega_b/k, \qquad (4.16)$$

where $\langle v\rangle$ is the average beam velocity. Using (1.6) for the spectrum $\omega = kc_0$, we then find $\langle v\rangle = c_0$. Consequently, the beam is ultimately slowed to the phase velocity of the electromagnetic wave, $c_0$.

The structure of the spectrum of plasma waves which is excited can be determined from the equality $\Delta_s = 0$ at any instant. To see this, we convert to a continuous spectrum in the equation

$$\frac{1}{2}\sum_s |a_s|^2=\frac{s-1}{s} \qquad (4.17)$$

(we replace the sum by an integral from 1 to $s$). We then find

$$|a_s|^2\equiv|a(s)|^2=2/s^2. \qquad (4.18)$$

The spectral density of the electromagnetic wave varies in the same way. The width of the spectrum at half-maximum corresponds to the $s$ interval from 1 to 2. In this range of wave numbers, the changes in the spectrum are completed over times on the order of $\nu^{-1}$.

Let us examine the dynamics of the discrete spectrum, specifying the following parameter values: $2^{-1/2}\nu = 0.03$, $\varepsilon_{0s} = 0.01$, $s = 0.8 + nh$, $h = 0.01$, $n = 0, 1,...,221$ $(0.8 \leqslant s \leqslant 3)$. For these values of $h$ and $\nu$, only about a tenth of the modes fall simultaneously in the resonant band.[6] Figure 4 shows the dynamics of the spectrum of weak turbulence. As time elapses, the spectrum broadens. By the time $\tau = 800$, the entire spectrum has been excited, and its evolution has basically been completed. The difference between the spectrum (4.18) and the calculated spectrum shown in Fig. 4 is the consequence of the discrete nature of the latter $(|a(s)|^2 h \approx |a_s|^2)$. Figure 5 shows the beam temperature. This temperature increases until the entire spectrum is excited, at which point it becomes essentially constant. A particular feature of the solutions shown here is their irreversibility in time, in contrast with (3.11).

[1]In the case of a beam with a infinitely narrow cross section in a waveguide we would have $\Omega_b{}^2 = S_b\omega_b{}^2 f(k^2)$, $\alpha_n = f(n^2k^2)/f(k^2)$, where

$$f(x^2)=\sum_{m=1}^{\infty} x^2(k_{\perp m}^2+x^2)^{-1}\varphi_m{}^2(\mathbf{r}_b)\|\varphi_m\|^{-2},$$
$$\omega_b{}^2=4\pi e^2 n_b/m,$$

$S_b$ is the cross-sectional area of the beam, $k_{\perp m}{}^2$ and $\varphi_m$ are the eigenvalue and eigenfunction of the waveguide, $r_d$ is the coordinate of the beam in the cross section, and $\|\varphi_m\|^2$ is the square norm of the eigenfunction. In the opposite limit, $k^2S_b \gg 1$ we have $\Omega_b{}^2 = \omega_b{}^2$ and $\alpha_n = 1$ for all $n$.

[2]Explicit expressions for the parameter $\nu$ for various specific systems can be found in Refs. 4 and 5, which also contain a rigorous derivation of Eqs. (1.4) and (1.5) for these systems. For an electrostatic undulator,[5] for example, the expression is

$$\nu={}^1/_2|kz_E|[S_b\varphi_n{}^2(\mathbf{r}_b)\|\varphi_n\|^{-2}k_{\perp n}{}^2c^2/\omega\omega_b]^{1/2},$$

where $z_E$ is the amplitude of the oscillations of the electrons in the field of the electrostatic pump, and $c$ is the velocity of light.

[3]For $|a_1| > 1$ a charge-density wave breaks, and multistreaming arises in the plasma.

[4]The nonlinear frequency shift due to relativistic effects was studied in Ref. 9.

[5]In a magnetized plasma, a dispersion law of this sort is typical of the long-wave part of the spectrum. For example, we have $(\omega - ku)^2 = \omega_b{}^2k^2/(k_\perp{}^2 + k^2) \approx \omega_b{}^2k^2/k_\perp{}^2$ at $k^2 \ll k_\perp{}^2$.

[6]If a single mode falls in the resonant band, the solutions of system (4.2) reduce to (3.11).

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