Theory of dynamical-soliton relaxation in ferromagnets

V. G. Bar'yakhtar, B. A. Ivanov, T. K. Soboleva, and A. L. Sukstanskiĭ

Institute of Metal Physics, Academy of Sciences of the Ukrainian SSR
(Submitted 11 March 1986)
Zh. Eksp. Teor. Fiz. **91**, 1454–1465 (October 1986)

A system of equations describing the evolution of the integrals of motion of the dynamical soliton of the Landau-Lifshitz equations under the action of relaxation processes of both relativistic and exchange natures are derived within the framework of a phenomenological theory. The corresponding integral curves are constructed, and the time dependences of the soliton parameters at different stages of the relaxation are analyzed.

Recently there have been obtained in investigations of the dynamical properties of magnetically ordered crystals numerous exact solutions to the nonlinear equations of motion of magnetization—the Landau-Lifshitz equations.[1] Solutions describing dynamical and topological solitons, nonlinear periodic waves, etc., have been found for practically all the principal classes of magnetic materials (ferromagnets, ferrimagnets, and antiferromagnets). The exact integrability of the Landau-Lifshitz equations has been demonstrated for the simplest one-sublattice-ferromagnet models within the framework of the inverse-scattering problem method, and this has allowed[2] the construction of multisoliton solutions and the analysis of the soliton interaction processes (see also Ref. 1).

Of extreme importance for the application of soliton theory to the description of real experiments or their setup is the problem of the description of soliton relaxation. Allowance for the damping processes destroys the exact integrability of the equations, and leads to the relaxation of the nonlinear excitations ultimately to the ground state.

The relaxation of the kink-type topological solitons, which describe domain walls, has been studied in sufficiently great detail both phenomenologically[3,4] and with the aid of a detailed microscopic analysis of the processes of inelastic interaction of a kink with magnons, phonons, crystal-lattice defects, etc. The advantage of the microscopic approach is that we can, on its basis, find the temperature and defect-concentration dependence of the retarding force acting on the soliton (see, for example, Ref. 5). But the microscopic approach is rather complicated, and has in fact been applied only to the simplest one-dimensional solitons of the domain-wall type.[5] The transition to the description of more complex solitons of the two-parameter-bion type,[2] the generalization to multidimensional solitons, etc., within the framework of the method proposed in Ref. 5 are a highly nontrivial problem. In particular, the application of this method requires knowledge of the exact spectrum and wave functions of the magnons in the background of the soliton, but they are known only for a small number of one-dimensional problems.

The general relaxation picture not only for kinks, but also for arbitrary nonlinear excitations in magnetic materials can be described within the framework of the macroscopic approach in which the energy dissipation processes are taken into account through the introduction of additional relaxation terms into the basic equation of motion.[6] For an adequate description of the relaxation processes in magnetic materials, we must use an equation proposed in Ref. 4 by one of us, which contains relaxation terms of both relativistic and exchange natures, and allows a systematic description of the spatial dispersion of the relaxation. In the present paper we propose a simple version of these equations, which allows us to describe the relaxation of arbitrary nonlinear magnetization waves under the assumption that the damping is weak.

The smallness of the corresponding relaxation constants allows us to develop a perturbation theory for the description of the evolution of the soliton parameters. In particular, in Refs. 7 and 8 a specific form of the perturbation theory is constructed for exactly integrable systems on the basis of the inverse scattering problem method.

In the present paper we shall use a simpler perturbation-theory variant based on the construction of evolution equations for the integrals of motion of the unperturbed system. These equations describe the slow evolution of the parameters of the initial excitation under the influence of the dissipation. The simplest variant of this approach is used in Ref. 9 to study fluxon damping in Josephson junctions within the framework of the perturbed sine-Gordon equation and in Ref. 10 to describe the damping of the low-amplitude low-frequency solitons of the Landau-Lifshitz equation with a relativistic relaxation term. The advantage of this approach lies in the fact that it can be used even in the case when the basic equation is not exactly integrable, e.g., in the case of the analysis of three-dimensional magnetic solitons.[1)] We shall discuss the limitation on its application below.

In the present paper the indicated approach is developed for the description of the relaxation of the two-parameter soliton (bion) of the Landau-Lifshitz equations. The evolution of the dynamical-soliton parameters in a uniaxial ferromagnet with the "easy-axis" type of magnetic anisotropy is investigated with allowance made in the equations of motion for dissipative terms of both relativistic and exchange natures. It is shown that the relaxation picture for a bion is much more complicated than the corresponding picture for a domain wall. In particular, its parameters can, depending on the initial conditions, vary in time according to either an exponential or a power law, the velocity can decrease or increase, etc.

Let us note that it is not difficult to construct a general-

 0038-5646/86/100857-07$04.00

ization of the proposed computational scheme for the description of the relaxation of N-soliton excitations in exactly integrable systems, or of the relaxation of the multiparameter solutions to any nonlinear dynamical equations that are characterized by a finite number of integrals of motion.

1. THE EFFECTIVE EQUATIONS

We shall, in describing the nonlinear-excitation-relaxation processes, proceed from equations of motion for the magnetization vector $\mathbf{M}$ (Landau-Lifshitz equations) that contain a dissipative term with the form proposed in Ref. 4 on the basis of the Onsager relations[2)] and exchange symmetry:

$$\frac{\partial \mathbf{M}}{\partial t}=-g[\mathbf{M}\mathbf{H}_e]+gM\{\lambda_1(\mathbf{m}\mathbf{H}_e)\mathbf{m}+\lambda_r[\mathbf{m}[\mathbf{H}_e\mathbf{m}]]-\lambda_e a^2\Delta\mathbf{H}_e\}. \tag{1}$$

Here $M=|\mathbf{M}|$; $\mathbf{m}=\mathbf{M}/M$ is the unit magnetization vector; g is the gyromagnetic ratio; $\mathbf{H}_e=-\delta W/\delta\mathbf{M}$, W being the energy of the magnetic material; λ_1, λ_r, and $\lambda_e a^2$ are the relaxation constants; and a is the lattice constant. The choice of the dissipative term in such a form allows us to describe the relaxation of the excitations as a result of both relativistic and inhomogeneous-exchange interactions; to the latter interactions corresponds the constant $\lambda_e a^2$.

According to Eq. (1), the modulus of the vector $\mathbf{M}$ varies because of the relaxation terms. Indeed, scalar multiplying (1) by $\mathbf{M}$, we obtain

$$\partial M/\partial t=gM\{\lambda_1 H_\parallel-\lambda_e a^2(\mathbf{m}\Delta\mathbf{H}_e)\}, \tag{2}$$

where $H_\parallel=\mathbf{m}\mathbf{M}_e$ is the component of the effective field $\mathbf{H}_e$ in the direction of the vector $\mathbf{M}$. It is easy to show that $H_\parallel=-\delta W/\delta M$. On account of this condition, $H_\parallel=0$ in the static equilibrium case.

Writing the magnetization $\mathbf{M}$ in the form

$$\mathbf{M}=M\mathbf{m},\quad \mathbf{m}^2=1,$$

and using (2), we easily obtain the dynamical equation for the normalized (unit) magnetization vector $\mathbf{m}$:

$$\frac{\partial \mathbf{m}}{\partial t}=-g[\mathbf{m}\mathbf{H}_e]+g\lambda_r\mathbf{H}_\perp+g\lambda_e a^2[\mathbf{m}[\mathbf{m}\Delta\mathbf{H}_e]]. \tag{3}$$

Notice that this equation contains only one relativistic constant λ_r, whereas the original equation (1) contains two (λ_1 and λ_r). Without the exchange relaxation (i.e., for $\lambda_e a^2=0$), Eq. (3) contains only the perpendicular component of the effective field, namely, the component $\mathbf{H}_\perp=\mathbf{H}_e-\mathbf{m}(\mathbf{m}\cdot\mathbf{H}_e)$, and, in this approximation, it literally coincides with the classical Landau-Lifshitz equation for magnetization dynamics. If we limit ourselves in this equation with $\lambda_e a^2=0$ to the leading approximation in the relaxation constant, and substitute into the dissipative term

$$\mathbf{H}_\perp=g^{-1}[\mathbf{m},\ \partial\mathbf{m}/\partial t], \tag{4}$$

then we obtain an equivalent form of the Landau-Lifshitz equation with a relaxation term that has the Gilbert form. Thus, the only natural form of the dynamical equation for the normalized magnetization $\mathbf{m}$, when the exchange relaxation is neglected, is the Landau-Lifshitz equation.

The situation is much more complicated when the exchange relaxation is taken into account. If $\lambda_e a^2\neq 0$, then all the components of $\mathbf{H}_e$, including $H_\parallel\mathbf{m}$, are important in the relaxation term, since

$$[\mathbf{m}[\mathbf{m}\Delta(H_\parallel\mathbf{m})]\}\simeq -H_\parallel[\Delta\mathbf{m}+\mathbf{m}(\nabla\mathbf{m})^2]-2\nabla H_\parallel\nabla\mathbf{m}\neq 0.$$

Let us write the dynamical equation for $\mathbf{m}$ in the leading approximation in the λ constants in the form

$$\frac{\partial \mathbf{m}}{\partial t}=-g[\mathbf{m}\mathbf{H}_e]+\lambda_r\left[\mathbf{m}\frac{\partial\mathbf{m}}{\partial t}\right]+\lambda_e a^2\Big\{\mathbf{m}(\mathbf{m}\Delta(\mathbf{m}H_\parallel))-\Delta(\mathbf{m}H_\parallel)+\mathbf{m}\left(\mathbf{m}\Delta\left[\mathbf{m}\frac{\partial\mathbf{m}}{\partial t}\right]\right)-\Delta\left[\mathbf{m}\frac{\partial\mathbf{m}}{\partial t}\right]\Big\}. \tag{5}$$

Thus, when allowance is made for the exchange relaxation, the equation for the normalized magnetization $\mathbf{m}$ ceases to be a closed equation. The relaxation terms in it contain the quantity $H_\parallel$, which does not enter into the dynamical part of the equation, and should be found separately. In principle, we can get around this by using the equation for $\mathbf{M}$, and not the one for $\mathbf{m}$, but all soliton solutions are constructed on the basis of the dynamical equation for $\mathbf{m}$, and it is also more convenient in the analysis of their relaxation to proceed from such an equation after suitably modifying its relaxation part.

To construct this equation, let us turn to the explicit expression for the energy of a ferromagnet:

$$W=\int[f(M^2)+M^2w\{\mathbf{m}\}]d\mathbf{r}. \tag{6}$$

Here $f(M^2)$ gives the energy density for the exchange interaction determining the length of the magnetization vector and $w\{\mathbf{m}\}$ includes the anisotropy energy w_a and the inhomogeneous exchange energy:

$$w={}^1/_2\alpha(\nabla\mathbf{m})^2+w_a(\mathbf{m}). \tag{7}$$

From (6) we obtain

$$H_\parallel=-2M[w+df/dM^2].$$

Let us use the fact that, at low temperatures, $f(M^2)$ has a sharp minimum at $M^2=M_0^2$, where $M_0(T)$ is the equilibrium value of the magnetization (we assume that $w\{\mathbf{m}\}=0$ in the ground state). In this case we can assume that only the M values close to M_0, i.e., the quantities $\mu=M-M_0\ll M_0$, are essential, and write

$$M\,df/dM^2\approx\mu/\chi_\parallel,\qquad \chi_\parallel=4M_0{}^2d^2f(M_0{}^2)/(dM_0{}^2)^2,$$

where the quantity $\chi_\parallel\ll 1$ has the meaning of a longitudinal susceptibility of the ferromagnet. In this case

$$H_\parallel=-\mu_\parallel/\chi_\parallel-2M_0w.$$

Notice that the assumption that $\mu\ll M_0$ and $\chi_\parallel\ll 1$ is already implied in the energy formula (6), in which terms of, for example, the type $\alpha(\nabla M)^2$ have been discarded. But the assumption that $\chi_\parallel\ll 1$ is correct in the broad range of temperatures lower than the Curie temperature, and we shall assume that it is fulfilled.

In the static case $H_\parallel=0$ and the length of the magnetization is coordinate dependent $M = M_0(1 - 2\chi_\parallel w)$ even when the magnetization distribution is inhomogeneous (for example, in the presence of a stationary domain wall or an external inhomogeneous field). This follows from the well-known fact that in a consistent phenomenological theory M is a function of the local internal field even in the isothermal case. In the presence of a dynamical magnetization wave of fairly large amplitude the quantity $H_\parallel$ is, generally speaking, not equal to zero. To compute $H_\parallel$ or μ, we must use the relation (2). Taking the relation between $H_\parallel$ and μ into account, we obtain for $H_\parallel$ the linear inhomogeneous equation

$$(\chi_\parallel/gM_0)\,\partial H_\parallel/\partial t - \lambda_e a^2 \Delta H_\parallel + [\lambda_1 + \lambda_e a^2 (\nabla \mathbf{m})^2] H_\parallel = (\lambda_e a^2/g)\,(\mathbf{m}\Delta[\mathbf{m},\ \partial\mathbf{m}/\partial t]) - 2\chi_\parallel M_0 \partial w/\partial t. \quad (8)$$

The general solution to this equation without the right-hand side describes the relaxation of $H_\parallel$ to the equilibrium value $H_\parallel = 0$, the characteristic relaxation time being of the order of $(\lambda_1 g M_0/\chi_\parallel)^{-1}$ and very small for $\chi_\parallel \to 0$ (Ref. 4). The particular solution to Eq. (8) can be nonzero only when $\partial \mathbf{m}/\partial t \neq 0$, i.e., only in the presence of a dynamical magnetization wave.

Notice that Eq. (8) is valid up to first order in the small relaxation constants and the quantity $\chi_\parallel$. The equation (5) for the normalized magnetization $\mathbf{m}$ is also valid in this approximation, which is of interest to us here. Thus, Eqs. (5) and (8) constitute a closed system for the three variables: the value of $H_\parallel$ and the two independent quantities describing the unit vector $\mathbf{m}$. It is convenient to choose as these quantities the angle variables defined by the relations

$$m_z = \cos\theta, \quad m_x + i m_y = \sin\theta\, e^{i\varphi}. \quad (9)$$

In our opinion, the system (5), (8) is more convenient than the basic equation (1) for the purpose of analyzing the damping of nonlinear magnetization waves in media with small relaxation constants.

Let us proceed to the formulation of the equations for the soliton parameters, which we obtain from the integrals of motion of the unperturbed equation. The principal integral of motion is the energy W. The rate of change of the energy is determined by the dissipation function Q: $dW/dt = -2Q$ (see the formula (36) in Ref. 4). The soliton energy depends on parameters of the following types: the soliton velocity v, the magnetization precession frequency ω, etc.; in the non-dissipation approximation these quantities are constants.

Let us, on the one hand, find dW/dt as a linear combination of the rates of change, dv/dt, $d\omega/dt$, etc., of the soliton parameters and, on the other hand, compute the value of Q as a function of these parameters. By equating the corresponding quantities, we obtain the sought energy-balance equation. We can, under the assumption that $\chi_\parallel \ll 1$, ignore the variation in length of $\mathbf{M}$ in the expression for the energy W by setting $\mu = \chi_\parallel [H_\parallel + 2M_0 w] \to 0$: then, generally speaking,[3)] $H_\parallel \neq 0$. In this case, to which we limit ourselves (the effects stemming from the finite $\chi_\parallel$ value will be considered in a separate paper), $f(M^2) \sim \mu^2/\chi_\parallel \sim \mu \to 0$, and the energy-balance equation assumes the form

$$dE/dt = -2Q, \quad E = M_0^2 \int w\{\mathbf{m}\}\, d\mathbf{r},$$

and for Q we can, in the case when $\chi_\parallel \ll 1$, use the formula

$$Q = \frac{M_0}{2g} \int \Big\{ \lambda_r \left(\frac{\partial \mathbf{m}}{\partial t}\right)^2 + \lambda_e a^2 \Big(\frac{\partial}{\partial x_i} \Big[\mathbf{m} \frac{\partial \mathbf{m}}{\partial t} \Big] \Big)^2 + g H_\parallel \lambda_e a^2 \Big(\mathbf{m}\Delta \Big[\frac{\partial \mathbf{m}}{\partial t} \mathbf{m} \Big] \Big) \Big\}\, d\mathbf{r}. \quad (10)$$

The dissipation function (10) of a ferromagnet with $\chi_\parallel \ll 1$ depends on $H_\parallel$. To eliminate $H_\parallel$, and express Q in the required form in terms of only $\mathbf{m}(\mathbf{r},t)$, we must use Eq. (8). By eliminating $H_\parallel$, and using the specific structure of the solution to the dynamical equation, we can find the rate of energy dissipation in first order perturbation theory in terms of the relaxation constants.

Similarly, we can find the rates of change of the system's other integrals of motion. Let us, for definiteness, consider the uniaxial-ferromagnet model with energy

$$w = {}^1\!/_2 \alpha (\nabla \mathbf{m})^2 + {}^1\!/_2 \beta (m_x^2 + m_y^2), \quad (11)$$

the z axis being the axis of easy magnetization. Such a ferromagnet admits of the integral of motion I_z equal to the total deviation of the z component of the magnetization from its equilibrium value. Its magnitude can be expressed in terms of the number N of magnons in the soliton:

$$N = (1/2\mu_0) \int (M - M_z)\, d\mathbf{r}.$$

Computing with the aid of Eq. (5) the rate of change of this integral of motion in the case when $\chi_\parallel \ll 1$, we obtain

$$\frac{dN}{dt} = \frac{d}{dt}\Big[(M_0/2\mu_0) \int (1 - m_z)\, d\mathbf{r} \Big] = -(M_0/2\mu_0) \int \{\lambda_r H_{\perp z} + \lambda_e a^2 m_z (\mathbf{m}\Delta \mathbf{H}_e)\}\, d\mathbf{r}. \quad (12)$$

Equations (10) and (12) are adequate for the decription of the evolution of a two-parameter soliton (bion) in a uniaxial ferromagnet. The use of the momentum (P) integral of motion does not lead to new equations, on account of the relation $dE = \hbar\omega dN + v dP$, which is valid for any ferromagnet that admits of the existence of bions.[1] For the analysis of a biaxial ferromagnet, in which I_z is not an integral of motion, we can use the equations for dW/dt and dP/dt.

Let us consider the relaxation of a bion in a uniaxial ferromagnet. Let us, using the angle variables (9) for $\mathbf{m}$, rewrite (10) and (12) in the form

$$\dot{E} = -E_0 q, \quad \dot{N} = -(E_0/\hbar\omega_0)\,\eta, \quad (13)$$

where $E_0 = M_0^2 a^2 (\alpha\beta)^{1/2}$ is a characteristic value of the energy, a is the lattice constant, $\omega_0 = g\beta M_0$ is the ferromagnetic resonance frequency, and the dot denotes differentiation with respect to the dimensionless time $\tau = \omega_0 t$. The functions q and η are determined by the sum of the relativistic (q_r, η_r) and exchange (q_e, η_e) terms. The contribution of the exchange relaxation can, in its turn, be conveniently represented in the form of a sum of two terms, the first of which does not contain $H_\parallel$ and the second is linear in this function (see (10) and (12)). Finally, we obtain

$$q = q_r + q_e^{(1)} + q_e^{(2)}, \quad \eta = \eta_r + \eta_e^{(1)} + \eta_e^{(2)},$$

where

$$q_r=\lambda_r\langle\dot\theta^2+\dot\varphi^2\sin^2\theta\rangle,\quad \eta_r=\lambda_r\langle\dot\varphi\sin^2\theta\rangle; \tag{14}$$

$$\begin{aligned} q_e^{(1)}=\lambda_e'\langle(\dot\theta')^2+\dot\theta^2(\varphi')^2+(\theta')^2\dot\varphi^2+(\dot\varphi')^2\sin^2\theta \\ +\sin 2\theta[\dot\theta\varphi'\dot\varphi'+\theta'\dot\varphi\dot\varphi'-\dot\theta'\dot\varphi\varphi'] \\ +\dot\varphi^2(\varphi')^2\sin^2\theta\cos^2\theta+2\theta'\dot\theta\varphi'\dot\varphi\cos 2\theta\rangle, \end{aligned} \tag{15}$$

$$\begin{aligned} \eta_e^{(1)}=\lambda_e'\langle(\theta')^2\dot\varphi+\dot\theta\theta'\varphi'\cos 2\theta+\dot\varphi(\varphi')^2\sin^2\theta\cos^2\theta \\ +\sin\theta\cos\theta(\theta'\dot\varphi'-\dot\theta'\varphi')\rangle; \end{aligned} \tag{16}$$

$$\begin{aligned} q_e^{(2)}=(\lambda_e'/M_0)\langle H_\parallel\sin\theta[\dot\theta\varphi''-\theta''\dot\varphi+2\dot\theta'\varphi'-2\theta'\dot\varphi'] \\ -2H_\parallel\dot\varphi\cos\theta[(\theta')^2+(\varphi')^2\sin^2\theta]\rangle, \end{aligned} \tag{17}$$

$$\begin{aligned} \eta_e^{(2)}=-(\lambda_e'/M_0) \\ \times\langle H_\parallel[2(\theta')^2\cos\theta+\theta''\sin\theta+(\varphi')^2\sin^2\theta\cos\theta]\rangle. \end{aligned} \tag{18}$$

In these formulas $\lambda_e'=\lambda_e(a/x_0)^2$ and the prime and angle brackets respectively denote differentiation with respect to, and integration over, the dimensionless space variable $\xi=x/x_0$, where $x_0=(\alpha/\beta)^{1/2}$.

We shall use the above general formulas to analyze two-parameter solitons in a one-dimensional ferromagnet.

2. EVOLUTION OF THE PARAMETERS OF A ONE-DIMENSIONAL BION

The magnetization distribution in a bion of a one-dimensional ferromagnet is described by the relations[1]

$$\varphi=\Omega t+\psi(x-vt),\quad x_0(d\psi/dx)=-(v/v_m)(1/\cos^2\theta/2), \tag{19}$$

$$\mathrm{tg}^2\frac{\theta}{2}=\frac{\varkappa^2}{A\,\mathrm{ch}^2[\varkappa(x-vt)/x_0]+(B-A)/2},$$

where $v_m=2\omega_0x_0$, $\varkappa=[1-\Omega/\omega_0-(v/v_m)^2]^{1/2}$, $x_0/\varkappa$ is the effective width of the soliton,

$$A^2=B^2+4\varkappa^2(v/v_m)^2,\quad B=\Omega/\omega_0+2(v/v_m)^2.$$

The parameters determining the structure of the soliton are its velocity v and precession frequency Ω. The soliton solution (19) exists in the region $\varkappa^2>0$ or

$$\Omega/\omega_0+(v/v_m)^2<1. \tag{20}$$

The integrals of motion E and N corresponding to the soliton (19) are, when computed for an area of a^2, equal to[1]

$$E=4E_0(1-\omega-u)^{1/2},$$

$$N=(2E_0/\hbar\omega_0)\,\mathrm{arsh}\{2[(1-\omega-u)/(\omega^2+4u)]\}^{1/2},$$

where $\omega=\Omega/\omega_0$ and $u=(v/v_m)^2$ are the dimensionless soliton parameters, which are convenient for the subsequent calculations.

The evolution of the integrals of motion E and N is connected with the corresponding variations of the parameters u and ω. From the formulas (19) and (13) we obtain for the functions ω and u the system of equations

$$\dot\omega=f(\omega,u)=(4\varkappa)^{-1}\{(2-\omega)q-(\omega^2+4u)\eta\},$$

$$\dot u=g(\omega,u)=(4\varkappa)^{-1}\{-(\omega+2u)q+(\omega^2+4u)\eta\}.$$

Since the functions q and η are each represented in the form of a sum of three terms, the functions f and g, which govern the evolution of the bion parameters, can be written in the form

$$f=f_r+f_e^{(1)}+f_e^{(2)},\quad g=g_r+g_e^{(1)}+g_e^{(2)},$$

where f_r and g_r are determined by the relativistic relaxation, $f_e^{(1)}$ and $g_e^{(1)}$ are connected with those exchange relaxation terms which do not depend on $H_\parallel$, and $f_e^{(2)}$ and $g_e^{(2)}$ can be represented in the form of integrals of expressions linear in $H_\parallel$ (see (14)–(18)).

The explicit form of the relativistic terms and $f_e^{(1)}$, $g_e^{(1)}$ are easily obtained by substituting the explicit form of the magnetization distribution (19) in the soliton into the formulas (14) and (15), (17). As a result we obtain

$$\begin{aligned} f_r=\lambda_r(\omega^2+4u)[2(1-\omega)+(I_0/\varkappa)(\omega-\omega^2-2u)], \\ g_r=\lambda_r u(\omega^2+4u)[-2+(I_0/\varkappa)(2-\omega)]; \end{aligned} \tag{21}$$

$$\begin{aligned} f_e^{(1)}=\lambda_e'\{2\omega^2(1-\omega)+{}^2/_3u(3\omega^3-16\omega^2+4)+{}^{16}/_3u^2(\omega-5) \\ +(I_0/\varkappa)[-\omega^3(1-\omega)+u\omega(\omega^3-4\omega^2+10\omega-4) \\ +4u^2(\omega^2-4\omega+6)]\}, \\ g_e^{(1)}=\lambda_e'u\{{}^2/_3\omega(3\omega^2-\omega+4)+4u(\omega^2+{}^{10}/_3)+{}^{32}/_3u^2 \\ +(I_0/\varkappa)[\omega^2(\omega^2-\omega-2)+2u(\omega^3-2\omega-4)+8u^2(\omega-2)]\}, \end{aligned} \tag{22}$$

where $I_0=\mathrm{Artanh}[2\varkappa/(1-\omega)]$.

As shown in Ref. 12, similar formulas describe the evolution of the soliton parameters in the case of fairly high soliton frequencies ($\omega>\lambda_r,\lambda_e(a/x_0)^2$).

Analysis shows that dissipative terms of these two types have essentially different effects on the evolution of the soliton parameters. In the entire admissible region of the soliton parameters [$u+\omega<1$ (see (20))], $f_r>0$ and $g_r>0$, i.e., both the precession frequency and the velocity of the soliton increase during the relaxation process (the soliton is accelerated; this result is obtained in Ref. 10 for low-amplitude solitons).

The nature of the relativistic relaxation is graphically depicted by the integral curves of the equations $\dot\omega=f_r$ and $\dot u=g_r$, as obtained by numerical integration of these equations (see Fig. 1). As to the contribution of the exchange

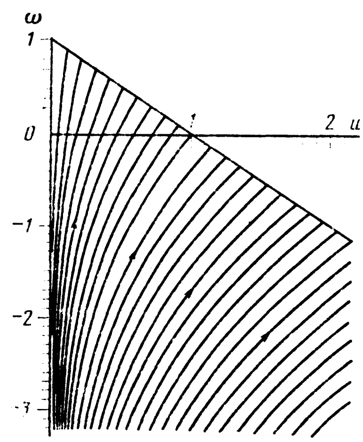

FIG. 1. Evolution of the soliton parameters as a result of the relativistic relaxation.

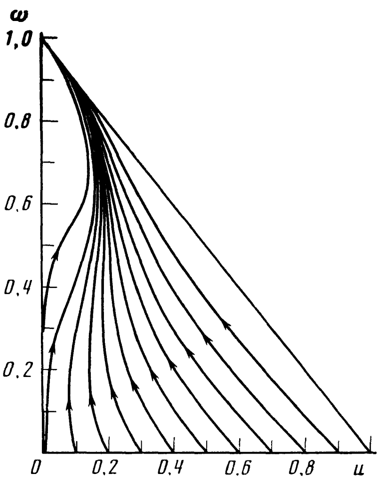

FIG. 2. Evolution of the soliton parameters when allowance is made for only the exchange relaxation in the case when $H_\parallel = 0$.

relaxation, $f_e^{(1)}$ is also greater than zero in the entire (u,ω) region, but the sign of $g_e^{(1)}$ can change, and in practically the entire region $g_e^{(1)} < 0$ (see Fig. 2). Thus, the contributions of the exchange and relativistic relaxations to $\dot u$ have opposite signs. If the contribution of $H_\parallel$ is small, and the terms with $f_e^{(2)}$ and $g_e^{(2)}$ can be neglected (for the corresponding condition, see below), then the evolution of the soliton parameters in the course of the relaxation is governed by the competition between these contributions: the increase of the contribution of the exchange relaxation leads to the increase of that parameter region in the (u,ω) plane where the soliton velocity decreases.

The investigation of the contribution of $f_e^{(2)}$ and $g_e^{(2)}$ to the exchange relaxation is a much more complicated problem, since it requires the computation of the field $H_\parallel$ on the basis of Eq. (8). Analysis shows that the nature of the relation between $H_\parallel$ and $\mathbf{m}(\mathbf{r},t)$ is determined by the relation connecting the parameter $\chi_\parallel$ to the parameters λ_1 and $\lambda_e a^2$. The functional relation between $H_\parallel$ and $\mathbf{m}$ in the case when $\chi_\parallel \gg \lambda_1$ is essentially different from the corresponding relation in the $\chi_\parallel \ll \lambda_1$ case. We are actually interested in the limiting value of $H_\parallel$ for $\chi_\parallel \to 0$ and $\lambda_1, \lambda_e a^2 \to 0$ ($f_e^{(2)}$ and $g_e^{(2)}$ themselves already contain the factor $\lambda_e a^2$, and we are ignoring the effect of a finite $\chi_\parallel$). Consequently, a nonanalyticity problem arises in the computation of $H_\parallel$, i.e., the value of $H_\parallel$ at small λ and $\chi_\parallel$ values depends on the order in which we take the $\chi_\parallel \to 0$ and $\lambda_1 \to 0$ limits. Let us consider two different cases: $\chi_\parallel \gg \lambda_1$ and $\chi_\parallel \ll \lambda_1$, which corresponds to two possible sequences in which the passage to the limit can be effected.

Let $\chi_\parallel \ll \lambda_1$. We can, for the purpose of analyzing this case, set $\chi_\parallel = 0$ in (8). The value of $H_\parallel$ is then given by the solution to the equation

$$\hat{L} H_\parallel = -\lambda_e a^2 \Delta H_\parallel + [\lambda_1 + \lambda_e a^2 (\mathbf{m}')^2] H_\parallel = \frac{\lambda_e a^2}{g x_0^2}\left(\mathbf{m}\left[\mathbf{m}, \frac{\partial \mathbf{m}}{\partial t}\right]''\right). \quad (23)$$

The operator $\hat{L}$ is positive-definite, and the solution that decreases in the region far from the soliton can be uniquely determined.

The solution to (23) for an arbitrary soliton can be written down in terms of the Green function for the operator $\hat{L}$, but we shall limit ourselves to the consideration of the simplest cases in which the relation connecting $H_\parallel$ to $\mathbf{m}$, $\mathbf{m}'$, and $(\partial \mathbf{m}/\partial t)$ is algebraic. This can be done for a low-amplitude soliton, for which $\varkappa = (1-\omega-u)^{1/2} \ll 1$. For this soliton

$$\theta \propto \varkappa/\mathrm{ch}[\varkappa(x-vt)], \quad x_0^2 (\nabla \mathbf{m})^2 \propto \varkappa^2 \ll 1.$$

Since the dimension of the region of localization of $H_\parallel$ is of the order of $1/\varkappa$, $\Delta H_\parallel \sim \varkappa^2 H_\parallel$, and the term with $\Delta H_\parallel$ can be neglected. As a result we obtain

$$H_\parallel = \frac{\lambda_e a^2}{\lambda_1 g x_0^2}\left(\mathbf{m}\left[\mathbf{m}\frac{\partial \mathbf{m}}{\partial t}\right]''\right). \quad (24)$$

This formula can be used also for a soliton of arbitrary amplitude if the constant λ_1 is formally considered to be sufficiently large, specifically, if $\lambda_1 \gg \lambda_e a^2/(\Delta x)^2$, where Δx is the region of localization of the soliton.

Let $\chi_\parallel \gg \lambda_1$. In this case the situation is slightly more complicated, since the result depends nonanalytically on the value of $\lambda/\varkappa_\parallel v$. The analysis shows that there appears in the problem a characteristic value v_0 for the soliton's forward speed: $v_0 \sim \lambda g M_0 (\Delta x)/\chi_\parallel \ll \omega_0 x_0$. For extremely low velocities, when $v < v_0$, we obtain

$$H_\parallel = 2M_0 \chi_\parallel v (dw/d\xi)/[\lambda_1 + \lambda_e (a^2/x_0^2)(\mathbf{m}')^2], \quad \xi = x - vt. \quad (25)$$

Naturally, $H_\parallel$ vanishes in the static case (i.e., in the case when $v \to 0$). In the more interesting $v > v_0$ case the quantity $H_\parallel$ does not depend on $\chi_\parallel$ and v:

$$H_\parallel = -2M_0 w = -\beta M_0 [\theta'^2 + \sin^2\theta (1 + (\varphi')^2)]. \quad (26)$$

The fact that the explicit form of $H_\parallel$ expressed in terms of $\mathbf{m}(x,t)$ is different demonstrates the above-indicated nonanalyticity of the problem at small values of the parameters $\chi_\parallel$ and λ_1, $\lambda_e a^2$. Consequently, the form of $f_e^{(2)}$ and $g_e^{(2)}$ for a specific system depends on the values of the longitudinal susceptibility $\chi_\parallel$ and the constant λ_1; the latter describes the rate of uniform relaxation of the length of the magnetization $\mathbf{M}$ to the equilibrium value. Notice that the constant λ_1 does not have the literal meaning of a relaxation constant in the dynamical equations for the vector $\mathbf{m}$. As can easily be seen from (24), as λ_1 increases, the value of $H_\parallel$ (and, consequently, the rate of energy dissipation) decreases.

Let us consider the estimate for the contribution of $H_\parallel$ to the evolution of the soliton parameters only in the above-indicated limiting cases. Analysis shows that, if $\chi_\parallel \ll \lambda_1$ and the value of λ_1 is large (i.e., if $\lambda_1 \gg \lambda_e a^2/x_0^2$), so that we can use the formula (24), then

$$f_e^{(2)}(g_e^{(2)}) \sim [f_e^{(1)}(g_e^{(1)})](\lambda_e a^2/\lambda_1 x_0^2)$$

and $f_e^{(2)} \ll f_e^{(1)}$, $g_e^{(2)} \ll g_e^{(1)}$ in the entire region of the soliton parameters. In this case we can neglect the contribution of $H_\parallel$, and assume that, for $\chi_\parallel \ll \lambda_1$, $\lambda_e a^2/x_0^2 \ll \lambda_1$,

$$\dot u = g_r + g_e^{(1)}, \quad \dot\omega = f_r + f_e^{(1)}.$$

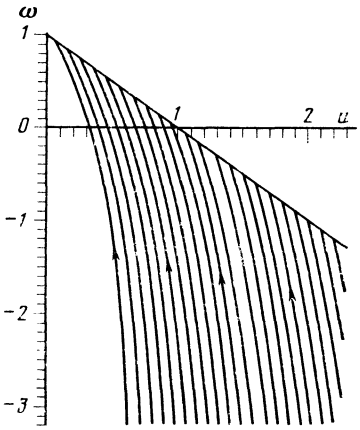

FIG. 3. Evolution of the soliton parameters as a result of the exchange relaxation in the case when $H_{\parallel}$ is given by the formula (26).

The corresponding integral curves are discussed above.

In the other limiting situation $\chi_{\parallel} \gg \lambda_1$ the value of $H_{\parallel}$ is not small (see (26)). In this case we can have $f_e^{(1)} \sim f_e^{(2)}$, $g_e^{(1)} \sim g_e^{(2)}$, so that the exchange relaxation is described by the sum of these two terms. Let us write out the explicit expressions for

$$f_e = f_e^{(1)} + f_e^{(2)}, \qquad g_e = g_e^{(1)} + g_e^{(2)}:$$

$$f_e = \lambda_e' \Big\{ \omega^2 \Big(\frac{5}{2}\omega^3 - \frac{53}{6}\omega^2 + \frac{5}{3}\omega + \frac{14}{3} \Big) + \frac{u}{3}(59\omega^3 - 192\omega^2 + 20\omega + 56) + \frac{4}{3}u^2(29\omega - 86) + \frac{I_0}{\varkappa}\Big[\omega^3 \Big(\frac{5}{4}\omega^3 - \frac{21}{4}\omega^2 + 11\omega - 7 \Big) + u\omega \Big(\frac{23}{2}\omega^3 - 32\omega^2 + 58\omega - 28 \Big) + 4u^2(9\omega^2 - 11\omega + 14) + 40u^3 \Big]\Big\}, \tag{27}$$

$$g_e = \lambda_e' u \Big\{ \omega^2 \Big(\frac{5}{2}\omega^2 - \frac{40}{3}\omega + \frac{64}{3} \Big) + \frac{2u}{3}(19\omega^2 - 80\omega + 128) + \frac{32}{3}u^2 + \frac{I_0}{\varkappa}\Big[\omega^2 \Big(\frac{5}{4}\omega^3 - \frac{15}{2}\omega^2 + 15\omega - 14 \Big) + 4u(2\omega^3 - 10\omega^2 + 15\omega - 14) + 4u^2(3u - 10) \Big]\Big\}. \tag{28}$$

It is interesting to note that f_e is, on account of (27) greater than zero, i.e., the soliton frequency increases, but that in this case $g_e < 0$, i.e., the soliton velocity decreases in the entire parameter region (see Fig. 3). Thus, in the $\chi_{\parallel} \gg \lambda_1$ case the "competition" between the exchange and relativistic contributions to the relaxation is even more intense. The sign of the derivative $\dot{u}$ and the forms of the functions $\omega(\tau)$ and $u(\tau)$ are determined by both the specific values of the parameters $u(0)$ and $\omega(0)$ and the relation between the relaxation constants λ_r and λ_e' (see Fig. 4).

Let us analyze in greater detail some limiting cases corresponding to the characteristic regions in the (u,ω)-parameter plane.

For low-amplitude solitons, to which corresponds the region $\varkappa \ll 1$ (the region of the parameters u and ω is located in the vicinity of the straight line $u + \omega = 1$ in the (u,ω) plane), the analysis is facilitated by the fact that, in both cases analyzed above, the contribution of $H_{\parallel}$ contains the next power of the small parameter $\varkappa$, and the nature of the soliton relaxation does not depend on the relation between $\chi_{\parallel}$ and λ_1.

1. At large values of the soliton velocity (i.e., for $u \gg 1$) the equation for u and ω have the asymptotic forms

$$\dot{u} = {}^8/_3(\lambda_r - \lambda_e')u\varkappa^2, \quad \dot{\omega} = 4((\lambda_r/3) + u\lambda_e')u\varkappa^2. \tag{29}$$

At room temperatures we can assume that $\lambda_e a^2/\alpha \sim \lambda_r/\beta$, i.e., $\lambda_r \sim \lambda_e'$. The solution to the system of equations (29) with the initial conditions $u = u(0)$ and $\omega = \omega(0)$ at $\tau = 0$ can be written in the form

$$u = u_1 + (u(0) - u_1)\exp\{-4\lambda_e'\tau u_1^2\},$$
$$\varkappa(\tau) = \varkappa(0)\exp\{-2\lambda_e' u_1^2 \tau\}, \tag{30}$$
$$u_1 = u(0) - {}^2/_3(\lambda_r - \lambda_e')\varkappa^2(0)/\lambda_e' u(0).$$

Thus, in the $\varkappa \ll 1$, $u \gg 1$ region under consideration the relaxation of the soliton is governed largely by the exchange interactions.

For $\tau \to \infty$, $u \to u_1$ and $\varkappa \to 0$, i.e., the integral curve $\omega = \omega(u)$ for Eq. (27) terminates (for $\tau \to \infty$) at that point on the straight line $u + \omega = 1$ where the soliton has zero amplitude and infinite effective width $(x_0/\varkappa)$ (the soliton degenerates into a homogeneous magnetization distribution).

Since the quantity $\varkappa$ is equal, apart from a dimensional factor, to the soliton energy E (see (19)), the exponential function $\varkappa(\tau)$ allows us to introduce an effective soliton lifetime $\tau_s = (2\lambda_e' u_1^2)^{-1}$, which, for fast low-amplitude solitons, is inversely proportional to the fourth power of the soliton velocity: $\tau_s \propto v^{-4}$.

2. At low velocities (i.e., for $u \ll 1$) the equations describing the relaxation of a low-amplitude $(\varkappa \ll 1)$ soliton have the form

$$\dot{u} = {}^8/_3(\lambda_r - \lambda_e')u\varkappa^2, \quad \dot{\omega} = 4\varkappa^2[\lambda_r + (\lambda_e'/3)(5u + \varkappa^2)]. \tag{31}$$

The solution to the system (29) for $\lambda_r \sim \lambda_e'$ has the form

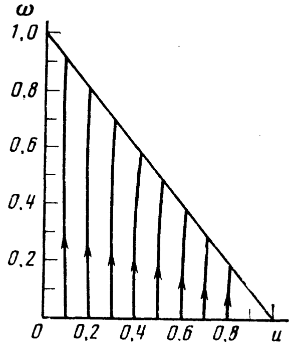

FIG. 4. Competition between the exchange and relativistic relaxations. When $\lambda_e = 0.5\lambda_r$, the soliton velocity almost does not vary in the course of the relaxation.

$$\varkappa(\tau)=\varkappa(0)\exp(-2\lambda_r\tau),$$

$$u(\tau)=u(0)\exp\left\{\frac{2}{3}\frac{\lambda_r-\lambda_e'}{\lambda_r}[1-\exp(-4\lambda_r\tau)]\right\}. \qquad (32)$$

For $\tau\to\infty$

$$u\to u_1=u(0)\exp\{2(\lambda_r-\lambda_e')/3\lambda_r\},\quad \varkappa\to 0.$$

In this case also the integral curve terminates on the line $u+\omega=1$.

It follows from (32) that the relaxation of a low-amplitude, low-velocity soliton is governed largely by the relativistic interactions. The soliton lifetime τ_s in this case is equal to $1/2\lambda_r$, and does not depend on the velocity. But the final value u_1 of the soliton velocity depends on λ_r and λ_e: $u_1>u(0)$ when $\lambda_r>\lambda_e'$.

It is not difficult to write out the solution to the equations describing the evolution of a low-amplitude soliton in the case of arbitrary values of the parameters u. We shall not do this because of the unwieldiness of the corresponding formulas. Let us only note that the exponential time dependence of the soliton parameters obtains everywhere in the vicinity of the line $u+\omega=1$ irrespective of the value of u. In the vicinity of the boundary of the existence region $N\propto\varkappa$; therefore, in the course of the relaxation process the number of magnons tends to zero according to an exponential law.

The stationary soliton. If $v=0$ at the initial moment of time, the soliton remains stationary at all subsequent moments of time. The integral curve in this case is a segment of the axis of ordinates.

For $|\omega|\ll 1$ we find from the formula for $\dot\omega$ that

$$\dot\omega=2\bar\lambda\omega^2,\quad \omega(\tau)=\omega(0)/[1-2\omega(0)\bar\lambda\tau], \qquad (33)$$

where $\bar\lambda=\lambda_r+\lambda_e'$ for $H_\parallel=0$, which is characteristic of the $v=0$ case.

It can be seen that, for $\omega(0)>0$, the precession frequency increases rapidly, and attains a value $\omega\sim 1$ over a finite period of time. If, on the other hand, $\omega(0)<0$, then for $\tau\to\infty$ the frequency tends asymptotically to zero in a power-law fashion. Thus, a stationary soliton with $\omega>0$ turns into a low-amplitude soliton over a finite period of time and then degenerates exponentially while a stationary soliton with $\omega<0$ becomes transformed into the singular solution of the Landau-Lifshitz equation with $\omega=0$, $v=0$, which describes two domain walls located infinitely far from each other.

A low-velocity soliton. If the initial value of the velocity is small, but nonzero, then, depending on the relation between λ_r and λ_e' and the one between $\varkappa_\parallel$ and λ_1, the soliton velocity can either decrease or increase in the course of the evolution. For $\omega\approx 1$, i.e., in the case of a low-amplitude soliton, the velocity increases when $\lambda_r>\lambda_e'$ and decreases when $\lambda_r<\lambda_e'$ (see (32)). But if $\omega<0$ and $|\omega|\gg 1$, then

$$\dot u={}^8/_3(\lambda_r-c\lambda_e')u|\omega|, \qquad (34)$$

where $c=\frac{13}{5}$ and $\frac{39}{7}$ for $H_\parallel=0$ and $H_\parallel=-2M_0w$, i.e., for $\chi_\parallel\ll\lambda_1$ and $\chi_\parallel\gg\lambda_1$ respectively. Thus, the soliton velocity decreases when $\lambda_r>c\lambda_e'$ and increases when $\lambda_r<c\lambda_e'$. In this case an almost stationary soliton accelerates in the course of its evolution.

The specific calculations carried out above demonstrate the fairly complicated nature of the evolution of the parameters of a magnetic bion in the course of its relaxation. Notice that an adequate description of the dissipation of the energy of a nonlinear excitation in a ferromagnet is impossible without allowance for the exchange relaxation processes. They not only affect the quantitative mobility estimates, as has been found in the case of the domain wall,[4] but can also lead to a qualitatively different soliton-parameter evolution picture. The role of the exchange mechanism of relaxation is most important at large values of the velocity of precession frequency of the soliton. This result is due to the fact that, for $\omega<0$ and $|\omega|\gg 1$, the effective soliton width $(x_0/\varkappa)$ is small, and the presence in the exchange relaxation term of the second derivative leads to the appearance of a large factor proportional to $\varkappa^2\gg 1$. But if this factor is small, then the dominant role is played by the relativistic relaxation processes.

In the principal soliton-parameter region, where $\omega\lesssim 1$, $u\lesssim 1$, i.e., $v\lesssim\omega_0x_0$, the contributions of the exchange and relativistic relaxations turn out to be comparable when we use the reasonable estimate[4] $\lambda_r\sim\lambda_e'$. The above-described complicated bion-parameter evolution picture is the result of the competitive natures of these interactions.

[1)] Such processes as coherent and Cherenkov radiation, the "disintegration" or collapse of a soliton as a result of its instability, etc., naturally cannot be considered within the framework of such an approach.

[2)] The Onsager relations are used in Ref. 11 to describe the linear dynamics of a ferromagnet.

[3)] The nonanalyticity of the problem in $\chi_\parallel$ (when $\chi_\parallel\ll 1$, the value of $\mu\to 0$, but then the component $H_\parallel$, the expression for which contains the term $\mu/\chi_\parallel$, is, generally speaking, finite) is one of the reasons why it is more convenient to write the supplementary equation (8) in terms of $H_\parallel$ than in terms of μ or M.

[1] A. M. Kosevich, B. A. Ivanov, and A. S. Kovalev, Nelineĭnye volny namagnichennosti (Nonlinear Magnetization Waves), Nauk. Dumka, Kiev, 1983.

[2] A. E. Borovik, Pis'ma Zh. Eksp. Teor. Fiz. **28**, 629 (1978) [JETP Lett. **28**, 581 (1978)]; E. K. Sklyanin, Preprint No. E-3, LOMI, Leningrad, 1979.

[3] A. P. Malozemoff and J. C. Slonczewski, Applied Solid State Science, Supplement I: Magnetic Domain Walls in Bubble Materials, Academic Press, New York, 1979 (Russ. Transl., Mir, Moscow, 1982).

[4] V. G. Bar'yakhtar, Zh. Eksp. Teor. Fiz. **87**, 1501 (1984) [Sov. Phys. JETP **60**, 863 (1984)].

[5] A. S. Abyzov and B. A. Ivanov, Zh. Eksp. Teor. Fiz. **76**, 1700 (1979) [Sov. Phys. JETP **49**, 865 (1979)]; B. A. Ivanov, Yu. N. Mitsaĭ, and N. V. Shakhova, Zh. Eksp. Teor. Fiz. **87**, 289 (1984) [Sov. Phys. JETP **60**, 168 (1984)].

[6] A. I. Akhiezer, V. G. Bar'yakhtar, and S. V. Peletminskiĭ, Spinovye volny (Spin Waves), Nauka, Moscow, 1967, 368 pages (Eng. Transl., North-Holland, Amsterdam; Wiley, New York, 1968).

[7] D. J. Kaup and A. C. Newell, Proc. R. Soc. London Ser. A **361**, 413 (1978).

[8] V. I. Karpman and E. M. Maslov, Zh. Eksp. Teor. Fiz. **73**, 537 (1977) [Sov. Phys. JETP **46**, 281 (1977)].

[9] D. McLaughlin and Alwyn C. Scott, in: *Solitons in Action* (Ed. by Karl Lonngren and Alwyn C. Scott), Academic Press, New York, 1978, (Russ. Transl., Mir, Moscow, 1981, p. 210).

[10] V. G. Bar'yakhtar, Pis'ma Zh. Eksp. Teor. Fiz. **42**, 49 (1985) [JETP Lett. **42**, 56 (1985)].

[11] I. E. Dzyaloshinskiĭ and B. G. Kukharenko, Zh. Eksp. Teor. Fiz. **70**, 2360 (1976) [Sov. Phys. JETP **43**, 1232 (1976)].

[12] V. I. Karpman, E. M. Maslov, and V. V. Solov'ev,. Zh. Eksp. Teor. Fiz. **84**, 289 (1983) [Sov. Phys. JETP **57**, 167 (1983)].

Translated by A. K. Agyei