Accumulation of the relaxation products of a stationary magnon packet

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It is shown that, when the magnon subsystem and heat bath are weakly coupled and the three-magnon interaction is weak compared to the exchange interaction, a low-power source of nonequilibrium in a ferromagnetic material leads to the appearance of a substantial number of nonequilibrium magnons in the material. The process by which the products of the relaxation of parametrically excited spin waves accumulate is analyzed in detail, and the results obtained are compared with the experimental data.

INTRODUCTION

Normally, investigations of the relaxation of a quasiparticle packet are limited to the computation of the lifetime characterizing the rate of departure of quasiparticles from the packet, no interest being taken in the subsequent evolution of the relaxation products. But if a source of nonequilibrium maintains the packet at a steady state level, and the system of quasiparticles is sufficiently well isolated adiabatically, then the relaxation products should accumulate, leading over long periods of time to significant changes in the thermal-quasiparticle distribution function.

The effects of the accumulation of a substantial number of non-equilibrium quasiparticles have been repeatedly observed in the course of the parametric excitation of magnons in ferromagnetic and antiferro-magnetic materials.[1–3] Evidently the anomalies reported in Refs. 4–6 in the relaxation of spin waves in the ferrites are also connected with the accumulation of the relaxation products (see Refs. 6 and 7).

In parametric excitation the source of nonequilibrium is the pump, which gives rise to a narrow packet of parametric spin waves (PSW) with frequency equal to half the pump frequency. The total number of PSW is determined by the amplitude of the pump,[8] and can be maintained under experimental conditions at a constant level. The PSW frequency normally lies in the microwave frequency region, which, on the temperature scale, is of the order of a fraction of a degree; the characteristic exchange interaction energy is usually equal to several score or several hundred degrees, while the temperature of the magnetic material in different experiments lies in the range from a few degrees to several hundred degrees. In this case the total number of PSW is small compared to the overall number of thermal magnons, but in the vicinity of a resonance surface the magnon occupation numbers deviate considerably from the equilibrium value and the PSW packet can be regarded as a source of nonequilibrium for the thermal magnons.

Usually, under experimental conditions the temperature of the medium surrounding the magnetic material stabilizes. Energy is exchanged between the nonequilibrium magnons and the heat bath through the lattice vibrations. Therefore, to the extent that the interaction with the thermal phonons is weak, the magnon subsystem can be considered to be almost adiabatically isolated. As a rule, the interaction with the thermal phonons does not change the number of magnons, but only redistributes them in **k** space; therefore, the interaction with the heat bath does not limit the number of nonequilibrium magnons. The number of magnons is limited in this case by the three-magnon processes resulting from the relatively weak magnetic-dipole interaction. As a result, a weak source of nonequilibrium can, over a long period of time, lead to the accumulation of a considerable number of nonequilibrium magnons. It should be noted that there is a definite analogy between the magnon accumulation process and the relaxation of the homogeneous magnetization of a ferromagnet, a relaxation which is also due to the relatively weak three-magnon and magnetoelastic interactions.[9]

The present paper is devoted to the study of the effects of the accumulation of the nonequilibrium magnons that arise as a result of the relaxation of a stationary PSW packet. In Sec. 1 we obtain for the magnons a kinetic equation linearized about thermodynamic equilibrium, and discuss the properties of its eigenfunctions. In Sec. 2 we determine the distribution function for the nonequilibrium magnons produced by a weak source. We also calculate there the number of nonequilibrium magnons in the case when their source is a PSW packet. In an experimental situation the intensity of the PSW packet is usually high, which results in a significant distortion of the magnon distribution function in the **k**-space region where the exchange relaxation is weak. Consequently the source of nonequilibrium in the linearized kinetic equation (LKE) is not only the PSW's, but also the highly nonequilibrium magnons participating in their relaxation. In Sec. 3 we determine the number of nonequilibrium magnons that accumulate under conditions of substantial PSW intensity, and show that the results obtained agree with the available experimental data.

1. KINETIC EQUATION FOR THE MAGNONS

The magnon distribution function $n_{\mathbf{k}}$ in **k** space is the solution to the kinetic equation

$$\partial n_{\mathbf{k}}/\partial t=I^{(3)}\{n_{\mathbf{k}}\}+I^{(4)}\{n_{\mathbf{k}}\}+f_{\mathbf{k}}+R_{\mathbf{k}}. \qquad (1.1)$$

Here $I^{(3)}\{n_{\mathbf{k}}\}$ and $I^{(4)}\{n_{\mathbf{k}}\}$ are the collision terms describing the three- and four-magnon processes, $f_{\mathbf{k}}$ is the probability for production of a magnon (i.e., the magnon source), and

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$R_{\mathbf{k}}$ is the relaxation term describing the interaction of the magnons with the equilibrium phonons (the heat bath).

In the majority of magnetic insulators the strongest interaction is the exchange interaction. When the magnon occupation numbers are not too large, the exchange interaction Hamiltonian expressed in terms of the magnon creation and annihilation operators $a_{\mathbf{k}}^+$ and $a_{\mathbf{k}}$ is equal to

$$\mathscr{H}_{ex}=\sum_{\mathbf{k}}\omega_{\mathbf{k}}a_{\mathbf{k}}^+a_{\mathbf{k}}+\frac{1}{2}\sum_{1234}T_{1234}a_1^+a_2^+a_3a_4\Delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4), \tag{1.2}$$

$$\omega_{\mathbf{k}}=\omega_0+\omega_{ex}(ak)^2,\quad T_{1234}=-(\omega_{ex}/4S)(\mathbf{k}_1\mathbf{k}_2+\mathbf{k}_3\mathbf{k}_4),$$

where ω_{ex} is the exchange frequency, S is the spin of the unit cell, a is the lattice constant, and ω_0 is the gap in the magnon spectrum, proportional to the magnetic field H_0.

As a result of this interaction a collision term $I^{(4)}\{n_{\mathbf{k}}\}$ appears which is equal to

$$I^{(4)}\{n_{\mathbf{k}}\}=(2\pi)^{-5}v_0^2\sum_{\mathbf{g}}\int|T_{\mathbf{k}123}|^2[(n_{\mathbf{k}}+1)(n_1+1)n_2n_3$$
$$-n_{\mathbf{k}}n_1(n_2+1)(n_3+1)]\delta(\omega_{\mathbf{k}}+\omega_1-\omega_2-\omega_3)$$
$$\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3-\mathbf{g})\,d\mathbf{k}_1\,d\mathbf{k}_2\,d\mathbf{k}_3. \tag{1.3}$$

Here v_0 is the volume of the unit cell and $\mathbf{g}$ is the reciprocal lattice vector.

The three-magnon processes are due to weaker interactions: the magnetic dipole and spin-orbit interactions. If we denote the amplitude of the three-magnon interaction by V_{123}, then

$$I^{(3)}\{n_{\mathbf{k}}\}$$
$$=(2\pi)^{-2}v_0\sum_{\mathbf{g}}\int\{|V_{\mathbf{k}12}|^2[(n_{\mathbf{k}}+1)n_1n_2-n_{\mathbf{k}}(n_1+1)(n_2+1)]$$
$$\times\delta(\omega_{\mathbf{k}}-\omega_1-\omega_2)\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{g})+2|V_{1\mathbf{k}2}|^2[n_1(n_{\mathbf{k}}+1)(n_2+1)$$
$$-(n_1+1)n_{\mathbf{k}}n_2]\delta(\omega_1-\omega_{\mathbf{k}}-\omega_2)\delta(\mathbf{k}_1-\mathbf{k}-\mathbf{k}_2-\mathbf{g})\}\,d\mathbf{k}_1\,d\mathbf{k}_2. \tag{1.4}$$

In the absence of a source the kinetic equation (1.1) has the steady state solution

$$n_{\mathbf{k}}^0=[\exp(\omega_{\mathbf{k}}/T)-1]^{-1}, \tag{1.5}$$

which describes the thermodynamic equilibrium with temperature T. If we neglect the relatively weak three-magnon interaction, then the class of equilibrium solutions broadens because of the conservation of the magnon number:

$$n_{\mathbf{k}}^0=\{\exp[(\omega_{\mathbf{k}}-\mu)/T]-1\}^{-1}, \tag{1.6}$$

where μ is an arbitrary chemical potential determined by the external conditions.

In the case when the power of the source is not too high the magnon distribution function is close to the equilibrium function:

$$n_{\mathbf{k}}=n_{\mathbf{k}}^0+\delta n_{\mathbf{k}}. \tag{1.7}$$

If we introduce in place of $\delta n_{\mathbf{k}}$ a new variable $\varphi_{\mathbf{k}}$ through the relation

$$\delta n_{\mathbf{k}}=n_{\mathbf{k}}^0(n_{\mathbf{k}}^0+1)\varphi_{\mathbf{k}}, \tag{1.8}$$

then we obtain for $\varphi_{\mathbf{k}}$ a LKE with a symmetric kernel[10]:

$$n_{\mathbf{k}}^0(n_{\mathbf{k}}^0+1)\frac{\partial\varphi_{\mathbf{k}}}{\partial t}+\int P_{\mathbf{k}12}(\varphi_{\mathbf{k}}-\varphi_1-\varphi_2)\,d\mathbf{k}_1\,d\mathbf{k}_2$$
$$+2\int P_{1\mathbf{k}2}(\varphi_{\mathbf{k}}+\varphi_2-\varphi_1)\,d\mathbf{k}_1\,d\mathbf{k}_2$$
$$+\int Q_{\mathbf{k}123}(\varphi_{\mathbf{k}}+\varphi_1-\varphi_2-\varphi_3)\,d\mathbf{k}_1\,d\mathbf{k}_2\,d\mathbf{k}_3=n_{\mathbf{k}}^0(n_{\mathbf{k}}^0+1)\frac{\partial\varphi_{\mathbf{k}}}{\partial t}$$
$$+\int A_{\mathbf{k}\mathbf{k}'}\varphi_{\mathbf{k}'}\,d\mathbf{k}'=f_{\mathbf{k}}+R_{\mathbf{k}}, \tag{1.9}$$

where

$$P_{123}=(2\pi)^{-2}v_0\sum_{\mathbf{g}}|V_{123}|^2(n_1^0+1)n_2^0n_3^0\delta(\omega_1-\omega_2-\omega_3)$$
$$\times\delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3-\mathbf{g}),$$

$$Q_{1234}=(2\pi)^{-5}v_0^2\sum_{\mathbf{g}}|T_{1234}|^2(n_1^0+1)$$
$$\times(n_2^0+1)n_3^0n_4^0\delta(\omega_1+\omega_2-\omega_3-\omega_4)$$
$$\times\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4-\mathbf{g}). \tag{1.10}$$

To determine the form of the relaxation term $R_{\mathbf{k}}$, let us consider the simplest form of magnon-number conserving magnon-phonon interaction

$$\mathscr{H}_{mph}=\sum_{\mathbf{q}\mathbf{k}\mathbf{k}'\mathbf{g}}[U_{\mathbf{q}\mathbf{k}\mathbf{k}'}b_{\mathbf{q}}^+a_{\mathbf{k}}^+a_{\mathbf{k}'}\Delta(\mathbf{k}+\mathbf{q}-\mathbf{k}'-\mathbf{g})+\text{H.c.}]. \tag{1.11}$$

Here $b_{\mathbf{q}}^+$ is the creation operator for a phonon with quasimomentum $\mathbf{q}$. In second-order perturbation theory this interaction gives rise to the following contribution to the kinetic equation:

$$R_{\mathbf{k}}=\frac{\pi v_0}{(2\pi)^3}\sum_{\mathbf{g}}\int\{|U_{\mathbf{q}\mathbf{k}\mathbf{k}'}|^2[(n_{\mathbf{k}}+1)(N_{\mathbf{q}}^0+1)n_{\mathbf{k}'}$$
$$-n_{\mathbf{k}}N_{\mathbf{q}}^0(n_{\mathbf{k}'}+1)]$$
$$\times\delta(\omega_{\mathbf{k}}+\Omega_{\mathbf{q}}-\omega_{\mathbf{k}'})\delta(\mathbf{k}+\mathbf{q}-\mathbf{k}'-\mathbf{g})+|U_{\mathbf{q}\mathbf{k}'\mathbf{k}}|^2[(n_{\mathbf{k}}+1)N_{\mathbf{q}}^0n_{\mathbf{k}'}$$
$$-n_{\mathbf{k}}(N_{\mathbf{q}}^0+1)(n_{\mathbf{k}'}+1)]\delta(\omega_{\mathbf{k}'}+\Omega_{\mathbf{q}}-\omega_{\mathbf{k}})\delta(\mathbf{k}'+\mathbf{q}-\mathbf{k}-\mathbf{g})\}\,d\mathbf{k}'\,d\mathbf{q}, \tag{1.12}$$

where $\Omega_{\mathbf{q}}$ is the phonon dispersion law and $N_{\mathbf{q}}^0=[\exp(\Omega_{\mathbf{q}}/T)-1]^{-1}$. For small deviations of the distribution function from the equilibrium function we have

$$R_{\mathbf{k}}=-\int M_{\mathbf{k}\mathbf{k}'}(\varphi_{\mathbf{k}}-\varphi_{\mathbf{k}'})\,d\mathbf{k}',$$

$$M_{\mathbf{k}\mathbf{k}'}=M_{\mathbf{k}'\mathbf{k}}=\frac{\pi v_0}{(2\pi)^3}\sum_{\mathbf{g}}\{|U_{\mathbf{k}'-\mathbf{k}-\mathbf{g},\mathbf{k},\mathbf{k}'}|^2(n_{\mathbf{k}'}^0+1)N_{\mathbf{k}'-\mathbf{k}-\mathbf{g}}^0n_{\mathbf{k}}^0$$
$$\times\delta(\omega_{\mathbf{k}}+\Omega_{\mathbf{k}'-\mathbf{k}-\mathbf{g}}-\omega_{\mathbf{k}'})+|U_{\mathbf{k}-\mathbf{k}'-\mathbf{g},\mathbf{k}',\mathbf{k}}|^2(n_{\mathbf{k}}^0+1)N_{\mathbf{k}-\mathbf{k}'-\mathbf{g}}^0n_{\mathbf{k}'}^0$$
$$\times\delta(\omega_{\mathbf{k}'}+\Omega_{\mathbf{k}-\mathbf{k}'-\mathbf{g}}-\omega_{\mathbf{k}})\}. \tag{1.13}$$

The steady state LKE, (1.9), has the form

$$\hat{A}\varphi_{\mathbf{k}}=f_{\mathbf{k}}+R_{\mathbf{k}},$$

where $\hat{A}$ is a symmetric integral operator. Its eigenfunctions

$$\hat{A}\varphi_{\mathbf{k}}^{(\lambda)}=\lambda\varphi_{\mathbf{k}}^{(\lambda)},$$

belonging to different eigenvalues are orthogonal:

$$\int \varphi_{\mathbf{k}}^{(\lambda)} \varphi_{\mathbf{k}}^{(\mu)} d\mathbf{k}=0, \quad \lambda\neq\mu. \tag{1.14}$$

On account of the H-theorem, these eigenvalues are nonnegative.

In the spatially homogeneous case the LKE possesses an eigen-function with zero eigenvalue:

$$\varphi_{\mathbf{k}}^{(0)}=\beta\omega_{\mathbf{k}} \quad (\beta=1/T), \tag{1.15}$$

which corresponds to a small change in the temperature in the thermodynamic-equilibrium solution (1.5). If we neglect the relatively weak three-magnon processes, then the LKE possesses another solution with zero eigenvalue, corresponding to a small change in the chemical potential in (1.6):

$$\varphi_{\mathbf{k}}^{(1)}=1. \tag{1.16}$$

Generally speaking, the LKE can also possess solutions with zero eigenvalue that arise when the energy fluxes and the magnon number are varied slightly over the spectrum (see Ref. 11). But, as shown in Ref. 12, such solutions for magnons do not exist in a Heisenberg ferromagnet because of the "ultraviolet" divergence in the LKE (the absence of locality).

As to the remaining eigenvalues of the LKE, they can be estimated by integrating the kernel Q_{1234} in (1.9). As a result we find that the characteristic value of λ is, in order of magnitude, equal to the mean value of the exchange relaxation rate (3.4) for the magnons:

$$\lambda \propto \int \gamma_{\mathbf{k}}^{ex} n_{\mathbf{k}}^0 \, d\mathbf{k} \Big/ \int n_{\mathbf{k}}^0 \, d\mathbf{k} \propto \frac{T^4 (ak_m)^3}{\omega_{ex}^3 S^2}, \tag{1.17}$$

where k_m is the characteristic wave vector of the thermal magnons:

$$ak_m \approx \begin{cases} (T/\omega_{ex})^{1/2}, & T/\omega_{ex}\ll 1, \\ 1, & T/\omega_{ex}\gg 1. \end{cases} \tag{1.18}$$

2. NONEQUILIBRIUM-MAGNON DISTRIBUTION FUNCTION

1. Let us consider the steady state solution to the LKE with a source and a relaxation term describing the interaction between the magnons and the heat bath. We shall seek this solution in the form

$$\varphi_{\mathbf{k}}=c_0\varphi_{\mathbf{k}}^{(0)}+c_1\varphi_{\mathbf{k}}^{(1)}+\sum_{\lambda\neq 0} c_\lambda \varphi_{\mathbf{k}}^{(\lambda)} \tag{2.1}$$

with the orthogonalized functions $\varphi_{\mathbf{k}}^{(0)}$ and $\varphi_{\mathbf{k}}^{(1)}$:

$$\varphi_{\mathbf{k}}^{(0)}=\beta\omega_{\mathbf{k}}-\langle\beta\omega_{\mathbf{k}}\rangle, \quad \varphi_{\mathbf{k}}^{(1)}=1, \tag{2.2}$$

where

$$\langle\beta\omega_{\mathbf{k}}\rangle=\frac{v_0}{(2\pi)^3}\int \beta\omega_{\mathbf{k}}\, d\mathbf{k}. \tag{2.3}$$

In the absence of relatively weak factors—the interaction with the heat bath and the three-magnon interactions—the coefficients c_0 and c_1 formally become infinite. (Indeed, in this case a nonstationary process will occur: the coefficients c_0 and c_1 grow at a constant rate until the LKE can be used to describe the behavior of the perturbation $\delta n_{\mathbf{k}}$.) Thus, the dominant contribution to the total number of accumulated nonequilibrium waves,

$$\delta n=\frac{v_0}{(2\pi)^3}\int \delta n_{\mathbf{k}}\, d\mathbf{k}=\frac{v_0}{(2\pi)^3}\int n_{\mathbf{k}}^0(n_{\mathbf{k}}^0+1)\varphi_{\mathbf{k}}\, d\mathbf{k}, \tag{2.4}$$

is determined by the large coefficients c_0 and c_1. Substituting (2.1) into the LKE (1.9), we obtain

$$\begin{gathered} c_0=f_0/M, \quad c_1=f_1/P, \\ f_0=\frac{v_0}{(2\pi)^3}\int f_{\mathbf{k}}\beta\omega_{\mathbf{k}}\, d\mathbf{k}, \quad f_1=\frac{v_0}{(2\pi)^3}\int f_{\mathbf{k}}\, d\mathbf{k}, \\ M=\frac{1}{2}\frac{v_0}{(2\pi)^3}\int M_{\mathbf{k}\mathbf{k}'}(\beta\omega_{\mathbf{k}}-\beta\omega_{\mathbf{k}'})^2\, d\mathbf{k}\, d\mathbf{k}', \\ P=\frac{v_0}{(2\pi)^3}\int P_{123}\, d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3. \end{gathered} \tag{2.5}$$

As a result the total number of nonequilibrium magnons is equal to

$$\delta n=c_0\frac{v_0}{(2\pi)^3}\int \beta\omega_{\mathbf{k}} n_{\mathbf{k}}^0(n_{\mathbf{k}}^0+1)\, d\mathbf{k}+c_1\frac{v_0}{(2\pi)^3}\int n_{\mathbf{k}}^0(n_{\mathbf{k}}^0+1)\, d\mathbf{k}, \tag{2.6}$$

which corresponds to the following effective changes in the temperature and chemical potential of the magnons:

$$\Delta T/T=c_0=f_0/M, \quad \Delta\mu/T=c_1=f_1/P. \tag{2.7}$$

The dominant contribution to $\delta n_{\mathbf{k}}$ is made by the first term in (2.1) when $\omega_{\mathbf{k}} < Tc_0/c_1$, and by the second term when $\omega_{\mathbf{k}} > Tc_0/c_1$. The c_0 and c_1 values are respectively established over time periods

$$t_0\propto M^{-1}, \quad t_1\propto P^{-1}. \tag{2.8}$$

At the earlier stages after the source has been switched on, i.e., at times $t < t_0, t_1$, the amplitudes c_0 and c_1 grow at the constant rate

$$\begin{gathered} \dot{c}_i=(B^{-1})_{ij}f_j, \quad i, j=0, 1 \\ B_{ij}=\frac{v_0}{(2\pi)^3}\int \varphi_{\mathbf{k}}^{(i)}\varphi_{\mathbf{k}}^{(j)} n_{\mathbf{k}}^0(n_{\mathbf{k}}^0+1)\, d\mathbf{k}, \end{gathered} \tag{2.9}$$

the steady state values (2.5) being established at $t > \max(t_0, t_1)$.

2. Let us consider in greater detail the process by which nonequilibrium magnons accumulate in a ferromagnet. We shall assume the spin wave spectrum in it to be a quadratic spectrum with a gap:

$$\omega_{\mathbf{k}}=\omega_0+\omega_{ex}(ak)^2 \quad (\omega_0\ll\omega_{ex}) \tag{2.10}$$

and neglect the relatively weak effects connected with the magnetic dipole interaction. In this case the integrals in (2.6) are equal to

$$\frac{v_0}{(2\pi)^3}\int \beta\omega_{\mathbf{k}} n_{\mathbf{k}}^0(n_{\mathbf{k}}^0+1)\, d\mathbf{k} \approx (2\pi)^{-2}\left(\frac{T}{\omega_{ex}}\right)^{3/2} \begin{cases} \Gamma(5/2)\zeta(3/2), & T\ll\omega_{ex}, \\ 2(\omega_{ex}/T)^{1/2}, & T\gg\omega_{ex}, \end{cases} \tag{2.11}$$

$$\frac{v_0}{(2\pi)^3}\int n_{\mathbf{k}}^0(n_{\mathbf{k}}^0+1)\, d\mathbf{k} \approx \frac{1}{8\pi}\left(\frac{T}{\omega_{ex}}\right)^2\left(\frac{\omega_{ex}}{\omega_0}\right)^{1/2} \tag{2.12}$$

As can be seen from (2.12), the integral in the second term in (2.6) contains the large parameter $(\omega_{ex}/\omega_0)^{1/2}$, and therefore makes a large contribution to the total number δn of accumulated magnons.

Let us now consider the quantity P given by (2.5), which determines the coefficient c_1. We shall compute it under the assumption that the amplitude V_{123} does not depend on the wave vectors (actually V_{123} depends on the directions of $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$, but not on their magnitudes, and so this is an approximation). This simplifying assumption allows us to compute

$$P=\frac{v_0^2}{(2\pi)^5}\int|V_{123}|^2(n_1^0+1)n_2^0n_3^0\delta(\omega_1-\omega_2-\omega_3)\times\delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\,d\mathbf{k}_1\,d\mathbf{k}_2\,d\mathbf{k}_3$$
$$\approx(2\pi)^{-3}\frac{\pi}{4}\frac{|V|^2}{\omega_{ex}}\left(\frac{T}{\omega_{ex}}\right)^3\ln\frac{\omega_{ex}a^2k_m^2}{\omega_0},$$
$$V\approx2^{-1/2}\omega_m/S,\quad \omega_m=4\pi(g\mu_B)^2S/v_0, \tag{2.13}$$

where k_m is the upper limit of the integration, and is given by (1.18), μ_B is the Bohr magneton, and $g\approx2$ is the gyromagnetic ratio. Here we have limited ourselves to this case, usually realized in experiment, of cubic symmetry, when the three-magnon processes are due only to the magnetic dipole interaction.

We can represent the magnon distribution function $n_\mathbf{k}$ over all of $\mathbf{k}$ space in the form

$$n_\mathbf{k}=n_\mathbf{k}^0+\delta n_\mathbf{k},\quad \mathbf{k}\in D$$
$$n_\mathbf{k}=\bar{n}_\mathbf{k},\quad \mathbf{k}\,\overline{\in}\,D\ (\mathbf{k}\in\bar{D}). \tag{2.14}$$

Here D is the $\mathbf{k}$-space region where $|\delta n_\mathbf{k}|\ll n_\mathbf{k}^0$; in the remaining relatively small part $\bar{D}$ of $\mathbf{k}$ space the deviation of the distribution functions from the equilibrium function is large (i.e., $|n_\mathbf{k}-n_\mathbf{k}^0|\gtrsim n_\mathbf{k}^0$).

In the case when the pump energy is not much higher than the parametric instability threshold, the deviation from equilibrium can be large only for a narrow PSW packet localized at the resonance surface $\omega_\mathbf{k}=\omega_p/2$ (where ω_p is the pump frequency). Under experimental conditions the PSW that get excited are usually those with quite small wave vectors, i.e., PSW for which the four-magnon interaction amplitude is small, and therefore

$$f_\mathbf{k}=f_\mathbf{k}'=\frac{v_0}{(2\pi)^3}\int\Big\{2|V_{1\mathbf{k}2}|^2n_\mathbf{k}^0n_2^0\frac{\bar{n}_1}{n_1^0}\delta(\omega_1-\omega_\mathbf{k}-\omega_2)\times\delta(\mathbf{k}_1-\mathbf{k}-\mathbf{k}_2)$$
$$+|V_{\mathbf{k}12}|^2(n_\mathbf{k}^0+1)\bar{n}_1\bar{n}_2\delta(\omega_\mathbf{k}-\omega_1-\omega_2)\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\Big\}\,d\mathbf{k}_1\,d\mathbf{k}_2. \tag{2.15}$$

It is not difficult to see that the quantity f_1', (2.5), computed for $f_\mathbf{k}=f_\mathbf{k}'$, (2.15), is nonzero only in the case when the conservation laws allow either the decay of a PSW or the merging of two PSW, because when a PSW merges with a thermal magnon a new thermal magnon is produced, i.e., the process conserves the magnon number.

3. ACCUMULATION OF THE PRODUCTS OF THE RELAXATION OF AN INTENSE PSW PACKET

1. In the fairly broad range of values of the parameters (the pump frequency ω_p and the magnetic field H) that has been most thoroughly investigated in experiment the wave vector k_p of the PSW is so small that the conservation laws allow only the merging of PSW with thermal magnons. The quantity f_1', defined in (2.5) and computed from Eq. (2.15), is then equal to zero, but experiment indicates that a substantial number of nonequilibrium magnons accumulate in this case as well. The point is that, in the case of the excitation of PSW with small wave vectors, the accumulation of nonequilibrium magnons is governed by the contribution to f_1 which is nonlinear in the PSW number. This contribution is large in the case when the distribution function for the magnons produced as a result of the merging of PSW with thermal magnons is far from an equilibrium distribution. Let us find out when this is possible.

For magnons with the spectrum (2.10) fusion with PSW is allowed when

$$\omega_\mathbf{k}\geqslant\omega_1(k_p)=\omega_0+\omega_0^2/4\omega_{ex}(ak_p)^2, \tag{3.1}$$
$$\omega_0+\omega_{ex}(ak_p)^2=\omega_p/2. \tag{3.2}$$

Magnon decay in which the PSW participate is allowed if

$$\omega_\mathbf{k}\geqslant\omega_2(k_p)=\omega_1(k_p)+\omega_p/2. \tag{3.3}$$

In order to find out when the magnon distribution function arising from merging with the PSW departs greatly from equilibrium, we should consider the behavior of the magnon relaxation rate $\gamma_\mathbf{k}$ as a function of the wave vector. The exchange scattering predominantes at k values greater than some value[13] k^*:

$$\gamma_\mathbf{k}\approx\gamma_\mathbf{k}^{ex}\frac{T^2\omega_\mathbf{k}(ak)^2}{24\pi^3S^2\omega_{ex}^2}\left[\ln^2\left(\frac{T}{\omega_\mathbf{k}}\right)-\frac{10}{3}\ln\left(\frac{T}{\omega_\mathbf{k}}\right)-0.3\right]. \tag{3.4}$$

In the region $k<k^*$ the damping is due to the magnetic dipole interaction (see, for example, Ref. 14):

$$\gamma_\mathbf{k}\approx\gamma_\mathbf{k}^m=\frac{|V|^2T}{\pi(ak)\omega_{ex}^2}\times\ln\left\{\frac{[\omega_\mathbf{k}+(\omega_\mathbf{k}-\omega_0)^{1/2}(\omega_\mathbf{k}-3\omega_0)^{1/2}](4\omega_\mathbf{k}^2-3\omega_0^2)}{[\omega_\mathbf{k}-(\omega_\mathbf{k}-\omega_0)^{1/2}(\omega_\mathbf{k}-3\omega_0)^{1/2}](4\omega_\mathbf{k}-3\omega_0)\omega_0}\right\}. \tag{3.5}$$

The quantity k^* is determined by equating the damping constants (3.4) and (3.5):

$$k^*a\approx\left[12\pi^2S\frac{\omega_m^2}{T\omega_{ex}}\frac{\ln(\omega_{k^*}/\omega_0)}{\ln^2(T/\omega_{k^*})-{}^{10}/_3\ln(T/\omega_{k^*})-0.3}\right]^{1/5}, \tag{3.6}$$

which, for $\omega_m\approx0.5$ °K, $\omega_{ex}\approx40$ °K, and $T=300$ °K, yields $ak^*\approx0.3$ (estimate for yttrium iron garnet, which was used in the experiments reported in Refs. 1, 2, and 4–6).

If the products from the merging of PSW with thermal magnons fall within the region where the exchange damping is important (i.e., in the region $\omega_1(k_p)\gtrsim\omega_{k^*}$), then the magnons produced rapidly thermalize, and cannot have a highly nonequilibrium distribution function. The magnons that fall with the magnetic-dipole relaxation region can be relatively easily brought out of thermodynamic equilibrium because of the weakness of the interaction and the smallness of the phase volume of the quasiparticles participating in these processes. In order for this to occur, the inequality

$$\omega_1(k_p) \leqslant \omega_{k^*}, \tag{3.7}$$

must be satisfied, which, for $k^* a \approx 0.3$ and $\omega_0 \approx 0.5$ °K, yields $k_p \gtrsim 0.02 a^{-1} \approx 2\times 10^5$ cm^{-1}. As a rule, this condition is fulfilled in experimental situations.

In the region $k < k^*$, where the magnetic-dipole interaction is the dominant interaction, the distribution function $n_{\mathbf{k}}$ is the solution to the kinetic equation (1.1) with the collision term (1.4). The distribution function is a combination of a narrow PSW packet $n_{\mathbf{k}}^p$ and a smooth function $\tilde{n}_{\mathbf{k}}$:

$$n_{\mathbf{k}} = n_{\mathbf{k}}^p + \tilde{n}_{\mathbf{k}}, \quad n_{\mathbf{k}}^p = 4\pi^2(\omega_{ex}/ak_p) n^p \delta(\omega_{\mathbf{k}} - \omega_p/2), \tag{3.8}$$

n^p being the total number of PSW.

Analysis of the kinetic equation shows that, in the region of wave vectors where $\omega_{\mathbf{k}} > \omega_2(k_p)$, the magnon merging and decay processes in which the PSW participate cancel each other out to a high precision; therefore, the distribution function in this region is close to the thermodynamic-equilibrium function $n_{\mathbf{k}}^0$. Substantial distortions of the magnon distribution occur at

$$\omega_1(k_p) \leqslant \omega_{\mathbf{k}} \leqslant \omega_2(k_p), \tag{3.9}$$

i.e., in the region where only merging with PSW is allowed. L'vov has obtained in this region an approximate solution to the kinetic equation, that correctly describes the limiting cases of large and small PSW numbers:

$$\tilde{n}(\omega_{\mathbf{k}}) - n^0(\omega_{\mathbf{k}} + \omega_p/2)$$
$$= [n^0(\omega_{\mathbf{k}}) - n^0(\omega_{\mathbf{k}} + \omega_p/2)] \gamma_{\mathbf{k}}^0 [\gamma_{\mathbf{k}}^0 + \eta(k, k_p) n^p]^{-1}, \tag{3.10}$$

where

$$\eta(k, k') = \pi |V|^2 (\omega_{ex} a^2 k k')^{-1}$$

and $\gamma_{\mathbf{k}}^0$ is the equilibrium magnon-damping constant. As $n^p \to 0$, the magnon-distribution function tends to the equilibrium function n_k^0; in the opposite limiting case, when $\eta(k,k_p) n^p \gg \gamma_{\mathbf{k}}^0$, the occupation numbers of the magnons participating in decay and merging with PSW are equal. The magnons with frequency $\omega_{\mathbf{k}} + \omega_p/2 > \omega_2(k_p)$ are in equilibrium; therefore, $\tilde{n}_{\mathbf{k}} \to n^0(\omega_{\mathbf{k}} + \omega_p/2)$. The presence of a group of nonequilibrium magnons in the region (3.9) leads to the following expression for f_1:

$$f_1 = f_1' + f_1'',$$
$$f_1'' = (2\pi)^{-5} v_0^2 \int |V_{123}|^2 (2\tilde{n}_2 n_3^p + \tilde{n}_2 \tilde{n}_3) \delta(\omega_1 - \omega_2 - \omega_3)$$
$$\times \delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \tag{3.11}$$

and f_1' is given by the formula (2.15). For $\eta(k,k_p) n^p > \gamma_{\mathbf{k}}^0$ and $\omega_1(k_p) > 3/2\omega_0$,

$$f_1'' \approx \frac{|V|^2 T n^p}{4\pi \omega_{ex}^2 (ak_p)} \ln\left[1 + \frac{\omega_p}{2\omega_2(k_p)}\right]. \tag{3.12}$$

In particular, for $\omega_{ex}(ak_p)^2 \ll \omega_0$

$$f_1 \approx |V|^2 T(ak_p) n^p / \pi \omega_{ex} \omega_0. \tag{3.13}$$

Substitution of (3.12) into (2.6) yields the final answer for the number of accumulated nonequilibrium magnons:

$$\frac{\delta n}{n^p} \approx \left(\frac{\omega_{ex}}{\omega_0}\right)^{1/2} \frac{\ln[1 + \omega_p/2\omega_2(k_p)]}{(ak_p)\ln(T/\omega_0)}. \tag{3.14}$$

2. The answer obtained agrees with the available experimental data on the accumulation of nonequilibrium magnons when spin waves are excited parametrically in ferrites. These data can be divided into two groups. First, the total number of nonequilibrium magnons has been directly measured in light scattering experiments,[1,2] and it has been found that this number is approximately proportional to the number of PSW, with the coefficient of proportionality ranging, as the wave vector k_p of the PSW increases, from 5 to 10. According to (3.14), $\delta n/n^p \approx 12$ under the conditions of Gall and Jamet's experiment,[1] i.e., for $ak_p \approx 3\times 10^{-2}$ and $T = 300$ °K, and this theoretical estimate is fairly close to the experimental $\delta n/n^p$ value, which ranged from 7 to 10 as the magnetic field orientation with respect to a crystal symmetry axis was varied. The other group of experiments is connected with the study of the anomalies in the relaxation of the PSW when the pump is switched on and off. It has been found that, in the first moments after the pump is switched on, the center of the PSW packet shifts in **k** space,[6] and, as a result, the packet is broadened.[5] The most probable cause of the drift of the PSW packet in **k** space is the accumulation of nonequilibrium magnons, the products of the PSW relaxation. In the first moments following the switching on of the pump (specifically, in the period of time $t \lesssim P^{-1}$) the number of such magnons increases at a constant rate, leading to the following change in the eigenfrequency:

$$\delta\omega_{\mathbf{k}}(t) = 2\frac{v_0}{(2\pi)^3} \int T_{\mathbf{k}\mathbf{k}'\mathbf{k}\mathbf{k}'} \delta n_{\mathbf{k}'}(t) d\mathbf{k}',$$
$$\frac{\partial}{\partial t}\left[\frac{v_0}{(2\pi)^3} \int \delta n_{\mathbf{k}'} d\mathbf{k}'\right] = f_1. \tag{3.15}$$

The shape and amplitude of the PSW packet under conditions of eigen-frequency drift are investigated in Ref. 7, where it is shown, in particular, that the PSW packet has a substantial eigenfrequency width:

$$\Delta\omega_{\mathbf{k}} \approx \left\{\frac{3}{2}\frac{\langle T\rangle}{\langle S\rangle}\frac{f_1}{\gamma_p n^p}(p-1)^{1/2}\ln[(p-1)^{1/2}/\xi]\right\}^{1/3}\gamma_p, \tag{3.16}$$

where $\langle T\rangle$ is the characteristic value of the quantity $T_{\mathbf{k}\mathbf{k}'\mathbf{k}\mathbf{k}'}$ in (3.15),

$$\langle S\rangle = \langle T_{\mathbf{k},-\mathbf{k},\mathbf{k}',-\mathbf{k}'}\rangle, \quad \xi = 4\pi^2(ak_p)\langle S\rangle/\omega_{ex}\omega_p,$$

ξ is a small parameter characterizing the effect of the thermal noise on the PSW, $p = h^2/h_c^2$ is the ratio of the pump power to the threshold value, and γ_p is the PSW damping constant. The quantity $\Delta\omega_{\mathbf{k}}/\gamma_{\mathbf{k}}$, as measured by Zhitnyuk, Melkov, and the present author,[4-6] depends weakly on the temperature and the wave vector k_p, and the dependence on the pump power is close to the dependence predicted by (3.16). Thus, the effects of the accumulation of nonequilibrium magnons considered above allow us to explain the available set of experimental facts connected with the excitation of a substantial number of nonequilibrium magnons in the ferrites.

L'vov and Fal'kovich[15,16] have also investigated the interaction of PSW with thermal magnons. They found the correction $\delta n_{\mathbf{k}}$ to the thermal-magnon distribution function in the τ approximation, in which the collision term in the LKE, (1.9), is replaced by the expression $\gamma_{\mathbf{k}}^{0}(n_{\mathbf{k}} - n_{\mathbf{k}}^{0})$, where $\gamma_{\mathbf{k}}^{0}$ is the equilibrium magnon damping constant. In this approximation the expression for $\delta n_{\mathbf{k}}$ has a very simple form:

$$\delta n_{\mathbf{k}} = f_{\mathbf{k}}/\gamma_{\mathbf{k}}^{0}. \qquad (3.17)$$

Strictly speaking, such a magnon distribution is established over a time period $\gamma_{\mathbf{k}}^{-1}$ after the source is switched on. It reflects those characteristics of the steady state distribution function $\delta n_{\mathbf{k}}$ which are dictated by the conservation laws, and describes qualitatively the shape of $\delta n_{\mathbf{k}}$ in the **k**-space region where $f_{\mathbf{k}}$ is large. In particular, the expression (3.17) allows us to determine the effect of the distortion of the thermal-magnon distribution function on the damping of the PSW.[15,16] Naturally, the solution (3.17) to the LKE in the τ approximation contains only part of the information about the perturbation $\delta n_{\mathbf{k}}$ of the distribution function, since it does not include the nontrivial solutions (1.15) and (1.16), and therefore cannot lead to the accumulation effects considered above; the estimate of the number δn from the formula (3.17) yields $\delta n \propto \delta n^{p}$, which is significantly smaller than (3.14). Summarizing, we can assert that the formula (3.17) is an approximate expression for that contribution to $\delta n_{\mathbf{k}}$ which is equal to the sum of the eigenfunctions with nonzero eigenvalues in (2.1). In the present paper we have determined the contribution to $\delta n_{\mathbf{k}}$ which is connected with the eigenfunctions belonging to the zero eigenvalue. This contribution is a smooth function of the wave vector, and substantially changes the total number of nonequilibrium magnons.

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