Bounds on inflationary models of the universe[1)]

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Bounds on inflationary models are obtained from the possible spatial inhomogeneity of the initial scalar field. It follows from the exponential estimates for the probability of formation of "our" universe obtained in the framework of the chaotic inflationary scenario that the very idea of this scenario is unrealistic for the minimally required degree of inflation.

1. In the recent papers Refs. 1 and 2 the bounds obtained on all inflationary models of the universe were stronger than in Ref. 3. These bounds are general in nature and do not depend on the details of the inflationary model. In the derivation of the bounds it was assumed that the scale factor is spatially homogeneous (the exponential expansion is spatially homogeneous). In particular, it follows from the results of Ref. 1 that the chaotic inflationary scenario[4,5] with potential $V(\varphi) = (\lambda/4)\varphi^4$ and $\lambda \lesssim 10^{-2}$ is unrealistic, since the bounds obtained in Ref. 1 lead to $\lambda \lesssim 10^{-11}$. It was shown independently in Ref. 6 that the chaotic model of Ref. 4 is unsatisfactory from the point of view of probability theory.

Below, using the bounds of Refs. 1 and 2, we obtain additional bounds on all inflationary models. These bounds follow from consideration of the admissible spatial inhomogeneity of the scale factor (i.e., of the spatial inhomogeneity of the exponential expansion). We also give exponential estimates for the probability of formation in the chaotic inflationary model of Ref. 4 of a universe with "our" homogeneous properties. These estimates lead to the conclusion that the very idea of the chaotic scenario is unrealistic for the minimally necessary degree of inflation. In this connection we note that, as is asserted in Ref. 5, the realization of the new scenario[7,8] in the framework of relic inflation in supergravity is possible only in the chaotic scenario of Ref. 4. But in Ref. 5 it is also asserted that quantum creation of the universe[9] can be realized in the chaotic scenario of Ref. 4.

2.[2)] As follows from Ref. 1, for all inflationary models a necessary condition on the admissible inhomogeneity of the initial scalar field $\varphi(x)$ must be satisfied:

$$(\nabla\varphi)^2 \leqslant V(\varphi) \to |\nabla\varphi| = \beta V^{1/2}(\varphi), \quad \beta \leqslant 1. \tag{1}$$

From Ref. 1 there follow an estimate for the duration Δt of the time of exponential expansion,

$$\Delta t \leqslant [24\pi V^3(\varphi)]^{1/2}/m_p^2[dV(\varphi)/d\varphi]^2 \tag{2}$$

and also the ordinary formula for the Hubble constant:

$$H = [(8\pi/3m_p^2)V(\varphi)]^{1/2}. \tag{3}$$

In all inflationary models (see, for example, Ref. 4), the scalar field varies little during the time interval Δt of the inflation. Therefore, in what follows, we shall assume that $\varphi(x,t) \approx \varphi(x,0) \equiv \varphi(x)$. We note immediately that the Hubble constant $H(3)$ is in fact spatially inhomogeneous because of its dependence on the scalar field $\varphi(x)$ and, through this, on the possible inhomogeneity of this field $\varphi(x)$. Similar considerations lead to a possible spatial inhomogeneity of the duration of the time interval $\Delta t(2)$ of exponential expansion.

Suppose (to simplify the expressions but without loss of generality of the results) that $V(\varphi)$ is polynomial:

$$V(\varphi) = \alpha_n\varphi^n. \tag{4}$$

Then from (2) and (3) we obtain

$$\Delta t \leqslant [24\pi\alpha_n^3\varphi^{3n}]^{1/2}/m_p^2(n\alpha_n\varphi^{n-1})^2, \tag{2a}$$

$$H = [(8\pi/3m_p^2)\alpha_n\varphi^n]^{1/2}. \tag{3a}$$

We denote $\varphi_0 = \varphi(x=0)$, where $x = 0$ is the central point of the spatial region measuring $l \approx 2H^{-1}$ that as a result of the exponential expansion is transformed into the visible universe with scale $\approx 10^{28}$ cm. As follows from (2a) and (3a), the scale factor R is determined by (see, for example, Ref. 10)

$$R = \exp(4\pi\varphi_0^2/nm_p^2). \tag{5}$$

From the estimates of the scale of the observed universe ($\approx 10^{28}$ cm) it follows necessarily that

$$R(x=0) = \exp(4\pi\varphi_0^2/nm_p^2) = \exp(64N^2). \tag{6}$$

On the basis of (6) we obtain

$$\varphi_0 = (16n/\pi)^{1/2}m_pN. \tag{7}$$

We now estimate (rather crudely but with sufficient accuracy for the following conclusions) the inhomogeneity of the initial scalar field $\varphi(x)$ in the spatial region measuring $l \approx 2H^{-1}$ that is transformed by the exponential expansion into the observed universe. We have

$$\varphi(x=0\pm l) \approx \varphi_0 \pm l|\nabla\varphi_0|. \tag{8}$$

On the basis of (1) and (3) we obtain

$$\varphi(x=0\pm l) \approx \varphi_0 \pm 2\beta(3/8\pi)^{1/2}m_p = \varphi_0[1 \pm 2\beta(3/8\pi)^{1/2}(m_p/\varphi_0)]. \tag{9}$$

From (9) we obtain on the basis of (7)

$$\varphi(x=0\pm l) \approx \varphi_0[1 \pm \beta(3/2n)1/8N]. \tag{10}$$

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Since in accordance with (5) the scale factor R depends explicitly on $\varphi(x)$, we obtain on the basis of (10) an estimate (also very rough) of the spatial inhomogeneity of the scale factor:

$$R(x=0\pm l)\approx \exp\left(4\pi\frac{\varphi_0^2}{nm_p^2}\right)$$
$$\times\exp\left[4\pi\frac{\varphi_0^2}{nm_p^2}\left(\pm\beta\left(\frac{3}{2n}\right)^{1/2}\frac{1}{4N}+\beta^2\frac{3}{32nN^2}\right)\right]$$
$$=R(x=0)\exp\left[64N^2\left(\pm\beta\left(\frac{3}{2n}\right)^{1/2}\frac{1}{4N}+\beta^2\frac{3}{32nN^2}\right)\right]. \quad (11)$$

Despite the crudity of the estimates (10) and (11), we have obtained an exponential amplification in the scale factor of the inhomogeneities of the scalar field $\varphi(x)$. On the basis of (3a) and (10), we obtain the corresponding spatial inhomogeneity of the Hubble constant H:

$$H(x=0\pm l)\approx H(x=0)\left[1\pm\beta\left(\frac{3}{2n}\right)^{1/2}\frac{1}{8N}\right]^{n/2}. \quad (12)$$

On the basis of (11), we obtain

$$\frac{\delta R}{R}=\frac{R(x=0\pm l)-R(x=0)}{R(x=0)}$$
$$=\exp\left(\pm\beta 16N\left(\frac{3}{2n}\right)^{1/2}+\beta^2\frac{6}{n}\right)-1. \quad (13)$$

Since $\delta R/R$ must be small, it follows that[3)]

$$\delta R/R\approx 16\beta N(3/2n)^{1/2}. \quad (14)$$

Similarly,

$$\frac{\delta H}{H}=\frac{H(x=0\pm l)-H(x=0)}{H(x=0)}\approx\pm\beta\left(\frac{3n}{2}\right)^{1/2}\frac{1}{16N}. \quad (15)$$

Thus, on the basis of initial inhomogeneity of the initial scalar field $\varphi(x)$ we have obtained spatial inhomogeneity (spatial inhomogeneity of the scale factor R and of the Hubble constant H). The spatial inhomogeneity corresponding to the spatial inhomogeneity of the scale factor and the Hubble constant corresponds, of course, to a definite choice of the coordinate system, namely, it is chosen such that for $\varphi(x)$ = const the metric for any x is, as follows from Ref. 12, given by (for the notation, see Ref. 12)

$$ds^2=dt^2-[e^{Ht}a_{\alpha\beta}(x)+b_{\alpha\beta}(x)+e^{-Ht}c_{\alpha\beta}(x)+\ldots]dx^\alpha dx^\beta. \quad (16)$$

Now suppose $N=1$, i.e., a region measuring $l\approx 2H^{-1}$ is transformed by the exponential expansion into the observed universe of scale $\approx 10^{28}$ cm. In this case, the expression (7) becomes

$$\varphi_0=4(n/\pi)^{1/2}m_p, \quad (7a)$$

and the expressions (14) and (15) become, respectively,

$$\delta R/R\approx\pm\beta 16(3/2n)^{1/2} \quad (14a)$$

and

$$\delta H/H\approx\pm\beta^1/_{16}(3n/2)^{1/2}. \quad (15a)$$

It follows from the latest experimental data[13] that the allowed spatial inhomogeneity of "our" universe is of the order $\approx 10^{-6}$. On the basis of (14a) and (15a) this makes it possible to obtain an estimate for the allowed inhomogeneity of the initial scalar field $\varphi(x)$ in a region measuring $l\approx 2H^{-1}$ that is exponentially inflated to the scale of the observed universe:

$$\beta\leqslant 10^{-6}. \quad (17)$$

Thus, we have obtained the following bounds for the permitted spatial inhomogeneity of the initial scalar field $\varphi(x)$:

$$|\nabla\varphi|\leqslant 10^{-6}\alpha_n^{1/2}[4(n/\pi)^{1/2}m_p]^{n/2}. \quad (18)$$

For the potential $V(\varphi)=\lambda\varphi^4/4$ (as in the model of Ref. 4), taking into account the bounds of Ref. 1, we obtain from (18)

$$|\nabla\varphi|\leqslant 10^{-11}m_p^2. \quad (19)$$

For the potential $V(\varphi)=\frac{1}{2}m^2\varphi^2$ (Ref. 1),[4)] taking into account the bound $m\lesssim 10^{-5}m_p$ (Ref. 1), we obtain from (18) the same (in order of magnitude) inequality as (19).

But if we consider the case $N\gg 1$, which corresponds in accordance with (7) to a large value of the initial scalar field φ_0 (as in Ref. 9), then in accordance with (14) and (15) the spatial inhomogeneity cannot be related to the spatial inhomogeneity of the observed universe,[13] since a region measuring $l\approx 2H^{-1}$ is transformed by exponential inflation into a universe with scale much greater than the observed ($\approx 10^{28}$ cm) universe. In this case a region corresponding to a scale much less than $l\approx 2H^{-1}$ is inflated to the size of the observed universe. And then instead of (17) we obtain much weaker bounds, even as weak as $\beta\approx 1$. However, it seems to us that consideration of the case $N\gg 1$ is "unphysical," since the predictions (14) and (15) that then follow for the inhomogeneity of a universe having a scale much greater than that of the observed universe ($\approx 10^{28}$ cm) cannot even in principle be verified by observations.

The bounds (18) and (19) obtained above on the permitted inhomogeneity of the initial scalar field $\varphi(x)$ appear unrealistic. It is much more important that these bounds are obtained in addition to the bounds of Refs. 1 and 2. The bounds of Refs. 1 and 2 are bounds on the parameters (λ, m) of the physical inflation model. In contrast, the new bounds (18) and (19) are bounds on the initial conditions [on the allowed inhomogeneity of the initial scalar field $\varphi(x)$]. These new bounds, which follow from the allowed spatial inhomogeneity of the observed universe, do not in general have any physical grounds and appear as in no way justified *a posteriori* bounds sufficient for the validity of the inflationary model of the occurrence of "our" universe. Of course, if we knew the theory of the universe in the preinflationary epoch (something that may be possible in the framework of a future quantum theory of gravity), one could take the initial data of the inflationary epoch to be results of the theory of the preinflationary universe. But even in this case we should have simply transformed the bounds (18) and (19) into corresponding bounds on the initial data of the preinflationary epoch. There exists just a single radical way of solving this problem of the initial conditions [if we do not take into account the hypothetical possibilities of a "universe without boundary" (Hawking), for which there are no initial condition problems at all], which is that allowed inflation obtains for almost all initial conditions (cf. the situation in the well-known ergodic theorem). This idea was in indeed the basis of the chaotic inflationary scenario,[4] in which the initial scalar fields $\varphi(x)$ were assumed to be deterministic realizations of random fields. However, it was shown in Ref. 6 that the literally understood chaotic scenario of Ref. 4, in which it was explicitly assumed (see the equations and text in Ref. 4b on p. 178) that the scalar field $\varphi(x)$ is spatially homogeneous in regions measuring $l\approx 2H^{-1}$, is incorrect from the point of view of probability theory. This result followed from the mathematical fact that for almost all natural continuous random fields the probability for the existence of differentiable realizations with a finite region of constancy is strictly equal to zero [the realizations—the scalar fields $\varphi(x)$—of the random field must be differentiable, since the necessary condition (1) already presupposes differentiability of $\varphi(x)$]. This mathematical fact is valid even for arbitrary geometry of the space on which the initial scalar field φ is defined. The unrealistic nature of the chaotic scenario of Ref. 4 is seen even more convincingly on the basis of the exponential estimates given below for the probability of the permitted inhomogeneity of the realizations of the random fields. These estimates are based on new results of the theory of random fields that were obtained by B. S. Tsirel'son (April 1985) and were induced by the problems of this work. Below, we formulate only the consequences of these general results[15] that are needed for the following exposition.

Theorem. let $\xi(x), x\in R^3$, be a random Gaussian homogeneous [homogeneity (spatial) of the random field follows from the corresponding invariance (with respect to a shift of the spatial coordinates) of the general theory of relativity] field with differentiable realizations $\varphi(x)$ and $f(\lambda)$, where $\lambda\in R^3$ is the spectral density of this field. Suppose that for all $k>0$ there is defined a function $m(k)$ (Z^3 is an integral lattice in R^3):

$$m(k)=\underset{\lambda\in R^3}{\operatorname{essinf}}\left[\left(\frac{2\pi}{k}\right)^3\sum_{z\in Z^3} f\left(\lambda+\frac{2\pi}{k}z\right)\right]. \tag{20}$$

Then for all $x\in S^3$, where S^3 is a sphere of radius $l\approx 2H^{-1}$, the following exponential estimate holds:

$$P\{x\in S^3,\ |\xi(x)-\varphi_0|\leqslant\Delta\}$$
$$\leqslant \inf_{k>0,\ m(k)>0} \exp\left\{-\frac{1}{k^3}\frac{4}{3}\pi l^3\eta\left[\frac{\Delta}{m^{1/2}(k)}\right]\right\}, \tag{21}$$

$$\eta\left[\frac{\Delta}{m^{1/2}(k)}\right]=-\ln\left\{2\Phi\left[\frac{\Delta}{m^{1/2}(k)}\right]-1\right\},$$

$$\Phi(x)=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{x}\exp\left(-\frac{1}{2}y^2\right)dy,$$

where $\varphi_0=\text{const}(x)$ was defined earlier, and Δ is a certain constant $\Delta=\text{const}(x)$.

Using this theorem, one can in turn prove[5)] the following:

Corollary. Let $\xi(x)$, φ_0, and Δ be as in the theorem. Let $a>0$ and $Q>0$ be such that

$$f(\lambda)=a/|\lambda|^{5+\varepsilon},\quad 1\gg\varepsilon>0 \text{ for } |\lambda|\geqslant Q \tag{22}$$

(the condition (22) guarantees differentiability of the scalar fields $\varphi(x)$—the realizations of the homogeneous random Gaussian field $\xi(x)$). For $\Delta\geqslant 0$, we define the quantity q:

$$q=2\pi e^{1/3}(2\Delta^2/\pi a)^{1/3+\varepsilon}. \tag{23}$$

Using the definition of the spectral density $f(\lambda)$ in terms of the Fourier transform of the correlation function of the random field $\xi(x)$, we can readily show that $q(23)$ has the dimensions of a length. We now suppose that $Qq\leqslant\pi$. Then for all $x\in S^3$ we have the following exponential estimate:

$$P\{x\in S^3,\ |\xi(x)-\varphi_0|\leqslant\Delta\}\leqslant\exp\left(-\frac{2+\varepsilon}{6}\frac{4}{3}\pi\frac{l^3}{q^3}\right)$$
$$\approx\exp\left(-\frac{4}{9}\pi\frac{l^3}{q^3}\right). \tag{24}$$

The estimates (21) and (24) give estimates of the probability of realizations with given inhomogeneity Δ in the sphere S^3 of the Gaussian random field. These estimates depend explicitly on a functional of the spectral density of the Gaussian field and thus are not universal. In this connection, we emphasize that the homogeneous Gaussian random fields with differentiable realizations constitute a very small subset of all inhomogeneous random fields. At the same time, we emphasize that the Gaussian random fields are, naturally, associated with quantum scalar fields. For non-Gaussian random fields in the general situation there are no hopes at all of obtaining any good estimates of the type (21) and (24).

It follows from (24) that for $\Delta=0$ the corresponding probability of realizations that are constant (homogeneous) in the sphere S^3 is strictly equal to zero. In Ref. 6, this trivial mathematical fact was proved not only for Gaussian but for almost all natural random fields. The estimates (21) and (25) are, of course, of greatest interest for $\Delta\neq 0$. For sufficiently small Δ, the estimates (21) and (24) are sufficiently accurate. We now recall the estimates obtained earlier for the permitted inhomogeneity of the initial scalar fields, namely, (10) and (17). Then

$$\Delta\approx 10^{-6}\varphi_0. \tag{25}$$

As follows from (22), a is proportional to the variance $\overline{\varphi_0^2}$. In the chaotic scenario,[4] it is assumed that $\varphi_0/(\overline{\varphi_0^2})^{1/2}\approx 1$. Then from (24), on the basis of (25), taking into account that $l\approx 2H^{-1}$, we obtain

$$P\{x\in S^3,\ |\xi(x)-\varphi_0|\leqslant 10^{-6}\varphi_0\}\leqslant\exp(-10^{34}). \tag{26}$$

The estimate (26) corresponds to $N=1$, a case that, naturally, is distinguished. For $N\gg 1$ [see however the discussion following Eq. (19)] the corresponding estimate of the type (26) may be much less restrictive. It follows from (26) that the probability of formation of "our" universe with permitted spatial inhomogeneity is fantastically small ($\exp(-10^{34})$) in the framework of the chaotic scenario of Ref. 4. And from the estimates (21) and (24), which are sufficiently accurate for small Δ, it follows that the probability of formation of "not our" universes with impermissible spatial inhomogeneity is much greater (fantastically greater) than the probability of formation of "our" universe. As we know from quantum theory, especially after Bell's well-known inequalities[16] (see also Refs. 17–19), "God plays dice," but we cannot expect him to do so with such fantastic accuracy.

In a separate paper we shall use Tsirel'son's results to estimate fluctuations of quantum (and not classical) scalar fields.

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[1)]The main content of this paper was presented at the USSR Academy of Sciences Nuclear Physics Section Conference on Nuclear Interactions at the Institute of Theoretical and Experimental Physics in November 1985, Moscow.

[2)]The estimates of Sec. 2 were presented in a paper at the USSR Academy of Sciences Nuclear Physics Section Conference on Strong, Weak, and Gravitational Interactions at the P. N. Lebedev Physics Institute in April 1985, Moscow.

[3)]A formula analogous to (14) was obtained independently in Ref. 11.

[4)]The most complete investigation of the classical dynamics of this model was made in Ref. 14.

[5)]The mathematical proof of the Corollary, which we omit, is based on one purely geometrical lemma of B. S. Tsirel'son.

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