Relaxation of a resonantly driven nonlinear oscillator

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The behavior of a nonlinear oscillator in a resonant external field, and coupled to a heat bath, is studied. The coupling to the heat bath is assumed to be weak, and therefore the oscillator motion is diffusive in quasienergy space. It is shown that in such a system it is possible for the state with the larger oscillation amplitude to have an excess population. The time required to establish a stationary distribution is estimated. An expression for the average response of the oscillator to an auxiliary weak field at a near-resonance frequency is found.

INTRODUCTION

The question of the relaxation of a nonlinear oscillator located in an external resonance field is one of the examples of a large class of problems concerning the behavior of nonequilibrium systems having several stable states interacting with a heat bath. A change in the properties of the medium under the action of radiation can manifest itself in different nonlinear effects.[1] A nonlinear oscillator located in an external resonance field can serve as model for the interaction of a molecular gas with laser radiation: a small number of impurity-gas molecules are in resonance with an external field, and the buffer gas plays the role of a heat reservoir.

It is convenient to characterize the state of the nonlinear oscillator with the aid of "slow" variables u and v.[2,3] If the amplitude of the external force is less than some critical value, then the phase diagram of the nonlinear oscillator in the space of the slow variables has, in the absence of dissipation, the form shown in Fig. 1. To each phase trajectory in the (u, v) space corresponds a definite value of H, the "quasienergy," and $T(H)$, the period of the motion along the phase trajectory.

If we ignore the rapidly oscillating terms in the initial Hamiltonian of the oscillator, then the quasienergy H is an integral of the motion.[4–6] The stable vibrational states 1 (with the smaller oscillation amplitude) and 2 (with the larger amplitude) are separated by an unstable state s, through which the separatrix passes.[4]

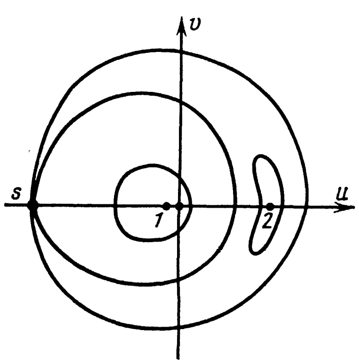

FIG. 1. Phase diagram of the nonlinear oscillator in the plane of the slow variables in the presence of two stable states.

As shown in Ref. 4, in the presence of interaction with the heat bath, the fluctuation-induced transitions between the various stable states can cause an excess population to appear in the state with the greater oscillation amplitude.

The purpose of the present paper is, first, to obtain the stationary oscillator-state distribution function in the presence of a weak interaction with the heat bath, and, second, to describe the relaxation to the stationary distribution and estimate the time required to establish this distribution. In Sec. 3 we consider the response of the oscillator to an auxiliary weak field at a near-resonance frequency in the case when the heat bath has a nonzero temperature.

§1. DERIVATION OF THE ONE-DIMENSIONAL KINETIC EQUATION

Let us consider the classical nonlinear oscillator with the Hamiltonian

$$H_0=\frac{\omega_0^2x^2+p^2}{2}+\frac{\tilde{\gamma}x^4}{4}-f_0x\cos\omega t. \tag{1}$$

It is convenient to introduce the slow variables u and v:

$$x=u\cos\omega t+v\sin\omega t=pe^{-i\omega t}+ge^{i\omega t}. \tag{2}$$

In terms of the slow variables we have for the oscillator the slow-time equations

$$\begin{aligned}\dot{u}&=-\frac{\partial H}{\partial v}=\xi v-\gamma(u^2+v^2)v,\\ \dot{v}&=\frac{\partial H}{\partial u}=-\xi u+\gamma(u^2+v^2)u-f,\end{aligned} \tag{3}$$

where

$$\xi=(\omega^2-\omega_0^2)/\omega^2,\quad \gamma=\tilde{\gamma}/\omega^2,\quad f=f_0/\omega^2.$$

The equations (3) describe the motion of a particle with the Hamiltonian

$$H=-\frac{\xi}{2}(u^2+v^2)+\frac{\gamma}{4}(u^2+v^2)^2-fu, \tag{4}$$

which has the meaning of quasienergy, and is an integral of the motion described by the equations (3).

Now let the oscillator interact with the heat bath. This leads to the appearance of dissipation and some "random force" acting on the oscillator. If $T(H)\ll 1/\vartheta$, where $T(H)$

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is the period of the motion along the phase trajectory and ϑ is the damping constant due to the interaction with the medium, then information about the oscillator state will be carried by the quasienergy H. Naturally, in this case the behavior of the oscillator can be regarded as some random walk in "quasienergy" space, and its state will be characterized by a one-dimensional distribution function $F(H, t)$.

Within the framework of the present model, the oscillator motion is a Markovian process; therefore, $F(H, t)$ satisfies the equation[7]

$$\frac{\partial F}{\partial t}=\sum_{n=1}^{\infty}(-1)^n\frac{1}{n!}\frac{\partial^n}{\partial H^n}[K_n(H,t)F(H,t)],$$
$$K_n(H,t)=\lim_{\Delta t\to 0}\frac{1}{\Delta t}[H(t+\Delta t)-H(t)]^n. \tag{5}$$

Let us compute K_n. We shall assume that the medium is made up of harmonic oscillators (with frequencies ω_k), which are linearly coupled to the oscillator in question[8]:

$$V_{int}=\sum_k \tilde{c}_k x_k x, \tag{6}$$

where $\tilde{c}_k$ is the coupling constant.

Let us set

$$x_k=p_k e^{-i\omega_k t}+g_k e^{i\omega_k t}. \tag{7}$$

We consider the system of slow-time equations

$$\dot{g}=i\frac{\partial H}{\partial p}+i\sum_k c_k p_k \exp\{i(\omega_k-\omega)t\},$$
$$\dot{p}=-i\frac{\partial H}{\partial g}-i\sum_k c_k p_k \exp\{-i(\omega_k-\omega)t\}; \tag{8}$$

$$\dot{p}_k=-ic_k p\exp\{-i(\omega-\omega_k)t\},$$
$$\dot{q}_k=ic_k g\exp\{i(\omega-\omega_k)t\}, \tag{9}$$

where $c_k=\tilde{c}_k/\omega^2$. By substituting (8) into (9), we can find Δg and Δp, the changes caused in g and p by the interacting with the heat bath during the period of time Δt. Let us, assuming Δp and Δg to be small, consider

$$\Delta H=\frac{\partial H}{\partial p}\Delta p+\frac{\partial H}{\partial g}\Delta g. \tag{10}$$

Since the state of the system is determined (in our approximation) by the quantity H, we must, in computing $\langle\Delta H\rangle/\Delta t$, average over the distribution of the "random quantities"—the initial coordinates and momenta of the oscillators of the heat bath—and over a period $T(H)$. The oscillators of the heat bath are in equilibrium; therefore, $\langle g_k(0)\rangle=0$ and $\langle p_k(0)\rangle=0$:

$$K_1=\lim_{\Delta t\to 0}\frac{\langle\overline{\Delta H}\rangle}{\Delta t}=2\pi c_\omega^2\oint_{T(H)}\left(\frac{\partial H}{\partial p}p+\frac{\partial H}{\partial g}g\right)dt$$
$$=-2\pi i c_\omega^2\oint_{C(H)}(p\,dg-g\,dp)=-i\vartheta\oint(p\,dg-g\,dp). \tag{11}$$

Let us proceed to the computation of $\langle(\Delta H)^2\rangle$:

$$(\Delta H)^2=\left(\frac{\partial H}{\partial p}\right)^2(\Delta p)^2+\left(\frac{\partial H}{\partial g}\right)^2(\Delta g)^2+2\frac{\partial H}{\partial p}\frac{\partial H}{\partial g}\Delta p\,\Delta g. \tag{12}$$

It is not difficult to verify that

$$\langle(\Delta g)^2\rangle_T=2\pi c_\omega^4 g^2(\Delta t)^2,$$
$$\lim_{\Delta t\to 0}\frac{\langle(\Delta g)^2\rangle_T}{\Delta t}\to 0. \tag{13}$$

Similarly,

$$\lim_{\Delta t\to 0}\frac{\langle(\Delta p)^2\rangle_T}{\Delta t}\to 0.$$

Let us consider

$$\frac{\langle\Delta p\Delta g\rangle_T}{\Delta t}=\frac{1}{\Delta t}\sum_k c_k^2\langle p_k(0)g_k(0)\rangle_T \times\left(\int_t^{t+\Delta t}\exp\{-i(\omega-\omega_k)t'\}dt'\right)$$
$$\times\left(\int_t^{t+\Delta t}\exp\{i(\omega-\omega_k)t''\}dt''\right)=\frac{1}{\Delta t}\int_{-\Delta}^{\Delta}\int_t^{t+\Delta t}d\tilde{\omega}\,\langle p_{\tilde{\omega}}(0)g_{\tilde{\omega}}^{(0)}\rangle_T$$
$$\times c_{\tilde{\omega}}^2\exp\{-i\tilde{\omega}(t''-t')\}\,dt'\,dt''=c_\omega^2\langle p_\omega(0)g_\omega(0)\rangle_T,$$
$$\tilde{\omega}=\omega-\omega_k. \tag{14}$$

To compute K_2, we must again average over $T(H)$:

$$K_2=D=\lim_{\Delta t\to 0}\frac{\langle\overline{(\Delta H)^2}\rangle}{\Delta t}=c_\omega^2\langle p_\omega(0)g_\omega(0)\rangle_T\oint_{T(H)}\frac{\partial H}{\partial p}\frac{\partial H}{\partial g}dt$$
$$=Q\oint_{T(H)}\frac{\partial H}{\partial p}\frac{\partial H}{\partial g}dt,\qquad Q=\vartheta kT/\omega^2. \tag{15}$$

The higher moments are computed in similar fashion. Only the terms containing

$$c_\omega^{2k}[p_\omega(0)g_\omega(0)]^k\Delta t\sim c_\omega^{2k}$$

do not vanish when averaged over the states of the heat bath, but they are small, since the interaction with the heat bath is weak, and $c_\omega^2\ll 1$.

Thus, the nonstationary distribution function satisfies the diffusion equation

$$\frac{\partial F}{\partial t}=-\frac{\partial}{\partial H}(\vartheta K(H)F)+Q\frac{\partial}{\partial H}\left(D(H)\frac{\partial}{\partial H}F\right). \tag{16}$$

The steady-state solution can be determined from the condition

$$\vartheta KF-QD\frac{\partial}{\partial H}F=S=\text{const}. \tag{17}$$

Because of the absence of sources, it is natural to assume that $S=0$.

$$F=\exp\left[-\frac{\vartheta}{Q}\int_{H_i}^{H}\left[\left(\oint p\,dg-g\,dp\right)\left(\iint\Delta H\,dp\,dg\right)^{-1}\right]_i dH\right]A_i, \tag{18}$$

where $i = \mathrm{I}$ in the region I, $i = \mathrm{II}$ in the region II, and $i = S$ in the region III. But equation (16), like the expression found with exponential accuracy for $F(H)$, is valid in each of the regions of phase space, except the neighborhood of the separatrix, a region with a width of order $\sim\vartheta$. The probability of finding the oscillator in the state 2 is $F(H_2) = A_{\mathrm{II}}$, while the probability of finding it in the state 1 is $F(H_1) = A_{\mathrm{I}}$.

We require $F(H)$ to be continuous on the separatrix, whence we obtain the following relation between A_{I} and A_{II}:

$$\frac{A_{\mathrm{I}}}{A_{\mathrm{II}}} = \left\{\exp\left[-\frac{\vartheta}{Q}\int_{H_1}^{H_S}\left(\frac{K}{D}\right)_{\mathrm{I}} dH\right]\right\}^{-1}\exp\left[-\frac{\vartheta}{Q}\int_{H_2}^{H_S}\left(\frac{K}{D}\right)_{\mathrm{II}} dH\right], \tag{19}$$

i.e., the ratio of the probabilities of finding the oscillator in the states 1 and 2 coincides with the result obtained in Ref. 4 through integration along the trajectories. The oscillator can be found with overwhelming probability in either the state with the smaller oscillation amplitude, or the state with the greater amplitude. In each region of phase space $F(H)$ attain its maximum value at $H = H_{1(2)}$.

Using the results obtained in Ref. 4, we shall assume that

a) for $\beta<\beta_0 \quad F_{\mathrm{I}}(H_1)\gg F_{\mathrm{II}}(H_2), \quad \beta=\gamma f^2/\xi^3$,

b) for $\beta>\beta_0 \quad F_{\mathrm{I}}(H_1)\ll F_{\mathrm{II}}(H_2), \quad \beta=\beta_0 : F_{\mathrm{I}}(H_1)=F_{\mathrm{II}}(H_2)$.

Then we can expand $\ln F$ in a series, and retain only the terms linear in ΔH in the vicinity of H_1 in the case a) and in the vicinity of H_2 in the case b). This will be useful in the computation of the mean susceptibility of the oscillator. Since in the regions I and II different values of H correspond to different phase trajectories, $F(H(p,g))$ can, up to the preexponential function, be regarded as the probability for finding the oscillator in the state with the given H, i.e., as a function of one variable H that uniquely defines the trajectory in phase space, with $\partial F/\partial H > 0$ when $H = H_1$; $\partial F/\partial H < 0$ when $H = H_2$.

§2. ESTIMATE OF THE TIME REQUIRED TO ESTABLISH THE STATIONARY DISTRIBUTION

The one-dimensional Fokker-Planck equation (16) obtained above can be transformed into the Schrödinger equation with the aid of the substitution

$$F(H,t) = \exp\left[-\frac{\vartheta}{2Q}\int\left(\frac{K}{D}\right)dH\right]\Psi(H,t). \tag{20}$$

The equation for Ψ has the following form:

$$Q\frac{\partial\Psi}{\partial(-t)} = -Q^2D(H)\frac{\partial^2\Psi}{\partial H^2} + V(H)\Psi, \tag{21}$$

where

$$V(H) = \frac{\vartheta^2K^2}{4D} - \frac{2K'D-D'K}{2D}Q\vartheta,$$

since $Q\ll 1$, $V(H)\simeq K^2\vartheta^2/4D$.

An arbitrary state Ψ can be represented in the form

$$\Psi(H,t) = \sum_i c_i e^{-\lambda_i t}\Psi_{\lambda_i}(H), \tag{22}$$

where the λ_i are the eigenvalues of Eq. (21). To the zeroth eigenvalue $\lambda_0 = 0$ corresponds the stationary distribution. the characteristic time for the establishment of the stationary distribution is $\tau\sim 1/\lambda_1$, where λ_1 is the smallest nonzero eigenvalue.

For H corresponding to the region II of phase space, Eq. (21) describes a particle with positive mass in the potential $V_{\mathrm{II}}(H)$; in the region I, a particle with negative mass in the potential $V_{\mathrm{I}}(H)$, or else a particle with a positive mass in the potential $V_{\mathrm{I}}(H)$ with $\hat{p}$ replaced by $-\hat{p}$.

The expression for $V(H)$ is not valid in the vicinity of H_s, which has a width $\sim\vartheta$; therefore, we can replace $V(H)$ approximately by a continuous function, since $V(H)$ in the vicinity of H_s is determined by the matching conditions (see Fig. 2). In the case when allowance is made for the fluctuation-induced transitions, we can compute λ_1 with exponential accuracy by considering the tunneling between the two minima in the semiclassical—in Q—approximation (see Ref. 9):

$$\lambda_1\sim\frac{1}{\tau}\sim\exp\left[-\frac{\vartheta}{Q}\left\{\int_{H_1}^{H_s}\left(\frac{K}{D}\right)_{\mathrm{I}} dH + \int_{H_2}^{H_s}\left(\frac{K}{D}\right)_{\mathrm{II}} dH\right\}\right]. \tag{23}$$

The time required for the establishment of the stationary distribution in all phase space is exponentially long: it is determined by the time characterizing the fluctuation-induced transitions between the various regions of phase space.

§3. RESPONSE OF THE OSCILLATOR TO A WEAK AUXILLIARY FIELD

Let us consider the response of the oscillator to a weak auxiliary field at a frequency ω' near resonance:

$$\varepsilon=\varepsilon_0(e^{i\omega' t}+e^{-i\omega' t}),$$

where $\omega' - \omega \ll \omega$. Let

$$p=p_0(H,t)+p', \qquad g=g_0(H,t)+g', \tag{24}$$

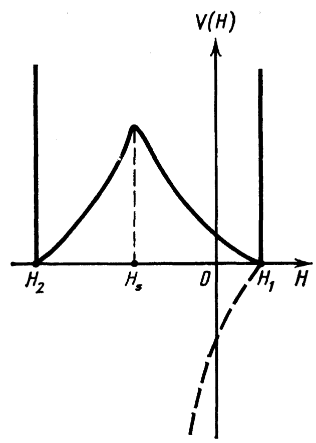

FIG. 2. Qualitative form of the potential $V(H)$: $V_{\mathrm{I}}(H_1)$ and $V_{\mathrm{II}}(H_2)$ are the minima of the potential. The dashed curve depicts the potential $V_{\mathrm{I}}(H)$ for a particle with negative $D(H)$.

where $p_0(H,t)$ and $g_0(H,t)$ are the unperturbed solutions to the abridged equations.

It is convenient to go over to the dimensionless variables

$$z=(\gamma/\xi)^{1/2}u,\quad y=(\gamma/\xi)^{1/2}v,\quad \tilde{H}=H\gamma/\xi^2,\quad \tau=\xi t,\quad \tilde{\varepsilon}_0=\varepsilon_0/\xi. \tag{25}$$

Then

$$\tilde{\vartheta}=\vartheta/\xi,\quad \tilde{Q}=\tilde{\vartheta}(KT\gamma/\xi^2\omega^2),\quad \xi'=\xi'/\xi.$$

The slow-time equations assume the form

$$\dot{z}=-\frac{\partial\tilde{H}}{\partial y}-\tilde{\vartheta}z,\quad \dot{y}=\frac{\partial\tilde{H}}{\partial z}-\tilde{\vartheta}y$$

or else

$$\dot{\tilde{p}}=-i\frac{\partial\tilde{H}}{\partial\tilde{g}}-\tilde{\vartheta}\tilde{p},\quad \dot{\tilde{g}}=i\frac{\partial\tilde{H}}{\partial\tilde{p}}-\tilde{\vartheta}\tilde{g}, \tag{26}$$

where

$$\tilde{p}=z+iy,\quad \tilde{g}=z-iy,$$

$$\tilde{H}=-\frac{(z^2+y^2)}{2}+\frac{(z^2+y^2)^2}{4}-\beta^{1/2}z.$$

Linearizing the equations in the vicinity of $\tilde{p}_0(\tilde{H},t)$, $\tilde{g}_0(\tilde{H},t)$, and averaging them over $T(\tilde{H})$, we obtain

$$\begin{aligned}\dot{\tilde{p}}'&=B(\tilde{H})\tilde{g}'+C(\tilde{H})\tilde{p}'+i\tilde{\varepsilon}e^{i\xi'\tau},\\ \dot{\tilde{g}}'&=B^*(\tilde{H})\tilde{p}'+C^*(\tilde{H})\tilde{g}'-i\tilde{\varepsilon}e^{-i\xi'\tau}.\end{aligned} \tag{27}$$

The expressions for $B(\tilde{H})$ and $C(\tilde{H})$ are given in the Appendix.

The approximate equations describing the response of the system to a near-resonant external field that is a low-frequency field in the slow variables are valid up to terms $\sim T(\tilde{H})\,\xi'\ll 1$. Indeed,

$$\tilde{\varepsilon}_0e^{i\xi'\tau_0}-\tilde{\varepsilon}_0e^{i\xi'\tau_0}\left\{\frac{e^{i\xi'T(\tilde{H})}-1}{i\xi'T(\tilde{H})}\right\}\sim\tilde{\varepsilon}_0e^{i\xi'\tau_0}\frac{\xi'T(\tilde{H})}{2}\ll 1. \tag{28}$$

Let us seek the solution in the form

$$\begin{aligned}\tilde{p}'&=Xe^{i\xi'\tau}+Ye^{-i\xi'\tau},\\ \tilde{g}'&=Y^*e^{i\xi'\tau}+X^*e^{-i\xi'\tau}.\end{aligned} \tag{29}$$

The response to the external field is given by the matrix

$$\hat{G}=\begin{vmatrix}\dfrac{-B[(\xi'^2-\lambda^2)+2i\xi'\tilde{\vartheta}]}{(\xi'^2-\lambda^2)^2+4\xi'^2\tilde{\vartheta}^2} & \dfrac{(i\xi'+C^*)[(\lambda^2-\xi'^2)-2i\xi'\tilde{\vartheta}]}{(\xi'^2-\lambda^2)^2+4\xi'^2\tilde{\vartheta}^2}\\ \dfrac{-(i\xi'-C)[(\xi'^2-\lambda^2)+2i\xi'\tilde{\vartheta}]}{(\xi'^2-\lambda^2)^2+4\xi'^2\tilde{\vartheta}^2} & \dfrac{-B[(\xi'^2-\lambda^2)-2i\xi'\tilde{\vartheta}]}{(\lambda^2-\xi'^2)^2+4\xi'^2\tilde{\vartheta}^2}\end{vmatrix}, \tag{30}$$

where $\lambda^2=|C|^2-|B|^2$. The coefficient of absorption of the weak auxiliary field has the following form:

$$\chi\sim\frac{\lambda^2+\xi'^2-2\xi'\,\mathrm{Im}\,C(\tilde{H})}{(\xi'^2-\lambda^2)^2+4\xi'^2}\frac{1}{\tilde{\vartheta}}. \tag{31}$$

For $\tilde{H}=\{\tilde{H}_1,\tilde{H}_2\}$, the expression (31) coincides with the result obtained in Ref. 4.

To find the response of the system to the auxiliary field, we must average $\hat{G}(\hat{H})$, which is given by the formula (30), the oscillator distribution function $F(\tilde{H})$:

$$\langle G\rangle=\int\hat{G}(\tilde{H})F(\tilde{H})d\tilde{H}$$

$$=\begin{cases}N_{\mathrm{I}}\displaystyle\int d\tilde{H}\hat{G}(\tilde{H})\exp\Big[-\frac{\xi^2\omega^2}{\gamma kT}\int_{\tilde{H}_1}^{\tilde{H}}\Big(\frac{K}{D}\Big)_{\mathrm{I}}dH\Big] & \text{in the region I}\\ N_{\mathrm{II}}\displaystyle\int d\tilde{H}\hat{G}(\tilde{H})\exp\Big[-\frac{\xi^2\omega^2}{\gamma kT}\int_{\tilde{H}_2}^{\tilde{H}}\Big(\frac{K}{D}\Big)_{\mathrm{II}}dH\Big] & \text{in the region II}\end{cases} \tag{32}$$

We obtain explicit expressions for $\langle G\rangle$ if $\beta=\gamma f^2/\xi^3\ll 1$. In that case

$$F(\tilde{H})=\begin{cases}A_{\mathrm{I}}\exp\Big\{-\dfrac{\xi^2\omega^2}{2\gamma kT}[1-(1+4\tilde{H})^{1/2}]\Big\} & \text{in the regionI}\\ A_{\mathrm{II}}\exp\Big[-\dfrac{\xi^2\omega^2}{\gamma kT}(\tilde{H}-\tilde{H}_2)\Big] & \text{in the region II}\end{cases} \tag{33}$$

On account of the exponential decrease of $F(\tilde{H})$, to determine $\langle G\rangle$, we only need to know $C(\tilde{H})$ and $B(\tilde{H})$ in the neighborhoods of $\tilde{H}=\tilde{H}_1$ and $\tilde{H}=\tilde{H}_2$.

We can assume that $F(\tilde{H})$ is nonzero only in the region I if $\beta<\beta_0$, and only in the region II if $\beta>\beta_0$.

If $\beta\ll 1$, then

$$\tilde{H}_1\approx 0,\quad \tilde{H}_2\approx-1/4-\beta^{1/2}. \tag{34}$$

$$C(\tilde{H})\approx-\tilde{\vartheta}+i\{1-2[1-(1+4\tilde{H})^{1/2}]\},\quad B(\tilde{H})\approx 0. \tag{35}$$

In the case of small deviations from the stable vibrational state in the region II we have for $C(\tilde{H})$ and $B(\tilde{H})$ the following expressions:

$$\begin{aligned}C(\tilde{H})&=-\tilde{\vartheta}-i(1+\beta^{1/2}),\\ B(\tilde{H})&=i[(1+\beta^{1/2})-4(\tilde{H}-\tilde{H}_2)\beta^{-1/2}].\end{aligned} \tag{36}$$

For $\beta<\beta_0$

$$\langle\hat{G}\rangle=\begin{vmatrix}\dfrac{1}{(\xi'-1+4\theta)+i\tilde{\vartheta}} & 0\\ 0 & \dfrac{1}{(\xi'-1+4\theta)-i\tilde{\vartheta}}\end{vmatrix}, \tag{37}$$

where $\theta = \gamma kT/\xi^2\omega^2$. The susceptibility of the oscillator in this case has the same structure as the susceptibility of the harmonic oscillator, but because of the nonlinearity it is temperature-dependent.

When $\beta \ll 1$, but $\beta > \beta_0$, the oscillator is found with an overwhelming probability in the region II. The response to the auxiliary field is given by the expression (30), where

$$B = i[(1+\beta^{1/2}) - 4\theta\beta^{-1/2}], \quad C = -\tilde{\vartheta} - i(1+\beta^{1/2}),$$

$$\lambda^2 = \vartheta^2 + 8(1+\beta^{1/2})\theta\beta^{-1/2} \tag{38}$$

up to terms quadratic in θ. The mean absorption coefficient can be computed in exactly the same way.

If

$$\operatorname{Im} C(\tilde{H}) - (|B|^2 - \tilde{\vartheta}^2)^{1/2} < \xi' < \operatorname{Im} C(\tilde{H}) + (|B|^2 - \tilde{\vartheta}^2)^{1/2},$$

where

$$\tilde{H} = \tilde{H}_i + (\gamma kT/\xi^2)[-1 + 2(z_i^2 + y_i^2)], \quad i = 1, 2$$

depending on the value of β, then $\langle \chi \rangle < 0$, i.e., the weak auxiliary field can be enhanced because of the presence of the stronger field.

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APPENDIX

The coefficients $B(\tilde{H})$ and $C(\tilde{H})$ have the following form:

$$B(\tilde{H}) = \frac{1}{T(\tilde{H})} \int_0^{T(\tilde{H})} \frac{\partial^2 \tilde{H}}{\partial \tilde{g}^2} d\tau = -\frac{i}{T(\tilde{H})} \int_0^{T(\tilde{H})} \tilde{p}_0^2(\tilde{H}, \tau)\, d\tau,$$

$$C(\tilde{H}) = -\tilde{\vartheta} + \frac{i}{T(\tilde{H})} \int_0^{T(\tilde{H})} (1 - 2\tilde{p}_0\tilde{g}_0)\, d\tau, \tag{A1}$$

where $p_0(\tilde{H}, \tau)$ and $g_0(\tilde{H}, \tau)$ are the solutions to the equations with a given $\tilde{H}$.

Then

$$\lambda^2 = \tilde{\vartheta}^2 + \frac{1}{T^2(\tilde{H})} \left\{ \left| \int_0^{T(\tilde{H})} d\tau\, (1 - 2\tilde{p}_0\tilde{g}_0) \right|^2 - \left| \int_0^{T(\tilde{H})} p_0^2\, d\tau \right|^2 \right\}. \tag{A2}$$

Taking account of the fact that $F(\tilde{H})$ decreases exponentially as $|\tilde{H} - \tilde{H}_1|$ increases in the region I and $|\tilde{H} - \tilde{H}_2|$ increases in the region II, we can derive for $\langle G \rangle$ up to terms quadratic in the temperature in the case when $\beta < \beta_0$ the expression

$$\langle G(\tilde{H}) \rangle = \hat{G}(\tilde{H}_1) + \hat{G}'(H_1)\langle \Delta \tilde{H} \rangle + \dots \tag{A3}$$

$$= \hat{G}(\tilde{H}_1) + \hat{G}_H'(\tilde{H}_1) \int_0^\infty \Delta H \exp[-\Delta H/\theta(-1 + z_1^2)]$$

$$\approx \hat{G}[\tilde{H}_1 + \theta(-1 + z_1^2)].$$

Similarly, for $\beta > \beta_0$, $\hat{G}(\tilde{H}) = \hat{G}[\tilde{H}_2 + \theta(-1 + 2z_2^2)]$.

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