

# Semiclassical processes in a high-frequency field

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The increase in the probability for a classically forbidden process under the action of a time-varying perturbation is investigated. As is well known, semiclassical tunneling through a potential barrier  $V(x)$  can be described in terms of complex classical trajectories satisfying Newton's equation, which perforce leads to the concept of motion in imaginary time. A homogeneous alternating field  $\mathcal{E} \cos \Omega t$  is transformed, when  $t$  is replaced by  $i\tau$ , into a field  $\mathcal{E} \cosh \Omega t$  that increases with  $\tau$ , the characteristic  $\tau$  values being determined by the period during which the motion takes place in the classically forbidden region. As a result, the effective field determining the change in the index of the tunneling exponential function is of the order of  $\mathcal{E} \exp(\Omega \tau_s)$ , where  $\tau_s$  is the imaginary part of the time it takes the particle to get to a singular point of the potential after emerging from under the barrier. A phenomenological picture of tunneling in an alternating field is described, and a general approach to the computation of the probabilities for semiclassical processes occurring under non-steady-state conditions is presented. The probabilities for penetration through a potential barrier, decay of a metastable state, and above-barrier reflection are found in the weak-field limit. The complete solution to the problem of tunneling through a triangular potential barrier (field emission) in an arbitrary alternating field is presented. The simplest band-structure model for a semiconductor is used to investigate the interband breakdown in an arbitrary field  $\mathcal{E}_0 + \mathcal{E} \cos \Omega t$  (the nonlinear Franz-Keldysh effect). The possibility of an experimental observation of the investigated phenomena is discussed.

## 1. INTRODUCTION

The processes of subbarrier transmission and above-barrier reflection, which are forbidden by classical mechanics, acquire finite probabilities when the quantum effects are taken into account. As a rule, these probabilities decrease exponentially as the barrier width-to-particle wavelength ratio increases. The computation of the probability for a classically forbidden process has a certain peculiarity from the mathematical standpoint: there necessarily arises here the concept of motion in imaginary time or along a complex trajectory.<sup>1</sup> This characteristic of semiclassical processes makes their computation quite difficult, and because of this no detailed investigation of, in particular, the effect of a variable perturbation on classically forbidden processes has thus far been published. The problems that have been solved either assume that the variable perturbation is weak, as obtains in the case of the Franz-Keldysh effect,<sup>2</sup> or are limited by the stipulation that the static potential is a short-range one, a condition which, fortunately, turns out to be adequate for the investigation of the practically important problem of many-photon ionization of an atom.<sup>3–5</sup> Thus, a large number of phenomena that arise when semiclassical processes occurring in realistic potential fields are exposed to the nonlinear action of a high-frequency field have as yet not been investigated.

In the the present paper we consider the effect of an alternating field on the quantum tunneling of a particle through a potential barrier, interband tunneling in a semiconductor, and above-barrier reflection. These effects are encountered in field emission, interband breakdown, charge

exchange between deep-lying impurity centers in semiconductors,<sup>6</sup> tunneling chemical reactions,<sup>7</sup> and the destruction of the adiabatic invariants in classical mechanics.<sup>8</sup> In all situations of this kind a variable perturbation substantially increases the probabilities for the forbidden processes.

Let us illustrate the subject of the present paper by the process of tunneling. If the amplitude of the alternating field is small, then the passage through the barrier will be determined largely by ordinary tunneling, and the alternating field can be taken into account within the framework of perturbation theory. This means that the probability for tunneling accompanied by absorption of one or several photons is small compared to the probability for tunneling in zero field. But if the alternating-field strength exceeds a certain value, then the tunneling will be insignificant, and the passage will occur as a result of the absorption by the particle of that number of photons which it needs in order to get to the top of the potential barrier.

These limiting cases are separated by a broad range of alternating field-strength values at which the passage through the barrier is a process of the mixed type, in the sense that it is advantageous for the particle to absorb a certain number of field quanta so as to tunnel in a higher energy region, where the barrier is more transparent. The optimum number of absorbed quanta is determined in this case by the competition between the growth of the tunneling probability and the decrease of the absorption probability as the number of quanta increases.

This type of problem is solved in Ref. 9 in the weak alternating field approximation. It is shown there that the

amplitude of the alternating field  $\mathcal{E} \cos \Omega t$  enters into the answer in the combination  $\mathcal{E} \exp(\Omega \tau_s)$ , where the  $\tau_s$  is connected with the particle motion in the forbidden region. Thus, in the high-frequency limit  $\Omega \tau_s \gg 1$  the barrier-penetration probability depends anomalously strongly on the amplitude and frequency of the field. The most promising in respect of an experimental observation of the indicated effects are Josephson junctions, for which we have theoretically investigated the voltage-state lifetime in a weak alternating field.<sup>10,11</sup>

In spite of the fact that the problem under discussion is a pressing one, in view of the latest experimental advances in the study of the decay of the voltage states of Josephson junctions,<sup>12-16</sup> the question of the effect of an alternating field on the tunneling processes has almost not been touched upon in the literature. In Refs. 17-19 the effect of an alternating field on the motion of a particle in the classically allowed region is considered. In Ref. 20 an attempt is made to take into account the effect of an alternating field on subbarrier motion, but the general arguments do not lead to specific results.

The purpose of the present paper is to investigate semiclassical processes in a high-frequency field, which is not necessarily considered to be weak. In the next section we qualitatively consider tunneling in an alternating field, and determine the typical orders of magnitude of the quantities involved in the process. In Sec. 3 we formulate the problem in a relatively general situation, and indicate a procedure for solving it with the use of the method of complex trajectories. Since Newton's equation in variable and spatially inhomogeneous fields can be solved only in specific cases, the remaining part of the paper is devoted to the analysis of a number of specific problems. In Sec. 4 we investigate the effect of a weak alternating field on tunneling, i.e., the case in which it is sufficient to take account of the linear—in the field—correction to the argument of the tunneling exponential function. This in no way implies that the effect in question is weak. On the contrary, the condition of applicability of the semiclassical approximation requires that the indicated correction be much greater than unity, so that the transmission coefficient should increase by several orders of magnitude. To elucidate more fully the physical picture of the phenomenon in question, we consider the effect of spatially inhomogeneous perturbations in the same linear approximation. In Sec. 5 we study the decay of metastable states in an alternating field, and show that the oscillating dependence of the argument of the tunneling exponential function on the frequency is connected not only with the normal classical, but also with a specific quantum, resonance whose frequency is determined by the particle motion in the forbidden region. In Sec. 6 we briefly investigate the effect of an alternating field on above-barrier reflection. Allowance in the argument of the exponential function for the terms nonlinear in the field is possible only for relatively simple potentials; therefore, here we shall not generalize the results obtained in Refs. 10 and 11 for a sinusoidal potential to the nonlinear case. Below we present the solution to the nonlinear problem in two situations. In Sec. 7 we obtain and investigate in detail

the exact solution to the problem of tunneling through a triangular barrier in an alternating field. These results have a direct bearing on the phenomenon of field emission. With the aid of the simplest band-structure model for a semiconductor, we carry out in Sec. 8 a detailed investigation of the interband breakdown in constant and alternating electric fields (the nonlinear Franz-Keldysh effect). In the Conclusion we discuss the results obtained in the paper and the possibility of their experimental observation.

## 2. QUALITATIVE ANALYSIS

Let us consider the problem of subbarrier tunneling in a uniform alternating field from a phenomenological point of view in order to elucidate the physics of the matter without laying any claims to quantitative results. We shall assume that a particle of energy  $E$  is incident from the left on a potential barrier (of height  $V$ ) of the type shown in Fig. 1. In the absence of an alternating field the probability for penetration through the barrier is, with exponential accuracy, equal to  $\exp[-A_0(E)]$ , where  $A_0$  is the imaginary part of the corresponding action.<sup>1</sup> In the general case, for a barrier of width of the order of  $a$ , and for  $E \sim V$ , the quantity  $A_0 \sim V/\omega$ , where  $\omega = (V/ma^2)^{1/2}$  is the characteristic oscillation frequency in the inverted potential. On the basis of the semiclassical approximation condition  $A_0 \gg 1$ .

When the alternating field is taken into account within the framework of perturbation theory, the transmission probability increases by a term proportional to the square of the field:

$$D \approx \exp[-A_0(E)] + (\mathcal{E}/\tilde{\mathcal{E}})^2 \exp[-A_0(E + \Omega)], \quad (1)$$

where  $\tilde{\mathcal{E}}$  is some internal field, the magnitude of which will be discussed later. The second term in (1) corresponds to a single-photon absorption and the subsequent tunneling with the increased energy  $E + \Omega$ . We shall assume everywhere below that the alternating-field frequency is small compared to the height of the potential barrier and the initial energy of the particles:  $\Omega \ll V, E$ . Taking into account the relation

$$\partial A_0 / \partial E = -2\tau_0,$$

where  $i\tau_0$  is the imaginary time of the motion under the barrier between the turning points, we obtain in place of (1) the expression

$$D(\mathcal{E}) \approx D(0) \{1 + [\mathcal{E} \exp(\Omega \tau_0) / \tilde{\mathcal{E}}]^2\}. \quad (2)$$

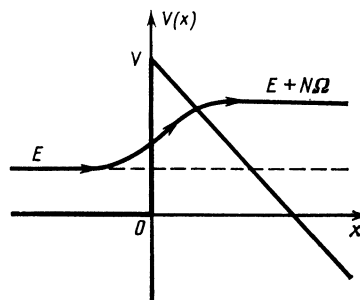


FIG. 1. Subbarrier transmission of a particle with absorption of  $N$  quanta of an alternating field.

It can be seen from this that perturbation theory ceases to be applicable even in very weak fields, specifically, in fields  $\sim \mathcal{E} \exp(-\Omega\tau_0)$ , when we are considering the high-frequency limit  $\Omega\tau_0 \gg 1$ . Let us note that  $\tau_0 \sim \omega^{-1}$ ; therefore, for semiclassical potentials (i.e., for  $V \gg \omega$ ), the semiclassicality condition  $V \gg \Omega$  can be fulfilled simultaneously with the condition  $\Omega\tau_0 \gg 1$ .

As the amplitude of the alternating field increases, the many-photon absorption processes, i.e., the next terms of the expansion in powers of  $(\mathcal{E}/\tilde{\mathcal{E}})^2$  in the expression (1), become more and more important. The term corresponding to the absorption of  $N$  photons has the form

$$D_N \approx (e\tilde{\mathcal{E}}/\tilde{\mathcal{E}}N)^{2N} \exp[-A_0(E+N\Omega)], \quad (3)$$

where  $n\Omega$  is the energy received by the particle from the field. The change in the perturbation-theory parameter is due to the fact that the  $N$ th term of the perturbation-theory series contains in the denominator the factor  $(N!)^4$  (Ref. 5), which has been included in (3) after being approximated by the Stirling formula, which is applicable when  $N \gg 1$ . We find the optimum number of quanta by maximizing (3) with respect to  $N$ , as a result of which we obtain

$$D \approx \exp[-A_0(E) + 2\tilde{\mathcal{E}} \exp(\Omega\tau_0)/\tilde{\mathcal{E}}], \quad (4)$$

$$N_m \approx \tilde{\mathcal{E}} \exp(\Omega\tau_0)/\mathcal{E}.$$

This expression is applicable when the correction to the argument of the exponential function is much greater than unity, a condition which coincides with the condition for the process to be a many-photon one. The effective field  $\tilde{\mathcal{E}} \exp(\Omega\tau_0)$  should therefore be stronger than the characteristic internal field  $\mathcal{E}$ . Moreover,  $N_m \Omega$  should be at least significantly smaller than the barrier height  $V - E$ .

When the field  $\mathcal{E}$  is increased further, and the energy transfer  $N_m \Omega$  becomes comparable in order of magnitude to the barrier height  $V - E$ , the passage occurs without the participation of tunneling. Retaining only the preexponential factor in the expression (3) with  $N\Omega$  replaced by  $V - E$ , we obtain in this limit the expression

$$D \approx \exp \left[ -\frac{2(V-E)}{\Omega} \ln \frac{\tilde{\mathcal{E}}(V-E)}{\mathcal{E}\Omega} \right]. \quad (5)$$

The exact limits of applicability of the expressions (2) and (5) will be indicated in Sec. 7, after the problem has been rigorously solved.

The expressions (2) and (5) have been obtained on the basis of simple physical arguments. The parameter  $\tilde{\mathcal{E}}$  entering into them can easily be estimated by computing with the aid of perturbation theory the linear—in the field amplitude—correction to the wave function of the transmitted particle. In the case of, for example, the triangular barrier shown in Fig. 1, this calculation yields

$$\tilde{\mathcal{E}} \sim \Omega^2 [m/(V-E)]^{1/2}.$$

Naturally, for potentials of general form, the foregoing results are only of a qualitative nature, for the same reason that the total probability cannot in the general case be

represented in the form of a product of separate probabilities for absorption and tunneling. Nevertheless, for potential barriers having artificial singularities, for example, kinks, the qualitative results obtained here are, as shown below, exact in the quantitative sense as well. Only the regions of applicability and the numerical coefficients of  $\tilde{\mathcal{E}}$  in the various limiting case are determined more accurately. Thus, the correctness of the expression (4) for triangular and rectangular barriers can easily be verified.

### 3. SEMICLASSICAL DESCRIPTION OF TUNNELING IN AN ALTERNATING FIELD

Let us proceed to the derivation of the general expression for the probability for tunneling of a particle through an arbitrary time-dependent semiclassical potential barrier  $V(x,t)$ . As is well known, in the semiclassical limit the wave functions can be sought with exponential accuracy in the form

$$\psi(x,t) = \exp[iS(x,t)],$$

where  $S(x,t)$  is the classical action, and  $x$  and  $t$  lie on the particle's classical trajectory, which can be found from Newton's equation

$$m d^2x/dt^2 + \partial V(x,t)/\partial x = 0. \quad (6)$$

Let the particle be incident on the barrier from the left. The problem is to find a relation between the values of the wave function at points  $x_1$  and  $x_2$  lying on opposite sides of the barrier. The forbiddenness of the tunneling process in classical mechanics implies, however, that there does not exist an ordinary trajectory connecting such points. For this reason, we shall consider the trajectories in complex time along the contour  $C_+$  in Fig. 2. On the symmetrically located contour  $C_-$  we have  $x(t^*) = x^*(t)$ . To the right on  $C_{\pm}$  the quantities  $x$  and  $t$  are real,  $x$  lies to the right of the barrier, and the solution to Eq. (6) depends on two arbitrary real parameters. We shall assume that the particle emerges from under the barrier at the moment of time  $t_2$  at the point  $x_2$ . This means that  $x(t_2) = x_2$  and  $(dx/dt)_{t_2} = 0$ . Then as the two real parameters, we can take the point  $x_2$  and instant  $t_2$  at which the particle emerges from under the barrier.

Let us assume that the nonstationary part of the poten-

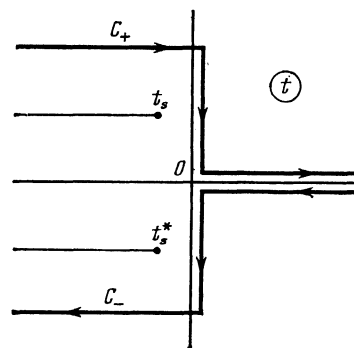


FIG. 2. The integration contour, singular points of the trajectory  $x(t)$ , and branch cuts for the computation of the subbarrier transmission coefficient.

tial is at least adiabatically switched off at  $t \rightarrow -\infty$ , i.e., that  $V(x,t) \rightarrow V(x)$ . Then at points far to the left on the contour  $C_+$  the function  $x(t)$  is a solution to the steady-state equation (6), and depends on two parameters, which we can take to be the time shift  $t_1$  and the conserved quantity  $E = m(dx/dt)^2/2 + V(x)$ , i.e.,  $x(t) = x(t - t_1, E)$ . It is significant that the quantities  $E$  and  $t_1$  are, generally speaking, complex, and depend on the point  $x_2$  and instant  $t_2$  at which the particle emerges from under the barrier.

The physically justified formulation of the problem consists in our prescribing the initial particle energy  $E$  and measuring the particle flux emerging from under the barrier at the instant  $t_2$ . In the general case there are no grounds for assuming that the trajectory  $x(t)$  defined by the real parameters  $E$  and  $t_2$  (the latter enters into the problem through the condition  $(dx/dt)_{t_2} = 0$ ) will itself be real. A more detailed analysis shows, however, that the time-averaged probability for semiclassical processes is determined solely by the real trajectories. The point is that the imaginary part of the action  $A(x,t) = 2 \operatorname{Im} S(x,t)$  obtained from the Hamilton-Jacobi equation is, on the basis of the semiclassicality condition, large, and its variations should also be much greater than unity. Accordingly, the particles should pass through the barrier at instants lying in a narrow neighborhood of that instant at which the function  $A(t)$  has its minimum value. With allowance for the condition  $(dx/dt)_{t_2} = 0$ , the condition for  $A(t_2)$  to be the minimum value of  $A(t)$  has the form

$$\partial A / \partial t_2 = -2 \operatorname{Im} V(x_2) = 0,$$

from which the reality of  $x_2$  follows. Below we limit ourselves to the computation of the minimum values of the function  $A(t)$ , and we shall therefore consider the trajectory  $x(t)$  to be real. The integration contour  $C_+$  in this case consists of a vertical section and two horizontal sections, as shown in Fig. 2.

The value of the wave function on the remote left section of the contour  $C_+$  differs from the value of the function on the real time axis by the quantity  $E\tau_0$  in the index of the exponential function. With allowance for this contribution, the effective Lagrangian has the form

$$L = \frac{1}{2} m (dx/dt)^2 - V(x, t) + E, \quad (7)$$

using which, we obtain for the transmission coefficient the expression

$$D = e^{-A}, \quad A = -i \int_C L dt, \quad (8)$$

where the contour  $C \equiv C_+ + C_-$ . For specific calculations it may turn out to be convenient to shift the integration path, allowing for its linkage behind the singular points of the trajectory.

Thus far we have considered the problem of tunneling of a particle with energy  $E$  prescribed at  $t \rightarrow -\infty$ , when the potential is stationary. If the tunneling proceeds from a state of thermodynamic equilibrium, then the result should be averaged over the energy  $E$  with the Gibbs distributions:

$$\langle D \rangle \sim \int \exp[-E/T - A(E)] dE,$$

where by  $A(E)$  we mean the action computed earlier with the formulas (7) and (8). Here it is important that the energy derivative of the action  $A(E)$  be equal to  $-2\tau_0$ , as was the case in the absence of the variable perturbation. The point is that, because the action is an extremal quantity, only the term with  $E$  in the expression (8) makes a contribution to  $\partial A / \partial E$ , and  $2\tau_0$  is the distance between the remote ends to the contours  $C_{\pm}$ . Of course the quantity  $\tau_0$  may itself depend on the amplitude of the variable field. Thus, in thermodynamic equilibrium the barrier penetration factor is found by substituting into (8) the real trajectory satisfying the condition

$$\tau_0 = 1/2T,$$

which implicitly selects the energy of the particles tunneling through the barrier.

The scope of the general expressions obtained in the present section is revealed below in a number of specific examples.

#### 4. TUNNELING THROUGH ANALYTIC POTENTIALS IN A WEAK HOMOGENEOUS FIELD

Let us consider tunneling through potential barriers specified by analytic functions, taking account of only that correction to the action  $A_0$  which is linear in the monochromatic field. We shall consider the field to be homogeneous, so that

$$L = \frac{1}{2} m (dx/dt)^2 - V(x) + \mathcal{E} x \cos \Omega t + E.$$

We shall also assume that  $\Omega \ll V$ , a condition which allows us to use the semiclassical expressions obtained in the preceding section. As is well known, allowance for a small perturbation in the Lagrangian in the action calculation amounts to the substitution of the perturbation into the Lagrangian of the unperturbed trajectory. The action  $A$  can therefore be represented in the form

$$A = A_0 + A_1,$$

where

$$A_0 = 2(2m)^{1/4} \int_{x_1}^{x_2} [V(x) - E]^{1/4} dx, \quad (9)$$

$$A_1 = -i\mathcal{E} \int_C x(t) \cos(\Omega t + \chi) dt.$$

Here  $x_1$  and  $x_2$  are the turning points,  $V(x_{1,2}) = E$ , and  $x(t)$  is the zero-field classical trajectory, fixed, for example, by the requirement that the particle emerge from under the barrier at the instant  $t = 0$ , i.e., by the condition  $x(0) = x_2$ . The quantity  $\chi$  then denotes the relative phase of the field.

The action  $A_1$ , as a function of the two variables  $x$  and  $t$ , can be found by computing the linear—in  $\mathcal{E}$ —correction  $S_1(x,t)$  to the solution to the Hamilton-Jacobi equation:

$$\frac{\partial S}{\partial t} + \frac{1}{2m} \left( \frac{\partial S}{\partial x} \right)^2 + V(x) = \mathcal{E} x \cos \Omega t.$$

We then obtain for  $A_1 = 2 \operatorname{Im} S_1$  in the region to the right of the barrier the expression

$$A_1(x, t) = -i\mathcal{E} \int_c dt' x(t') \cos \left\{ \Omega(t+t') + \chi - \Omega \int_{x_1}^x \{2[E - V(y)]/m\}^{-1/2} dy \right\}, \quad (10)$$

which coincides with the expression (9) if  $t = 0$  and  $x = x_2$ .

The integration path in the expressions (9) and (10) is the rectangular contour in Fig. 2, where the horizontal straight line on the left is at a distance of  $i\tau_0$  from the real axis and corresponds to motion to the left of  $x_1$ , the vertical section from  $i\tau_0$  to 0 corresponds to subbarrier motion, and the positive half of the  $t$  axis corresponds to motion to the right of  $x_2$ . On such a contour  $x(t)$  is real.

To evaluate the integral (9), it is convenient to shift the contour  $C$  far to the left (where the field is adiabatically switched off), allowing for its linkage behind the singular points of the trajectory  $x(t)$ . The importance of studying the singularities of the unperturbed trajectory is apparent from this. As can easily be seen from the following implicit dependence:

$$t(x) = \int_{x_1}^x \{2[E - V(y)]/m\}^{-1/2} dy,$$

the singularities of the functions  $x(t)$  are connected with those of the function  $V(x)$  in the complex  $x$  plane.

Let us consider those barriers for which  $V(x)$  possesses power-law singularities at some points  $x_s$  and  $x_s^*$ , becoming infinite there:

$$V(x) \approx \kappa(x - x_s)^\alpha, \quad x \rightarrow x_s,$$

where  $\alpha < 0$ . Included here are singularities of the type  $V \approx \kappa x^\alpha$  for  $x \rightarrow \infty$  and  $\alpha > 0$ . In the vicinity of  $x_s$ , the solution to Eq. (6) has the form

$$x(t) = x_s + [-\kappa(2 - \alpha)^2(t - t_s)^2/2m]^{1/(2 - \alpha)},$$

where  $t_s$  is the complex time required for the motion from  $x_2$  to  $x_s$ :

$$t_s = \int_{x_2}^{x_s} \{2[E - V(y)]/m\}^{-1/2} dy. \quad (11)$$

The corresponding singular points and branch cuts are shown in Fig. 2. In order of magnitude,  $\tau_s \equiv \text{Im } t_s$  is equal to the time  $\tau_0$  of subbarrier motion. In the limit of high alternating field frequency, i.e., for  $\Omega\tau_s \gg 1$ , the dominant contribution to the integral (9) is made by the branch-cut sections close to the singular points  $t_s$  and  $t_s^*$ . For the transmission coefficient we finally obtain

$$D(\mathcal{E}, t) = D(0) \exp [a_1 \cos(\Omega t + \chi_1)], \quad (12)$$

where  $\chi_1$  is the phase shift, which is unimportant for what follows, and

$$a_1 = \frac{2\pi\mathcal{E}}{\Omega} \left| \Gamma\left(\frac{2}{\alpha-2}\right) \right|^{-1} \left[ \frac{|\kappa|(2-\alpha)^2}{2m\Omega^2} \right]^{1/(2-\alpha)} \exp(\Omega\tau_s). \quad (13)$$

This exact result is similar in structure to the expression (4), but, instead of  $\tau_0$ , it contains the time  $\tau_s$ , which, for analytic potentials, is always smaller than  $\tau_0$ . The maximum value of the field on a real trajectory is, of course, of the order of  $\mathcal{E} \exp(\Omega\tau_0)$ , but because of the field oscillations, its contribution to the action is greatly reduced, so that, for example, for even potentials  $\mathcal{E}_{\text{eff}} \sim \mathcal{E} \exp(\Omega\tau_0/2)$ .

Averaging (12) over time with allowance for the inequality  $a_1 \gg 1$ , we obtain

$$\overline{D(\mathcal{E})} = D(0) (2\pi a_1)^{-1/2} \exp a_1. \quad (14)$$

The use of perturbation theory makes it possible for us to compute the pre-exponential factor.

The results (12) and (14) are applicable so long as

$$A_0 \gg a_1 \gg 1,$$

where the limitation from above justifies the linear expansion in  $\mathcal{E}$ , while the limitation from below is due to the use of the semiclassical approximation. The criterion limiting the field amplitude from above becomes much more rigid in a more accurate calculation, as can be seen from the results pertaining to tunneling through a triangular barrier (Sec. 7) and interband tunneling (Sec. 8).

Let us illustrate the results obtained with potential barriers of specific form. If  $V(x) = V \cosh^{-2}(x/a)$ , then the trajectory with energy  $E$  is given by the relations

$$\text{sh}(x/a) = [(V-E)/E]^{1/2} \text{ch } \omega t, \quad \omega^2 = 2E/ma^2, \\ dx/dt = a\omega \text{ sh}(\omega t) [E/(V-E) + \text{ch}^2 \omega t]^{-1/2}.$$

The appearance of singularities of the solution in the complex time plane can be directly seen from these relations. Substituting into the general expression (13) the values

$$x_s = i\pi a/2, \quad \kappa = -Va^2, \quad \alpha = -2, \quad \tau_s = \pi/2\omega,$$

we obtain for the transmission coefficient the expression

$$D(\mathcal{E}) = D(0) \exp \left[ \frac{a\mathcal{E}}{\Omega} \left( \frac{2\pi\omega}{\Omega} \right)^{1/2} \left( \frac{V}{E} \right)^{1/2} \exp(\Omega\tau_s) \right].$$

For the potential  $V(x) = V(1 + x^2/a^2)^{-1}$  we have  $x_s = ia$ ,  $\kappa = iVa/2$ , and  $\alpha = -1$ , and from (13) and (14) we obtain

$$D(\mathcal{E}) = D(0) \exp \left[ \frac{4\pi a\mathcal{E}}{\Omega\Gamma(1/3)} \left( \frac{V}{6ma^2\Omega^2} \right)^{1/2} \exp(\Omega\tau_s) \right],$$

where  $\tau_s$  can be expressed in terms of an elliptic integral.

The considered potentials above are even; therefore,  $\tau_s$  is exactly equal to half the subbarrier-motion time  $\tau_0$ . We shall not carry out the analysis for other types of potential, noting only that among the potentials with power-law singularities the  $\alpha = 2$  case, in which  $\tau_s \rightarrow \infty$  and the expression (13) is inapplicable, is unique.

Another characteristic example is the case of those potentials in which the particle slides to the right from  $x_2$  along the real axis, and reaches the point  $x = \infty$  in finite real time  $t_s$ . In this case  $\text{Im } t_s = 0$ , and the exponential field intensification effect does not occur. This circumstance was noted in the investigation of the tunneling decay of the voltage states in a Josephson junction carrying a near-critical direct cur-

rent.<sup>10,11</sup> The depth of the potential well in this case is small compared to the overall scale of the potential, and the shape of the potential in the vicinity of the barrier is well described in the cubic approximation, which is quite adequate for the computation of  $A_0$ , but which, as has already been stated, leads to the result that  $\tau_s = 0$ . To find the finite  $\tau_s$  value, we must remember that in fact the potential energy of a Josephson junction is described by an oblique sinusoid, along which the particle slides for an infinitely long time. By turning the integration contour in the expression (11) in the direction of imaginary  $x$  values, we can make it possible for the particle to reach the point where  $V(x) = \infty$  in finite, but now complex, time, so that  $\tau_s = 1.177 \omega_p^{-1}$ , where  $\omega_p$  is the Josephson plasma frequency, i.e., the frequency of small oscillations of the particle in one of the troughs of the sinusoidal potential when it is not sloping. Let us emphasize that  $\tau_s$  tends to a finite value as the direct current approaches the critical value, whereas the subbarrier motion time  $\tau_0$  tends in this case to infinity. This fact sharply distinguishes tunneling in Josephson junctions from the examples considered in the present paper, and demonstrates the importance of the analytic structure of the potential as a whole, i.e. in the regions far from the barrier region.

The exponential field enhancement has been noted by V'yurkov and Ryzhii<sup>21</sup> in the case of electron tunneling between  $\delta$ -function wells, when  $\tau_s$  is equal to the time of flight across the forbidden region, and by Sumetskiĭ<sup>22</sup> in the case of tunneling through a triangular potential barrier.

Let us compute the energy distribution for the particles that have tunneled through. The time dependence of the wave function can be explicitly found from the solution to the Hamilton-Jacobi equation. In the  $\Omega\tau_s \gg 1$  limit we have

$$\psi(t) \propto \exp\{-iEt - 1/2 a_1 \exp(-i\Omega t)\},$$

since there remains out of  $\cos \Omega t$  only the term that increases in the region  $\text{Im } t > 0$ . Leaving out the structure of the energy spectrum at scales  $\sim \Omega$ , we find that the spectral envelope

$$P(\omega) \propto \left| \int \psi(t) \exp(i\omega t) dt \right|^2 \\ \propto \exp[-2(\omega - E - a_1 \Omega/2)^2 / a_1 \Omega^2].$$

Thus, the particles that have passed through the barrier gain on the average an energy of  $a_1 \Omega/2$ , and have a Gaussian energy distribution in an interval  $\sim a_1^{1/2} \Omega$ .

Let us now consider the situation in which the intensity of the alternating field is nonuniform in space:

$$V_1(x, t) = V_1(x) \cos \Omega t.$$

Then instead of (9), for the correction to the action we write

$$A_1 = i \int_c V_1(x(t)) \cos \Omega t dt, \quad (15)$$

where the singularities of both the unperturbed trajectory  $x(t)$  and the potential  $V_1(x)$  as a function of  $x$  must be taken into account. Without intending here an investigation of the general case, we consider the class of potentials  $V_1(x)$  that possess no singularities. Then in the expression (13), we

should simply substitute for  $\mathcal{E}$  the quantity  $|V_1'(x)|$ , i.e., the amplitude of the additional field at the singular point of the original potential. In the particular case when

$$V_1'(x) = \mathcal{E} \exp[-(x-x_0)^2/\gamma^2], \quad V(x) = V \text{ch}^{-2}(x/a),$$

we obtain

$$D(\mathcal{E}) = D(0) \exp \left[ \frac{a\mathcal{E}}{\Omega} \left( \frac{2\pi\omega}{\Omega} \right)^{1/2} \left( \frac{V}{E} \right)^{1/4} \right. \\ \left. \times \exp \left( \Omega\tau_s + \frac{\pi^2 a^2 - 4x_0^2}{4\gamma^2} \right) \right].$$

This example shows that the anomalous increase in the contribution of the perturbation to the tunneling exponential function can be the result of not only the rapid variations in time, but also the pronounced spatial inhomogeneity, of the perturbation. The effect of the perturbation weakens as the intensity of the alternating field in the barrier region is decreased.

Also of interest is the case when the potential is nonstationary as a result of jittering as a whole:

$$V(x, t) = V(x) (1 + \beta \cos \Omega t), \quad \beta \ll 1.$$

Then in the formula (15) we have  $V_1(x) = \beta V(x)$ , and it is convenient to rewrite it in the form

$$A_1 = -\frac{im\beta}{2} \int_c \left( \frac{dx}{dt} \right)^2 \cos \Omega t dt.$$

In place of (13) we obtain

$$a_1 = \pi\beta m \Omega \left( \frac{2}{\alpha - 2} \right)^2 \left[ \frac{|\kappa| (2 - \alpha)^2}{2m\Omega^2} \right]^{2/(2-\alpha)} \frac{\exp(\Omega\tau_s)}{|\Gamma(2\alpha/(\alpha-2))|},$$

from which we can find the answers for potentials of specific forms.

Notice that the tunneling problem in a high-frequency field ( $\Omega \gg \omega$ ) cannot be solved with the aid of Kapitza's pendulum method,<sup>8</sup> since in complex time the field is not an oscillating one, and the amplitude of the alternating field in the region of interest to us is much smaller than the intensity of the constant field.

## 5. TUNNELING DECAY OF A METASTABLE STATE IN A WEAK ALTERNATING FIELD

In the preceding section we investigated the effect of an alternating field on the transparency of an isolated potential barrier. If the initial state corresponds to a particle located in a potential well, the particle motion in zero field is finite and periodic:  $x(t + t_0) = x(t)$ , where  $t_0$  is the period of the particle vibrations in the well. Taking this into account, we can transform the contour  $C \equiv C_+ + C_-$  in Fig. 2 into a series of closed contours differing from each other by a  $t_0$  shift along the real  $t$  axis. The function  $x(t)$  has the same form on all the contours; therefore, the summation over the contours corresponds to our going over from (9) to the following expression:

$$A_1 = \frac{i\mathcal{E}}{2 \sin(\Omega t_0/2)} \oint dt x(t) \sin \Omega t,$$

where the integration is along the contour shown in Fig. 3.

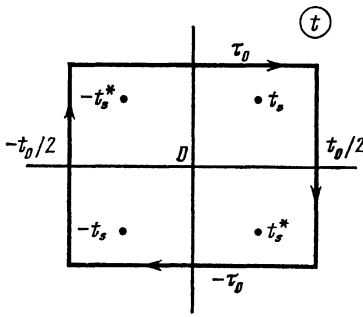


FIG. 3. The integration contour and singular points of the trajectory  $x(t)$  in the case of the computation of the probability for decay of a metastable state.

The arbitrary phase corresponding to the choice of the initial moment of time is included in the function  $x(t)$ , and should be chosen so as to minimize  $A_1$ .

We shall assume, as before, that  $V(x)$  is singular at the points  $x_s$  and  $x_s^*$ . In the  $t$  plane these points correspond not to two points  $t_s$  and  $t_s^*$ , as before, but to four points:  $\pm t_s$  and  $\pm t_s^*$ . The latter circumstance is due to the fact that, in contrast to the potential barrier problem, in which the sign of the root in the expression (11) is specified by the direction of the particle motion, roots of either sign are suitable for the problem with periodic motion. The summation of the residues at the indicated points leads to an expression differing from (13) by the factor

$$F(\Omega) = \left| \frac{\sin \Omega t_1}{\sin(\Omega t_0/2)} \right|,$$

where  $t_1 = \text{Re } t_s$ .

The result shows that the field-frequency dependence of the decay probability is of an oscillatory nature, and is determined by two periods. One of them is the period of the classical vibrations of the particle in the well at a given energy; the second one is determined by the quantum effects, and does not admit of a simple interpretation. The function  $F(\Omega)$  is infinite when the frequency  $\Omega$  is comparable to the frequency  $2\pi/t_0$  of the internal motion. The final answer in this case can be obtained by taking account of the nonlinear effects or the damping of the oscillations. The complete solution to the problem of the decay of a metastable state in an oblique sinusoidal potential and in a weak alternating field is given in Ref. 11.

## 6. ABOVE-BARRIER REFLECTION IN A WEAK ALTERNATING FIELD

In the static situation the coefficient for above-barrier reflection is given by the expression<sup>1</sup>

$$R(0) = \exp(-A_0), \quad A_0 = 4(2m)^{1/2} \text{Im} \int_{x_1}^{x_0} [E - V(x)]^{1/2} dx,$$

where  $x_1$  is the real and  $x_0$  the complex root of the equation  $V(x) = E$  determining the position of the turning point. The correction due to the alternating field is given, as before, by the formula (9), but the integration contour will now be different.

The reflection coefficient is given by the ratio of the amplitudes of the incident and reflected waves. Let the particle be incident on the barrier from the left. Then the classical trajectory  $x(t)$  describing the reflection of the particle from the barrier is specified in the complex  $t$  plane by the contour  $C_+$  in Fig. 4. The section 1 corresponds to the incident particle; the section 3, to the reflected particle; and the section 4, to the transmitted particle. At the remote ends of the sections 1 and 3 we have  $x < 0$ . The section 2 contains the turning point, where  $dx/dt = 0$  and the coordinate  $x$  is complex, in accordance with the fact that the turning point does not exist in classical mechanics. With allowance for the foregoing

$$R(\mathcal{E}) = R(0) \exp \left[ -i\mathcal{E} \int_{c_+ + c_-} x(t) \cos \Omega t dt \right].$$

As before, in the high-frequency limit the dominant contribution to the integral is made by the potential's singular points, the behavior of  $x(t)$  in the vicinity of which has already been investigated. It is clear that, for potentials with power-law singularities, the correction to the action will, as before, be given by the expression (13), with the only difference that we should now use for the imaginary part of the time of motion to the singular point  $x_s$  the expression

$$\tau_s = \text{Im} \int_{x_1}^{x_s} \{2[E - V(x)]/m\}^{-1/2} dx,$$

where  $x_1$  is an arbitrary real coordinate.

For the particular case of a potential of the form  $V(x) = V \cosh^{-2}(x/a)$ , we have

$$\begin{aligned} \text{sh}(x/a) &= [(E - V)/E]^{1/2} \text{sh } \omega t, \quad \omega^2 = 2E/ma^2, \\ dx/dt &= (2E/m)^{1/2} \text{ch}(\omega t) [\text{sh}^2 \omega t + E/(E - V)]^{-1/2}. \end{aligned}$$

As can be seen, the turning point  $t_R = i\pi/2\omega$  is located in the middle of the section 2 in Fig. 4, where  $x$  is pure imaginary. On sections 1 and 3 the quantity  $x$  is real. The correction to the argument of the exponential function in the limit  $\Omega\tau_s \gg 1$  has the same form as for the transmission coefficient in the case when  $E < V$ :

$$R(\mathcal{E}) = R(0) \exp \left[ \frac{a\mathcal{E}}{\Omega} \left( \frac{2\pi\omega}{\Omega} \right)^{1/2} \left( \frac{V}{E} \right)^{1/4} \exp(\Omega\tau_s) \right].$$

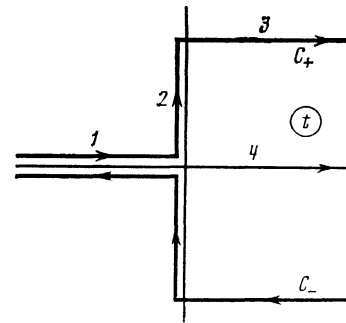


FIG. 4. Integration contour for the computation of the above-barrier reflection coefficient.

Let us note that the equality of  $\tau_s$  and  $\text{Im } t_R$  is due to the special choice of potential, and does not hold in the general case.

From the mathematical standpoint the above-barrier reflection effect is similar to the phenomenon of nonconservation of adiabatic invariants.<sup>8</sup> The results of the present section show that a weak variable perturbation should anomalously greatly change the adiabatic invariant if the frequency of the perturbation exceeds the characteristic reciprocal time of the variation of the system's parameters.

## 7. EXACT SOLUTION FOR A TRIANGULAR BARRIER: FIELD EMISSION IN AN ALTERNATING FIELD

In the preceding sections we computed the corrections to the argument of the tunneling exponential function that are linear in the alternating field. The exact solution in the nonlinear case is possible only for potentials of a specific form. Below we consider in detail the practically important case of the triangular barrier, a case which has a direct bearing on field emission.<sup>23</sup> We shall assume that, as shown in Fig. 1, a particle of energy  $E$  strikes a barrier of height  $V$  from the left, and that a constant,  $\mathcal{E}_0$ , and an alternating,  $\mathcal{E} \cos \Omega t$ , field exist beyond the barrier. The assumption that there is no electric field in the region to the left of the barrier not only simplifies the problem, but also corresponds to the realizable conditions in which the field is screened off in the vicinity of the metal surface. These assumptions reduce the problem to the problem of computing the imaginary part of the action

$$S = \int_{t_1}^{t_2} \left[ \frac{m}{2} \left( \frac{dx}{dt} \right)^2 - V + \mathcal{E}_0 x + \mathcal{E} x \cos \Omega t + E \right] dt, \quad (16)$$

where  $t_1$  and  $t_2$  are respectively the instants at which the particle goes, and emerges from, under the barrier. The trajectory  $x(t)$  can be found from the equation

$$m d^2 x / dt^2 = \mathcal{E}_0 + \mathcal{E} \cos \Omega t \quad (17)$$

with the boundary conditions

$$x(t_1) = 0, \quad (dx/dt)_{t_1} = i[2(V-E)/m]^{1/2}. \quad (18)$$

As stated in Sec. 3, to compute the minimum value of the imaginary part of the action, it is sufficient to limit ourselves to real trajectories. It is easy to verify that the real solution to Eq. (17) corresponds to the situation in which the particle emerges from under the barrier at the instant when the alternating field has its maximum intensity. For definiteness, we shall therefore assume that  $t_2 = 0$ . The solution of (17) with the boundary condition  $(dx/dt)_{t=0} = 0$  allows us to find  $t_1$  from the condition (18), the quantity  $t_1 = i\tau$  turning out to be pure imaginary. The real trajectory satisfying the indicated conditions has the form

$$x(t) = \mathcal{E}_0 (t^2 + \tau^2) / 2m + \mathcal{E} (\text{ch } \Omega \tau - \cos \Omega t) / m \Omega^2, \quad (19)$$

where  $t$  varies along the imaginary axis and the quantity  $\tau$  can be found from the equation

$$\mathcal{E}_0 \tau + (\mathcal{E} / \Omega) \text{sh } \Omega \tau = \kappa,$$

where  $\kappa = [2m(V-E)]^{1/2}$  is the initial momentum for the subbarrier motion.

Substituting the expression (19) into (16), and going over to imaginary time, we obtain for the quantity  $A = 2 \text{Im } S$  after a number of transformations the expression

$$A = A_0 v_s^{-3} [3v_s^2 v / 2 - v^3 / 2 - (3\mathcal{E} / \mathcal{E}_0) (v \text{ch } v - \text{sh } v) - (3\mathcal{E}^2 / 8\mathcal{E}_0^2) (\text{sh } 2v - 2v)], \quad (20)$$

$$v + (\mathcal{E} / \mathcal{E}_0) \text{sh } v = v_s, \quad (21)$$

where we have introduced the notation

$$v = \Omega \tau, \quad v_s = \Omega \tau_s = \Omega \kappa / \mathcal{E}_0,$$

and the action  $A_0$  takes in the absence of an alternating field the form

$$A_0 = 2\kappa^3 / 3m\mathcal{E}_0. \quad (22)$$

The dependence, as given by the relations (20) and (21), of  $A/A_0$  on the parameters  $\mathcal{E}/\mathcal{E}_0$  and  $\Omega\tau_s$  is depicted in Fig. 5. Simpler analytic expressions can be obtained for several limiting cases.

Let us consider the case of high frequencies, i.e., the case in which  $v, v_s \gg 1$ . The equation for  $v$  assumes the form

$$v + (\mathcal{E} / 2\mathcal{E}_0) e^v = v_s.$$

Ignoring the terms small in the parameters  $v_s^{-1}$  and  $\exp(-v)$ , we obtain for  $A$  the expression

$$A = A_0 v_s^{-3} [3v_s^2 v / 2 - v^3 / 2 - (3\mathcal{E} / 2\mathcal{E}_0) v e^v], \quad (23)$$

where the last term in the brackets is always small compared to the other two, but must be retained, since it is precisely the term that gives the linear—in  $\mathcal{E}$ —correction in the case of small  $\mathcal{E}$ . In the latter case we have

$$A = A_0 \left[ 1 - \frac{1}{(\Omega\tau_s)^2} \frac{3\mathcal{E}}{2\mathcal{E}_0} \exp(\Omega\tau_s) \right],$$

$$\frac{(\Omega\tau_s)^2}{A_0} \ll \frac{\mathcal{E} \exp(\Omega\tau_s)}{\mathcal{E}_0} \ll 1. \quad (24)$$

The limitation on  $\mathcal{E}$  from below is due to the fact that, according to the condition of applicability of the semiclassical approximation, the absolute value of the correction to the

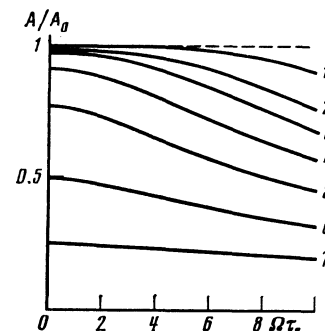


FIG. 5. Dependence of  $A/A_0$  on  $\Omega\tau_s$  for the following values of  $\mathcal{E}/\mathcal{E}_0$ : 1) 0.001; 2) 0.01; 3) 0.03; 4) 0.1; 5) 0.3; 6) 1; and 7) 3.



action should be much greater than unity. The limitation from above indicates that the linear—in the  $\mathcal{E}$  field—correction to the action  $a_1$  is correct so long as  $a_1 \ll A_0/(\Omega\tau_s)^2$ , which is a much more rigid criterion than the one indicated in Sec. 4.

In the region

$$(\mathcal{E}/\mathcal{E}_0) \exp(\Omega\tau_s) \sim 1$$

we obtain for  $A$  the expression

$$A = A_0 \{ 1 - (\Omega\tau_s)^{-2} f[\mathcal{E} \exp(\Omega\tau_s)/2\mathcal{E}_0] \}, \quad (25)$$

where the function  $f(x)$  is given by the relations

$$f = 3y + 3y^2/2, \quad (26)$$

$$ye^y = x, \quad (27)$$

and its plot is shown in Fig. 6. In the limit of small  $\mathcal{E}$  the expression (25) goes over into (24), while at large  $\mathcal{E}$  it tends to the expression

$$A = A_0 \left\{ 1 - \frac{3 \ln^2[\mathcal{E} \exp(\Omega\tau_s)/2\mathcal{E}_0]}{2(\Omega\tau_s)^2} \right\}, \quad (28)$$

which is applicable if

$$\Omega\tau_s \gg \ln[\mathcal{E} \exp(\Omega\tau_s)/2\mathcal{E}_0] \gg 1. \quad (29)$$

At still higher  $\mathcal{E}$  values we can neglect in the expression (23) the last term in the brackets, but the first two should be retained. Solving Eq. (27)  $y \gg 1$  by the iteration method, and introducing the variable  $z = 1 - y/\Omega\tau_s$ , we obtain

$$A = A_0 [3z/2 - z^2/2], \quad z = (\Omega\tau_s)^{-1} \ln[2\mathcal{E}_0\Omega\tau_s/\mathcal{E}]. \quad (30)$$

This expression is applicable if

$$\ln[\mathcal{E} \exp(\Omega\tau_s)/2\mathcal{E}_0] \gg 1, \quad \mathcal{E}/\mathcal{E}_0 \ll \Omega\tau_s,$$

where the latter inequality follows from the condition  $\nu \gg 1$ . The expression (30) is matched with (28) in the region (29). But if the condition

$$\ln(2\mathcal{E}_0\Omega\tau_s/\mathcal{E}) \ll \Omega\tau_s,$$

is fulfilled, then only the first term in the brackets in (30) should be retained. The field  $\mathcal{E}_0$  then drops out of the answer, and

$$A = 2[(V-E)/\Omega] \ln(2\Omega\kappa/\mathcal{E}), \quad (31)$$

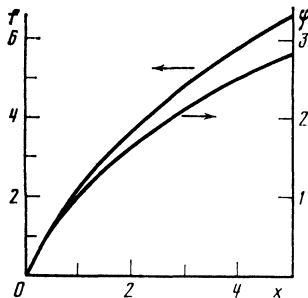


FIG. 6.

which corresponds to a multiquantum non-tunneling penetration of the barrier, when we have for the transmission coefficient

$$D \approx e^{-A} = (\mathcal{E}/2\kappa\Omega)^{2(V-E)/\Omega}.$$

An increase in the field amplitude  $\mathcal{E}$  leads to further reduction in the quantity  $\nu = \Omega\tau$ . When  $\nu \ll \nu_s$ , Eq. (21) should be written in the form

$$(\mathcal{E}/\mathcal{E}_0) \operatorname{sh} \nu = \nu_s,$$

which is valid in the  $\nu \sim 1$  case as well. In this situation it is necessary to return to the expression (20) for  $A$ , in which only the first and last terms in the square brackets should be retained. It is easy to see that the approximations made correspond to a field  $\mathcal{E}_0$  negligible compared with the strong alternating field, so that the problem reduces to the one solved earlier by Keldysh,<sup>3</sup> and has the solution

$$A = (\kappa^2/m\Omega) \{ (1 + \mathcal{E}^2/2\Omega^2\kappa^2) \operatorname{arsh}(\Omega\kappa/\mathcal{E}) - (1 + \mathcal{E}^2/\Omega^2\kappa^2)^{1/2}/2 \}. \quad (32)$$

In the region of still higher field intensities where  $\mathcal{E} \gg \kappa\Omega$ , we find from (32) that

$$A = 2\kappa^3/3m\mathcal{E}, \quad (33)$$

which corresponds to tunneling at the maximum field intensity in a time short compared to the period of the field, so that the tunneling occurs quasistatically.

Thus, as the amplitude  $\mathcal{E}$  of the high-frequency ( $\Omega\tau_s \gg 1$ ) field increases, the following tunneling regimes occur one after another: 1) the static regime (22), in which the field amplitude  $\mathcal{E}$  does not exceed the lower limit set by the inequality (24), and the field has no effect on the tunneling; 2) the regime in which the  $\mathcal{E}$  field has an anomalously strong effect on the tunneling, and the correction (of the order of  $(\Omega\tau_s)^{-2}$  to  $A_0$ ) depends on the combination  $\mathcal{E} \exp(\Omega\tau_s)/2\mathcal{E}_0$  (see (25)–(27) and Fig. 6); 3) the regime (30), in which  $A$  depends logarithmically on  $\mathcal{E}$ , i.e., varies much more slowly with  $\mathcal{E}$  than before; 4) the regime (31) of multiquantum non-tunneling penetration of the barrier; 5) the strong-alternating-field regime (32) investigated by Keldysh<sup>3</sup>; and 6) the quasistatic tunneling regime (33), in which, because of the high intensity of the variable field, the tunneling time is short compared to the period of the field, and the tunneling occurs at the instant when the barrier is narrowest.

## 8. THE FRANZ-KELDYSH EFFECT IN A STRONG VARIABLE FIELD

Let us consider the interband breakdown in a semiconductor in the presence of a constant and an alternating electric field. In the case of a very weak alternating field the probability for a one-photon transition can be computed with the aid of perturbation theory.<sup>2</sup> We shall be interested in the case of relatively strong fields, when the multiquantum transitions are important. In this case we must use the semiclassical approximation instead of perturbation theory. This method is used to solve the breakdown problem in Refs.

24 and 25, where the constant and alternating fields are considered separately.

For a one-dimensional two-band semiconductor with a spectrum given by

$$\varepsilon(p) = \pm (\varepsilon_g/2) [1 + (2pc/\varepsilon_g)^2]^{1/2}$$

the Lagrangian that takes account of the alternating,  $\mathcal{E} \cos \Omega t$ , and constant,  $\mathcal{E}_0$ , fields has the form

$$L = -(\varepsilon_g/2) [1 - (dx/dt)^2/c^2]^{1/2} + x(\mathcal{E}_0 + \mathcal{E} \cos \Omega t). \quad (34)$$

The classical trajectory can be found in its explicit form:

$$dx/dt = c[\mathcal{E}_0 t + (\mathcal{E}/\Omega) \sin \Omega t] \times \{ (\varepsilon_g/2c)^2 + [\mathcal{E}_0 t + (\mathcal{E}/\Omega) \sin \Omega t]^2 \}^{-1/2}. \quad (35)$$

The tunneling transition probability in the semiclassical limit can, according to (8), be computed with the Lagrangian (34), using as  $C_+$  and  $C_-$  the contours in Fig. 7. The square root in (35) has opposite signs on opposite banks of the branch cut, and the interband transition occurs at the instant  $i\tau$  corresponding to the zero of the radicand in (35). As in the preceding section, the instant at which the particle emerges from the forbidden region should correspond to the maximum value of the field. The argument of the tunneling exponential function then has the form

$$A = 4c \int_0^{\tau} \{ (\varepsilon_g/2c)^2 - [\mathcal{E}_0 \tau' + (\mathcal{E}/\Omega) \text{sh } \Omega \tau']^2 \}^{1/2} d\tau', \quad (36)$$

where the upper integration limit corresponds to the root of the integrand. Let us introduce the time

$$\tau_s = \varepsilon_g/2c\mathcal{E}_0,$$

which is the time required by the particle to gain in the static field  $\mathcal{E}_0$  an energy of the order of the forbidden-band width  $\varepsilon_g$ , and rewrite the action (36) in the form

$$A = A_0 (4/\pi v_s^2) \int_0^{\nu} \{ \nu_s^2 - [\mu + (\mathcal{E}/\mathcal{E}_0) \text{sh } \mu]^2 \}^{1/2} d\mu, \quad (37)$$

where

$$A_0 = \pi \varepsilon_g \tau_s / 2$$

is the action in the absence of an alternating field and  $\nu$  is the root of the equation

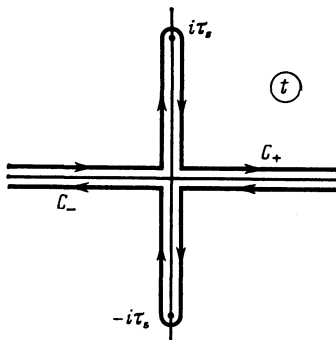


FIG. 7. Integration contour for the computation of the probability for interband tunneling in a semiconductor.

$$v + (\mathcal{E}/\mathcal{E}_0) \text{sh } \nu = v_s, \quad v_s = \Omega \tau_s. \quad (38)$$

The family of curves given by the expressions (37) and (38) is virtually indistinguishable from the family depicted in Fig. 5.

The analyses of the particular cases with simplified expressions for  $A$  is entirely similar to the corresponding analysis carried out in the preceding section. As before, we shall assume that  $\Omega \tau_s \gg 1$ .

In the limit of weak variable fields, i.e., for

$$\mathcal{E} \exp(\Omega \tau_s) \ll \mathcal{E}_0,$$

we obtain from (37) and (38) the expression

$$A = A_0 \left\{ 1 - \left( \frac{2}{\pi} \right)^{1/2} \frac{\mathcal{E} \exp(\Omega \tau_s)}{(\Omega \tau_s)^{1/2} \mathcal{E}_0} \right\}. \quad (39)$$

As in the preceding section, for the small, but nonlinear—in  $\mathcal{E}$ —correction to  $A_0$  we obtain

$$A = A_0 \{ 1 - (\Omega \tau_s)^{-1/2} \varphi[\mathcal{E} \exp(\Omega \tau_s)/2\mathcal{E}_0] \}, \quad (40)$$

where  $\varphi(x)$  is given by the parametric relations

$$\varphi = (2^{1/2}/\pi) \left\{ 2y^{3/2}/3 + \int_y^{\infty} [z^{1/2} - (z - ye^{y-z})^{1/2}] dz \right\}, \quad (41)$$

$$ye^{y-z} = x, \quad (42)$$

and its plot is shown in Fig. 6. The expression (40) goes over in the limit of weak fields into the expression (39), and tends in the limit of strong fields to

$$A = A_0 \left\{ 1 - \frac{2^{1/2}}{3\pi} \frac{\ln^2[\mathcal{E} \exp(\Omega \tau_s)/2\mathcal{E}_0]}{(\Omega \tau_s)^{1/2}} \right\}, \quad (43)$$

which is applicable in the region

$$\Omega \tau_s \gg \ln [\mathcal{E} \exp(\Omega \tau_s)/2\mathcal{E}_0] \gg 1. \quad (44)$$

For still higher values of  $\mathcal{E}$  we can neglect the hyperbolic sine in the integrand in (37), and solve Eq. (42) for  $y \gg 1$  by the interaction method, as a result of which we obtain

$$A = (2A_0/\pi) [\arcsin z + z(1-z^2)^{1/2}], \quad (45)$$

$$z = (\Omega \tau_s)^{-1} \ln (2\mathcal{E}_0 \Omega \tau_s / \mathcal{E}).$$

This expression is applicable if

$$\ln [\mathcal{E} \exp(\Omega \tau_s)/2\mathcal{E}_0] \gg 1, \quad \mathcal{E}/\mathcal{E}_0 \ll \Omega \tau_s,$$

where the latter inequality is a consequence of the condition  $\Omega \tau_s \gg 1$ . The expression (45) is matched with (43) under the conditions (44). In the region of still stronger  $\mathcal{E}$  fields where

$$\ln (2\mathcal{E}_0 \Omega \tau_s / \mathcal{E}) \ll \Omega \tau_s,$$

we have from (45) that

$$A = (2\varepsilon_g/\Omega) \ln (\varepsilon_g \Omega / c\mathcal{E}), \quad (46)$$

which corresponds to a multiquantum nontunneling surmounting of the forbidden band under conditions when the effect of the constant field can be ignored. Further increase of the field amplitude  $\mathcal{E}$  leads to the situation in which the

quantity  $\nu \ll \nu_s$ , and the action is given by the expression<sup>24,25</sup>

$$A = (2\varepsilon_g/\Omega) \int_0^{\nu} [1 - (2\mathcal{E}c/\varepsilon_g\Omega)^2 \operatorname{sh}^2 \mu]^{1/2} d\mu, \quad (47)$$

$$(\mathcal{E}/\mathcal{E}_0) \operatorname{sh} \nu = \nu_s,$$

where the effect of the constant field can be ignored if

$$\ln(\Omega\varepsilon_g/c\mathcal{E}) \ll \Omega\varepsilon_g/c\mathcal{E}_0.$$

In the region of relatively small  $\mathcal{E}$  the expression (47) goes over into (46), while for  $\mathcal{E} \gg \varepsilon_g \Omega/c$  the tunneling process is quasistatic, and

$$A = \pi\varepsilon_g/4\mathcal{E}c,$$

which is entirely similar to the expression (33) in the preceding section.

We shall not enumerate the tunneling regimes, since the corresponding field intensity and frequency regions coincide exactly with the regions given in the preceding section, and the entire difference amounts to some change in the dependences of the action  $A$  on the indicated parameters.

## 9. CONCLUSION

Our purpose in the present paper was to investigate in all detail the effect of an alternating field on semiclassical processes. The most striking effect here consists in the exponential enhancement of the high-frequency field during the motion of the particle in imaginary time. We have found that, in the particular cases of field emission and interband tunneling, the argument of the tunneling exponential function depends nonlinearly on the ratio  $\mathcal{E} \exp(\Omega\tau_s)/\mathcal{E}_0$  when the latter is of the order of unity. The relative correction to the action in this case is of the same order of smallness as the parameter  $(\Omega\tau_s)^{-2}$  or  $(\Omega\tau_s)^{-3/2}$ , but the absolute value of the correction should be large, if the semiclassical approximation is to be applicable. It is, apparently, in this regime that the qualitative difference between the variable and constant fields is most clearly manifested. As the field intensity is increased further, the dependence of the action on the field amplitude  $\mathcal{E}$  becomes much weaker, specifically, it becomes logarithmic. We have computed these dependences in the entire domains of the parameters, but the conditions of applicability of the corresponding analytic expressions are too rigid to be fulfilled in the case of real systems. Thus, we can hope for the fulfillment of the condition  $\Omega\tau_s \gg 1$ , but when the condition  $\ln(\Omega\tau_s) \gg 1$  is fulfilled simultaneously with the condition  $V \gg \Omega$ , the decay probability  $\exp(-\alpha V\tau_s)$  ( $\alpha$  is of the order of unity) will be so small that the decay will be unobservable.

In the case of charge exchange between deep-lying centers in semiconductors, when the distance between them is large, typical values of  $A_0 \sim \ln(1/D)$  are  $\sim 30$  (Ref. 6). We can then fulfill the condition  $A_0 \gg \Omega\tau_s$  by taking  $\Omega\tau_s \sim 6$ , which corresponds to a field enhancement ratio  $\exp(\Omega\tau_s) \sim 400$ . But if the action does not attain too large values, i.e., if  $A_0$  lies in the range from 10 to 15, as is possible in tunneling chemical reactions,<sup>7</sup> or in field emission,<sup>23</sup> then the exponential enhancement of the alternating field can be

observed in the case of the absorption of a small number  $N$  of quanta, when the quantity  $(V - E)/\Omega \gtrsim 1$ . For the triangular barrier

$$D(\mathcal{E}) \sim \sum_N C_N (\mathcal{E}/\tilde{\mathcal{E}})^{2N} \exp[-A_0(E + \Omega N)],$$

which corresponds to perturbation theory in the case of high field frequencies, when the semiclassical approximation is inapplicable. Nevertheless, as can be seen from the expression given above, the effective field turns out to be exponentially enhanced in this limit as well.

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