Estimate of the constant in "2/3 law" of turbulence obtained by reduction of the hydrodynamics equations

E. B. Gledzer

Institute of Atmosphere Physics, USSR Academy of Sciences
(Submitted 26 December 1985; resubmitted 17 April 1986)
Zh. Eksp. Teor. Fiz. **91**, 818–825 (September 1986)

The discrete cascade model of turbulence, obtained by reducing the Navier-Stokes equations, is used to obtain the dependence of the constant in the Kolmogorov-Obukhov "2/3 law" on the scale refinement coefficient. The range in which the constant is approximately universal is identified.

1. INTRODUCTION

The energy properties of the local structure of fully developed turbulent flow, based on the cascade mechanism of energy conversion, were statistically described by A. N. Kolmogorov[1,2] in terms of a structure function

$$B_{\alpha\alpha}(r)=\overline{v_\alpha^2(r)}=C\bar{\varepsilon}^{2/3}r^{2/3},$$

$$v_\alpha(\mathbf{r})=u_\alpha(\mathbf{x}+\mathbf{r},\ t)-u_\alpha(\mathbf{x},\ t),\quad \eta\ll r\ll L_0, \tag{1}$$

where u_α is the component of the velocity vector $\mathbf{u}$ in the direction of the vector $\mathbf{r}$, L_0 is the transverse scale of the flow, $\eta = \nu^{3/4}\bar{\varepsilon}^{-1/4}$ is the internal turbulence scale, determined by the kinetic viscosity and by the average energy dissipation $\bar{\varepsilon}$ per unit mass (the superior bar denotes statistical averaging). Confirming results were obtained by Obukhov[3,4] by a spectral approach. Equation (1) corresponds to a "5/3 law" for the spectral density of the turbulence kinetic energy $E(k)$ (k is the wave number) (see Refs. 5 and 6):

$$E(k)=C_1\bar{\varepsilon}^{2/3}k^{-5/3},\quad L_0^{-1}\ll k\ll\eta^{-1}. \tag{2}$$

C and C_1 in Eqs. (1) and (2) are universal constants, with

$$C_1=[55\Gamma(1/3)/27]C\approx0.76C. \tag{3}$$

According to Kolmogorov, the constant C cannot be obtained solely from scaling consideration, and has been therefore determined, starting with Ref. 2, from experiment. The experimental values of C obtained in a large number of studies were summarized in a review article by Yaglom,[7] the gist being that almost all the measurements lead to results that do not differ greatly from Kolmogorov's very first estimate ($C\approx 1.5$). According to present data, $C\approx2$ with an error that is apparently less than 10–15%.[6]

Since it follows from the well-known critical remark by Landau[8] that the statistical properties of the energy-dissipation field $\varepsilon(x,t)$ do not affect the probability distributions of the small-scale components of the turbulence, these distributions should change somehow when the turbulent-flow characteristics are altered, i.e., they cannot be absolutely universal (see Refs. 5 and 6).

An attempt is made in the present paper to obtain the "2/3 law" together with the constant C it contains on the basis of a definite reduction of the Navier-Stokes equations to the discrete cascade system proposed by Obukhov for the description of cascade processes in advanced turbulence.[9–11] Dealing with this subject, among others, is a paper by Kraichnan,[12] who obtained an estimate $C_1 = 1.4$ by using the "almost-Markovian-Galilean-invariant" model of turbulence and introducing in the theory an additional parameter. It is apparently also possible to use a diagram technique. Kuz'min and Patashinskiĭ[13] obtained a solution for the turbulence spectrum and for the numerical quantities involved, but only in the dissipative region of the wave numbers.

2. REDUCTION OF THE HYDRODYNAMICS EQUATIONS TO CASCADE SYSTEMS

We begin with the Fourier expansion of the velocity field of an incompressible liquid (in a cube of side L and with periodic boundary conditions)[14,15]

$$\mathbf{u}(\mathbf{x},t)=\sum_{\mathbf{k}}e^{i\mathbf{k}\mathbf{x}}\mathbf{v}(\mathbf{k},t),\quad \mathbf{k}=\frac{2\pi}{L}(m,n,p),\quad \mathbf{k}\mathbf{v}(\mathbf{k})=0, \tag{4}$$

$$\left(\frac{d}{dt}+\nu k^2\right)v_i(\mathbf{k})+\frac{i}{2}\Delta_{ijl}(\mathbf{k})\sum_{\mathbf{k}_1}v_j(\mathbf{k}_1)v_l(\mathbf{k}-\mathbf{k}_1)=0,$$

$$i,\ j,\ l=1,2,3,$$

$$\Delta_{ijl}(\mathbf{k})=\Delta_{il}(\mathbf{k})k_j+\Delta_{ij}(\mathbf{k})k_l,\quad \Delta_{ij}(\mathbf{k})=\delta_{ij}-\frac{k_ik_j}{k^2},\quad k=|\mathbf{k}|, \tag{5}$$

where m, n, and p are positive and negative integers, including zero. We single out in (5) the region where the sum over the vectors $\mathbf{k}_1$ satisfies the condition $|\mathbf{k}-\mathbf{k}_1|\ll k$, $k_1\ll k$. The contribution to the sum of (5) from this region is equal to

$$i(\mathbf{k}\mathbf{v}_0)v_i(\mathbf{k}),\quad \mathbf{v}_0=\sum_{k_0\ll k}\mathbf{v}(\mathbf{k}_0). \tag{6}$$

The velocity vector $\mathbf{v}_0$ depends little on k if k is large (in the inertial range), since the main contributions to (6) are those of the Fourier components $\mathbf{v}(\mathbf{k}_0)$ with small (containing the bulk of the energy) wave numbers. With this taken into account, the substitution

$$\mathbf{V}(\mathbf{k})=\mathbf{v}(\mathbf{k})\exp\left(i\mathbf{k}\int_0^t\mathbf{v}_0(\tau)d\tau\right) \tag{7}$$

transforms (5) into

$$\left(\frac{d}{dt}+\nu k^2\right)V_i(\mathbf{k})+\frac{i}{2}\Delta_{ijl}(\mathbf{k})\sum_{\mathbf{k}_1}{}'V_j(\mathbf{k}_1)V_l(\mathbf{k}-\mathbf{k}_1)=0, \tag{8}$$

where the primed summation sign means that the contribution of the specified summation region is excluded. The substitution (7) means a change to a new coordinate frame, in which perturbations of a given scale are transported without substantial distortion at a velocity higher than the large-scale motions. We note that, as shown in Refs. 16 and 17, it is just such a transport that causes the divergences in the diagrams of the formal perturbation theory for spectrum (2).

Since the field is incompressible, we represent the Fourier components $\mathbf{V}(\mathbf{k})$ as expansions in terms of two vectors $\mathbf{e}_1(\mathbf{k})$ and $\mathbf{e}_2(\mathbf{k})$ that are perpendicular to each other and to the wave vector $\mathbf{k}$:

$$\mathbf{V}(\mathbf{k},t)=a_1(\mathbf{k},t)\mathbf{e}_1(\mathbf{k})+a_2(\mathbf{k},t)\mathbf{e}_2(\mathbf{k}),$$
$$(\mathbf{e}_1\mathbf{k})=(\mathbf{e}_2\mathbf{k})=(\mathbf{e}_1\mathbf{e}_2)=0, \tag{9}$$
$$\mathbf{e}_\alpha(-\mathbf{k})=-\mathbf{e}_\alpha(\mathbf{k}),\quad a_\alpha(-\mathbf{k})=-a_\alpha(\mathbf{k}),\quad \alpha=1,2.$$

The kinetic energy per unit volume and the equations of motion take then the form

$$E=\frac{1}{2}L^{-3}\int_{L^3}\mathbf{u}^2(\mathbf{x})\,d\mathbf{x}=\frac{1}{2}\sum_{\mathbf{k}}[\,|a_1(\mathbf{k})|^2+|a_2(\mathbf{k})|^2]$$
$$=\int_0^\infty E(k)\,dk, \tag{10}$$

$$\left(\frac{d}{dt}+\nu k^2\right)a_\alpha(\mathbf{k})+i\sum_{\mathbf{k}_1}{}'\left\{\frac{\mathbf{k}-\mathbf{k}_1}{2}\mathbf{e}_\gamma(\mathbf{k}+\mathbf{k}_1)\right\}$$
$$\times\{\mathbf{e}_\alpha(\mathbf{k})\mathbf{e}_\beta(-\mathbf{k}_1)\}a_\gamma(\mathbf{k}+\mathbf{k}_1)a_\beta(-\mathbf{k}_1)=0. \tag{11}$$

The primed sum in (11) means that summation over the region $|\mathbf{k}+\mathbf{k}_1|\ll k$ is omitted. We divide the entire remaining region of summation over $\mathbf{k}_1$ into two regions with $k_1<k$ and $k_1>k$. We expand in these regions in terms of k_1/k and k/k_1, respectively. For the first factor in the sum of (11) we have approximately at $k_1<k$

$$\frac{\mathbf{k}-\mathbf{k}_1}{2}\mathbf{e}_\gamma(\mathbf{k}+\mathbf{k}_1)\approx\frac{\mathbf{k}-\mathbf{k}_1}{2}\left[\mathbf{e}_\gamma(\mathbf{k})+k_{1\beta}\frac{\partial\mathbf{e}_\gamma(\mathbf{k})}{\partial k_\beta}\right]$$
$$\approx-\frac{\mathbf{k}_1}{2}\mathbf{e}_\gamma(\mathbf{k})+\frac{k_{1\beta}}{2}\frac{\partial}{\partial k_\beta}\{\mathbf{k}\mathbf{e}_\gamma(\mathbf{k})\}-\frac{k_{1\beta}}{2}e_{\gamma\beta}(\mathbf{k})=-\mathbf{k}_1\mathbf{e}_\gamma(\mathbf{k}), \tag{12}$$

where the orthogonality condition (9) is used. For $k_1>k$, similarly,

$$\frac{\mathbf{k}-\mathbf{k}_1}{2}\mathbf{e}_\gamma(\mathbf{k}+\mathbf{k}_1)\approx\mathbf{k}\mathbf{e}_\gamma(\mathbf{k}_1). \tag{13}$$

From (11)–(13) we obtain approximate equations (which can be regarded as the zeroth terms of the corresponding perturbation-theory series)

$$\left(\frac{d}{dt}+\nu k^2\right)a_\alpha(\mathbf{k})-i\sum_{k_1<k}\{\mathbf{k}_1\mathbf{e}_\gamma(\mathbf{k})\}\{\mathbf{e}_\alpha(\mathbf{k})\mathbf{e}_\beta(-\mathbf{k}_1)\}$$
$$\times a_\gamma(\mathbf{k})a_\beta(-\mathbf{k}_1)+i\sum_{k_1>k}\{\mathbf{k}\mathbf{e}_\gamma(\mathbf{k}_1)\}\{\mathbf{e}_\alpha(\mathbf{k})\mathbf{e}_\beta(-\mathbf{k}_1)\}$$
$$\times a_\gamma(\mathbf{k}_1)a_\beta(-\mathbf{k}_1)=0. \tag{14}$$

We subdivide the regions where the vectors $\mathbf{k}$ are localized and the summation regions $k_1<k$ and $k_1>k$ into layers bounded by concentric spheres of radius $p_i=p_0q^i$, where the ith layer contains the vectors for which $p_i\leqslant k<p_{i+1}$. By virtue of the conservation of the integral E given by Eq. (10) by the system (14) (for $\nu=0$), the same layers correspond to partition of the total energy into a sum of the energies contained in the layers. We note that if the "5/3 law" (2) is valid, the "2/3 law" holds for the energy E_i of the ith layer:

$$E_i=\int_{p_i}^{p_{i+1}}E(k)\,dk=C_1\frac{3}{2}\bar{\varepsilon}^{2/3}p_i^{-2/3}(1-q^{-2/3}). \tag{15}$$

We next narrow down the regions where the vectors $\mathbf{k}$ are defined and the summation regions for each ith layer in (14) to vectors located on concentric spheres of radius p_i, and leave on each sphere only six vectors $\mathbf{k}$ representing the directions of the coordinate axes in three-dimensional space. This narrowing relates the equations in question to the "shell" models which have been widely used recently, for example, to investigate the properties of MHD turbulence.[18]

We introduce the unit vectors $\mathbf{S}_1=(1,0,0)$, $\mathbf{S}_2=(0,1,0)$, $\mathbf{S}_3=(0,0,1)$. The chosen vectors on the sphere of radius p_i are then

$$\mathbf{k}_1^i=p_i\mathbf{S}_1,\quad \mathbf{k}_2^i=p_i\mathbf{S}_2,\quad \mathbf{k}_3^i=p_i\mathbf{S}_3,$$
$$\mathbf{k}_4^i=-p_i\mathbf{S}_1,\quad \mathbf{k}_5^i=-p_i\mathbf{S}_2,\quad \mathbf{k}_6^i=-p_i\mathbf{S}_3. \tag{16}$$

The unit vectors $\mathbf{e}_1(\mathbf{k}_m^i)$ and $\mathbf{e}_2(\mathbf{k}_m^i)$, $m=1,\ldots,6$, in Eqs. (9) and (14) can be defined in accordance with the following table:

$\mathbf{k}_m^i$:	$\mathbf{k}_1^i$	$\mathbf{k}_2^i$	$\mathbf{k}_3^i$	$\mathbf{k}_4^i$	$\mathbf{k}_5^i$	$\mathbf{k}_6^i$
$\mathbf{e}_1(\mathbf{k}_m^i)$:	$\mathbf{S}_2$	$\mathbf{S}_3$	$\mathbf{S}_1$	$-\mathbf{S}_2$	$-\mathbf{S}_3$	$-\mathbf{S}_1$
$\mathbf{e}_2(\mathbf{k}_m^i)$:	$\mathbf{S}_3$	$\mathbf{S}_1$	$\mathbf{S}_2$	$-\mathbf{S}_3$	$-\mathbf{S}_1$	$-\mathbf{S}_2$

(17)

Using (16) and (17) to calculate the interaction coefficients in (14), and also the conditions (9), we obtain equations for $a_\alpha(\mathbf{k}_m^i)$, $\alpha=1,2$, $m=1,2,3$ (see Refs. 14 and 15). We write down the equations for the variation of $a_1(\mathbf{k}_1^i)$ and $a_2(\mathbf{k}_1^i)$:

$$\left(\frac{d}{dt}+\nu p_i^2\right)a_1(\mathbf{k}_1^i)+i\sum_{r=1}^{\infty}p_{i-r}a_2(\mathbf{k}_1^i)\,(a_2(\mathbf{k}_3^{i-r})$$
$$-a_2^*(\mathbf{k}_3^{i-r}))+ip_i\sum_{r=1}^{\infty}(a_1(\mathbf{k}_3^{i+r})a_2^*(\mathbf{k}_3^{i+r})+\text{c.c.})=0,$$
$$\left(\frac{d}{dt}+\nu p_i^2\right)a_2(\mathbf{k}_1^i)+i\sum_{r=1}^{\infty}p_{i-r}a_1(\mathbf{k}_1^i)\,(a_1(\mathbf{k}_2^{i-r})$$
$$-a_1^*(\mathbf{k}_2^{i-r}))+ip_i\sum_{r=1}^{\infty}(a_1(\mathbf{k}_2^{i+r})a_2^*(\mathbf{k}_2^{i+r})+\text{c.c.})=0. \tag{18}$$

The index r takes into account here the contributions made to (18) by the sequence of spherical summation regions, $k_1=p_{i-r}$ and $k_1=p_{i+r}$, respectively. The equations for the functions with the remaining vectors $\mathbf{k}_m^i$, $m=2,3$, are similar.

We confine ourselves in (18) to the interactions of regions adjacent to the ith one, and consider the particular case of the system (18) when

$$a_\alpha(\mathbf{k}_1^i)=a_\alpha(\mathbf{k}_2^i)=a_\alpha(\mathbf{k}_3^i)=i\eta_i,\quad \alpha=1,2. \tag{19}$$

We obtain the following discrete nonlinear system for η_i:

$$\dot{\eta}_i=2p_{i-1}\eta_{i-1}\eta_i-2p_i\eta_{i+1}^2-\nu p_i^2\eta_i. \tag{20}$$

For the specified number of vectors in the ith layer, the energy in this layer is obviously

$$E_i=\frac{1}{2}\sum_{k=p_i}(|a_1(\mathbf{k})|^2+|a_2(\mathbf{k})|^2)=6\eta_i^2. \tag{21}$$

The system (20) is equal, apart from a constant, to the Obukhov chain that is one of the branches of a multilevel nonlinear system that simulates the cascade mechanism of energy conversion in a turbulent stream:

$$\begin{gathered}\dot{v}_0=p_0(v_{1,1}^2-v_{1,2}^2)-\lambda_0 v_0+f_0,\\ \dot{v}_{i,2k-1}=-p_{i-1}v_{i-1,k}v_{i,2k-1}+p_i(v_{i+1,4k-3}^2-v_{i+1,4k-2}^2)-\lambda_i v_{i,2k-1},\\ \dot{v}_{i,2k}=p_{i-1}v_{i-1,k}v_{i,2k}+p_i(v_{i+1,4k-1}^2-v_{i+1,4k}^2)-\lambda_i v_{i,2k},\\ i=1,2,\ldots,\quad k=1,2,3,\ldots,2^{i-1},\quad \lambda_i=\nu p_i^2.\end{gathered} \tag{22}$$

At $f_0=0$ and $\lambda_i=0$ the system (22) possesses the fundamental properties of the hydrodynamics equations: it is quadratically nonlinear, conserves the phase volume $(\partial\dot{v}_{i,j}/\partial v_{i,j}=0)$, and has a quadratic integral of motion $E=\Sigma v_{i,j}^2/2$ (Ref. 19). Note that discrete (mesh) models have been extensively developed in connection with the study of nonlinear systems. We mention as examples the "Langmuir chain" [20] or the one-dimensional anharmonicity model (Toda chain).

In the reduction performed, we chose, in each of the spherical layers of summation over the wave numbers, the minimum possible number (for 3D space) of vectors. It is of interest to consider how the equations obtained and the results derived from them are changed if another distribution of the vectors in the layer is chosen, and in particular if the vectors are continuously distributed over the spheres $|\mathbf{k}|=p_i$.

We choose as the starting point in this case Eq. (8), which we rewrite, using approximations similar to (12) and (13), in the form

$$\left(\frac{d}{dt}+\nu k^2\right)V_i(\mathbf{k})-i\Delta_{ij}(\mathbf{n})A_{jl}(k)V_l(\mathbf{k})+i\Delta_{ij}(\mathbf{n})k_lB_{lj}(k)=0, \tag{23}$$

$$A_{jl}(k)=\sum_{k_1<k}V_j(-\mathbf{k}_1)\mathbf{k}_{1l},$$

$$B_{lj}(k)=\sum_{k_1>k}V_l(\mathbf{k}_1)V_j(-\mathbf{k}_1),\qquad \mathbf{n}=\mathbf{k}/k.$$

Note that in (23) $A_{jl}(k)$ and $B_{lj}(k)$ depend only on the modulus k of the vector $\mathbf{k}$. This enables us to solve (23) in the form

$$V_i(\mathbf{k})=\Delta_{ij}(\mathbf{n})n_sF_{sj}(\mathbf{k}). \tag{24}$$

For $F_{sj}(\mathbf{k})$ we have the equation

$$\begin{aligned}\left(\frac{d}{dt}+\nu k^2\right)&F_{sj}(\mathbf{k})+\frac{i}{2}\sum_{k_1<k}k_1n_{1m}\Delta_{j\nu}(\mathbf{n}_1)n_{1\mu}[F_{\mu\nu}(\mathbf{k}_1)\\ &-F_{\mu\nu}^*(\mathbf{k}_1)]F_{sm}(\mathbf{k})\\ &+ik\sum_{k_1>k}\Delta_{s\nu}(\mathbf{n}_1)n_{1\mu}n_{1\mu_1}F_{\mu\nu}(\mathbf{k}_1)\dot{F}_{\mu_1j}(\mathbf{k}_1)-ik\sum_{k_1>k}\Delta_{s\nu}(\mathbf{n}_1)\\ &\times n_{1\mu}n_{1\mu_1}n_{1j}n_{1\nu_1}F_{\mu\nu}(\mathbf{k}_1)F_{\mu_1\nu_1}^{\bullet}(\mathbf{k}_1)=0.\end{aligned} \tag{25}$$

We represent the solution of (25) as a series in powers of the products $\varepsilon_{\alpha\beta}$ of the direction cosines of the vector $\mathbf{n}$: $\varepsilon_{\alpha\beta}=n_\alpha n_\beta$, $|\varepsilon_{\alpha\beta}|\leqslant 1$. The last sum in (25) is then the term of lowest order. The zeroth approximation of the perturbation theory series for

$$F_{sj}(\mathbf{k})=iX_{sj}^0(k)+[n_{\alpha_1}n_{\alpha_2}F_{sj,\alpha_1\alpha_2}^{(1)}(k)+iX_{sj}^{(1)}(k)]+\ldots$$

depends then only on the modulus k of the vector $\mathbf{k}$ and satisfies the equation

$$\begin{gathered}\left(\frac{d}{dt}+\nu k^2\right)X_{lj}^0(k)-X_{l\mu_1}^0(k)S_{j\nu,\mu\mu_1}\\ \times\sum_{k_1<k}k_1X_{\mu\nu}^0(k_1)+kS_{l\nu,\mu\mu_1}\sum_{k_1>k}X_{\mu\nu}^0(k_1)X_{\mu_1j}^0(k_1)=0,\\ S_{j\nu,\mu\mu_1}=\sum_{|\mathbf{n}|=1}\Delta_{j\nu}(\mathbf{n})n_\mu n_{\mu_1}.\end{gathered} \tag{26}$$

The quantity $S_{j\nu,\mu\mu_1}$ can be calculated for the specified distribution of the vectors $\mathbf{n}$. For a continuous distribution on the sphere $|\mathbf{n}|=1$ we have

$$S_{j\nu,\mu\mu_1}=\frac{4\pi}{15}[4\delta_{j\nu}\delta_{\mu\mu_1}-(\delta_{j\mu}\delta_{\nu\mu_1}+\delta_{j\mu_1}\delta_{\nu\mu})].$$

Equations (26) have particular solutions

$$X_{sj}^0(k)=\delta_{sj}\xi_j(k),\qquad \xi_1=\xi_2=\xi,\quad \xi_3=0, \tag{27}$$

obtained with allowance for the fact that the tensor $\Delta_{ij}(\mathbf{n})$ is solenoidal. Transforming from (26) and (27) to the discrete system, we have

$$\begin{gathered}\left(\frac{d}{dt}+\nu p_i^2\right)\xi_i=\frac{4\pi}{15}\{p_{i-1}\xi_{i-1}\xi_i-p_i\xi_{i+1}^2\},\\ E_i=\frac{4\pi}{15}\xi_i^2,\end{gathered} \tag{28}$$

where E_i is the energy of the ith level of the continuous distribution of the wave vectors on a sphere of radius p_i. Using the variables $\zeta_i=(2\pi/15)\xi_i$ we reduce (28) to the form (20):

$$\left(\frac{d}{dt}+\nu p_i^2\right)\zeta_i=2p_{i-1}\zeta_{i-1}\zeta_i-2p_i\zeta_{i+1}^2. \tag{29}$$

The energy of the ith layer is

$$E_i=6\frac{5}{2\pi}\zeta_i^2, \tag{30}$$

and differs from (21) by the near-unit factor $(5/2\pi)$. The results that follow from the two systems obtained are further compared in the next section. Note that a reduction, different from ours, of the Navier-Stokes equations to discrete chains was carried out also in Ref. 21.

3. EQUILIBRIUM REGIME OF CASCADE SYSTEM AND ESTIMATE OF THE CONSTANT IN THE "2/3 LAW"

We consider the system (20), (21) in the inertial range, for which we neglect the viscous terms, $p_{i-1}|\eta_{i-1}| \gg \nu p_i^2$. For the change of the energy E_i of the ith level we have from (20)

$$\dot{E}_i = 24 p_{i-1} \eta_{i-1} \eta_i^2 - 24 p_i \eta_i \eta_{i+1}^2,$$

where the first term in the right-hand side represents the inflow of energy to the ith level, and the second the outflow. Under stationary conditions the energy flux over the spectrum is $\varepsilon = 24 p_i \eta_i \eta_{i+1}^2 = \mathrm{const}$, so that the ensuing stationary distribution of the amplitudes takes the form[9–11]

$$\eta_i^0 = (\varepsilon/24)^{1/3} q^{2/9} p_i^{-1/3}. \tag{31}$$

The energy E_i of the ith level is then, according to (21)

$$E_i = 6(24)^{-2/3} \varepsilon^{2/3} q^{4/9} p_i^{-2/3}. \tag{32}$$

Comparing (32) with (15) we find that the stationary solution (31) of the system (20) corresponds to the Kologorov-Obukhov law (1), (2) and the constant C_1 in (15) is expressed in terms of the scale-refinement coefficient q:

$$C_1 = q^{4/9} \frac{q^{2/3}}{q^{2/3}-1} 3^{-2/3} \equiv C_1(q). \tag{33}$$

The function $C_1(q)$ (see Fig. 1) has at $q_m = (5/2)^{3/2} \approx 3.95$ a minimum equal to $C_m = (5/3)(5/6)^{2/3} \approx 1.476$. At values of q in a large vicinity of this minimum (from 2.2 to 10), the function $C_1(q)$ varies very little, by less than 15% of C_m. The experimentally obtained Kolmogorov constant $C = 2$ compares with $C_1 \approx 1.52$, and according to (33) the corresponding values of q are approximately $q_1 = 8^{1/2}$ and $q_2 = 5.8$. The first of these values is critical for the stability of solution (31) of the chain (20), in which $\lambda_i = \nu p_i^2 = 0$ at $i < n$, $\lambda_n = p_{n-1}(\eta_{n-1}^0/\eta_n^0)\eta_n$, $\eta_{n+1} = 0$ (chain with quadratic friction)[22]: at $q \geqslant q_1$ the solution becomes stable.[23]

The constant in the Kolmogorov-Obukhov law, expressed in terms of the scale-refinement coefficient q, is a function that varies little when q is varied in a wide range (in this sense it can be regarded as a universal constant), and the corresponding values are close to the available experimental data. Note that the refinement coefficient q, which is the factor by which the characteristic sizes of the perturbations of successive levels vary, can take on values from 2–2.2 to ~ 10 in the interval in which C is universal. It can therefore be assumed that a cascade processes with a large number of levels is realized in flows with very differing perturbation sales, such as geophysical flows.

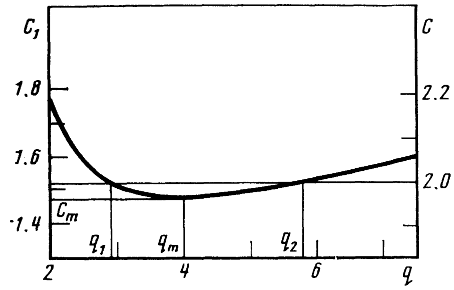

FIG. 1.

We note in conclusion that when the model (29), (30) is used the value of C_1 differs from (33) by a factor $(5/2\pi)^{1/3} \approx 0.93$, i.e., the results obtained with the aid of the two approaches described above are close to one another.

The author thanks A. M. Obukhov for posing the problem, and V. I. Tatarskiĭ, V. M. Ponomarov, F. V. Dolzhanskiĭ, and V. M. Gryanik for a discussion of the results.

[1]A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **30**, 299 (1941).
[2]A. N. Kolmogorov, *ibid.* **32**, 19 (1941).
[3]A. M. Obukhov, *ibid.* **32**, 22 (1941).
[4]A. M. Obukhov, Izv. AN SSSR, Izv. AN SSSR, Ser. Gegor. i Geofiz. **5**, 443 (1941).
[5]A. S. Monin and A. M. Yaglom, *Statistical Hydromechanics*, MIT Press, 1971, part 2.
[6]A. M. Yaglom, Comments on A. N. Kolmogorov's work on turbulence, in: A. N. Kolmogorov, Collected Works. Mathematics and Mechanics [in Russian], Nauka, 1985, p. 421.
[7]A. N. Yaglom, Izv. AN SSSR, Fiz. Atmosf. i Okeana **17**, 1235 (1981).
[8]L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, 2nd ed., Pergamon, 1959 [p. 157 of original Russian ed.].
[9]A. M. Obukhov, Izv. AN SSSR, Fiz. Atmosf. i Okeana **7**, 695 (1971).
[10]A. M. Oboukhov, Gerlands Beitr. Geophys. **82**, 282 (1973).
[11]A. M. Obukhov, Usp. Mat. Nauk **38**, 101 (1983).
[12]R. H. Kraichnan, J. Fluid Mech. **47**, 525 (1971).
[13]G. A. Kuz'min and A. Z. Patashinskiĭ, Zh. Eksp. Teor. Fiz. **76**, 2075 (1979) [Sov. Phys. JETP **49**, 1050 (1979)].
[14]E. B. Gledzer, F. V. Dolzhanskiĭ, and A. M. Obukhov, *Systems of Hydrodynamic Type and Their Applications*. Nauka, 1981, p. 196.
[15]E. B. Gledzer, Izv. AN SSSR. Mekh. Zhidk. i Gaza **1**, 26 (1981).
[16]E. A. Kuznetsov and V. S. L'vov, Phys. Lett. **64A**, 157 (1977).
[17]S. S. Moiseev, A. V. Tur, and V. V. Yanovskiĭ, Dokl. Akad. Nauk SSSR **279**, 96 (1984) [Sov. Phys. Dokl. **29**, 926 (1984)].
[18]C. Gloaguen, J. Leorat, A. Pouquet, and R. Grappin, Physica (Utrecht) **17D**, 154 (1985).
[19]A. M. Obukhov, Dokl. Akad. Nauk SSSR **184**, 309 (1969) [Sov. Phys. Dokl. **14**, 32 (1969)].
[20]V. E. Zakharov, S. L. Musher, and A. M. Rubenchik, Pis'ma Zh. Eksp. Teor. Fiz. **19**, 249 (1974) [JETP Lett. **19**, 151 (1974)].
[21]V. D. Zimin, Izv. AN SSSR. Fiz. Atmosf. i Okeana **17**, 1265 (1981).
[22]E. B. Gledzer and A. L. Makarov, *ibid.* **21**, 899 (1985).
[23]A. B. Glukhovskiĭ, *ibid.* **13**, 917 (1977).

Translated by J. G. Adashko