Acoustic turbulence in superfluid helium

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We study the problem of stochastic wave fields in first and second sound in superfluid helium when there is random pumping of wave energy. We find exact scale-invariant solutions of the kinetic equations for the pair correlators of the complex amplitudes in the stationary isotropic case. Using the solutions obtained we calculate the damping and dispersion of sound waves propagating in helium containing random wave fields.

Following Ref. 1 we call an ensemble of interacting, mutually uncorrelated acoustic waves acoustic turbulence (AT). Let there be a source of wave energy (an instability mechanism may, e.g., serve as such) which produces harmonics with characteristic wave numbers of order k_+. As a rule the value of k_+ will be of the same order of magnitude as the reciprocal size of the system, $k_+ \approx L^{-1}$. Due to the nonlinear interactions there occur in the system harmonics with larger values of k and these, in turn, generate even higher harmonics. For very large values of k of the order of k_- the viscous terms in the equations of motion come into play, and waves with momenta $k \gtrsim k_-$ are rapidly damped. Ultimately a wave distribution is established in **k** space and is characterized by an energy flux from large- to small-scale motions. This scenario is typical of turbulent phenomena and, since we are dealing with sound waves, it is called AT.

In the present paper we study AT in He II, whose distinctive feature is that apart from the interaction between waves described there is also a cross-interaction between first and second sounds. We obtain in the first section a stationary scale-invariant solution of the kinetic equations (KE) for the pair correlators of the complex amplitudes (*vide infra*). In the second section we describe the acoustic properties of turbulent He II. The last, third, section is devoted to a discussion of the criteria used and to numerical estimates.

We study the wave fields in Hamiltonian variables $a_{\mathbf{k}}^{\nu}(t)$ which are alternatively called the complex sound amplitudes.[2] The upper index $\nu = \pm 1,2$ identifies the wave mode, the minus sign indicates complex conjugation. The equations of motion of superfluid helium have in the variables $a_{\mathbf{k}}^{\nu}(t)$ the following form:[1)]

$$\partial a_{\mathbf{k}}^{\nu}/\partial t - i \operatorname{sign}\nu\,(\delta H/\delta a_{\mathbf{k}}^{-\nu}) = 0, \quad \nu = \pm 1, 2. \tag{1}$$

Up to terms of third order (which corresponds to the quadratic approximation in the equations of motion) the Hamiltonian H has the following form:

$$H = \sum_{\nu=1,2} \int \omega_{\mathbf{k}}^{\nu} a_{\mathbf{k}}^{\nu} a_{\mathbf{k}}^{-\nu}\, d\mathbf{k} + \sum_{\nu_1,\nu_2,\nu_3=\pm 1,2} \int V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}^{\nu_1\nu_2\nu_3} a_{\mathbf{k}_1}^{\nu_1} a_{\mathbf{k}_2}^{\nu_2} a_{\mathbf{k}_3}^{\nu_3} \times \delta\left(\sum_j \mathbf{k}_j \operatorname{sign}\nu_j\right) d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3. \tag{2}$$

Here $\omega_{\mathbf{k}}^{\nu} = c_\nu k$ are the first and second sound frequencies. The quantities $V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}^{\nu_1\nu_2\nu_3}$ are called the matrix elements or vertex parts (vertices) of the nonlinear processes. We shall not write down the cumbersome expressions for the quantities $V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}^{\nu_1\nu_2\nu_3}$ but merely note the fact, which is important for the following exposition, that they have the following identical structural dependence on the arguments $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$:

$$V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}^{\nu_1\nu_2\nu_3} = (k_1k_2k_3)^{1/2}\left[P_1^{\nu_1\nu_2\nu_3} + P_2^{\nu_1\nu_2\nu_3}\frac{\mathbf{k}_1\mathbf{k}_2}{k_1k_2} + P_3^{\nu_1\nu_2\nu_3}\frac{\mathbf{k}_2\mathbf{k}_3}{k_2k_3} + P_4^{\nu_1\nu_2\nu_3}\frac{\mathbf{k}_1\mathbf{k}_3}{k_1k_3}\right], \tag{3}$$

i.e., they are homogeneous functions of power 3/2. For the various forms of nonlinear processes we shall use the terminology introduced in Ref. 2. If two indices ν out of three equal ± 2 we call such processes decay processes. If two of the indices ν are equal to ± 1 we call the corresponding processes Cherenkov processes. Finally, if all indices are equal to ± 1 or to ± 2 we call such processes nonlinear eigenprocesses in the first or second wavemode.

The regular approach to describe random wave fields is based on Wyld's diagram technique[3] which was developed by him for hydrodynamic turbulence. The canonical variant of this technique described in Ref. 4 is more convenient for a study of waves. Following Ref. 4 we introduce for the description of random wave fields the following averages: the spectral density tensor $n_{q_1q_2}^{\nu_1\nu_2}$ and the Green tensor $G_{q_1q_2}^{\nu_1\nu_2}$ which we define as follows:

$$\langle a_{q_1}^{\nu_1} a_{q_2}^{\nu_2}\rangle = n_{q_1q_2}^{\nu_1\nu_2}\,\delta(q_1 \operatorname{sign}\nu_1 + q_2 \operatorname{sign}\nu_2), \tag{4}$$

$$\langle \delta a_{q_1}^{\nu_1}/\delta f_{q_2}^{\nu_2}\rangle = G_{q_1q_2}^{\nu_1\nu_2}\,\delta(q_1 \operatorname{sign}\nu_1 + q_2 \operatorname{sign}\nu_2). \tag{5}$$

Here $q = (\mathbf{k},\omega)$ is a four-dimensional wave vector, a_q^{ν} is the temporal Fourier component of the complex amplitude $a_{\mathbf{k}}^{\nu}$, and f_q^{ν} the Fourier component of the external random (Langevin) force f (see Ref. 4). The renormalized quantities (taking the interactions into account) $n_{q_1q_2}^{\nu_1\nu_2}$, $G_{q_1q_2}^{\nu_1\nu_2}$ satisfy a Dyson set of equations:

$$G_{q_1q_2}^{\nu_1\nu_2} = G_{0q_1}^{\nu_1} + G_{0q_1}^{\nu_1}\Sigma_{q_1q'}^{\nu_1\nu'}G_{q'q_2}^{\prime\nu_2}, \tag{6}$$

$$n_{q_1q_2}^{\nu_1\nu_2} = G_{q_1q'}^{\nu_1\nu'}\,\Phi_{q_1q''}^{\nu'\nu''}\,G_{q''q_2}^{\nu''\nu_2}. \tag{7}$$

Here G_{0q}^{ν} is the bare Green function equal to $G_{0q}^{\nu} = (\omega - \omega_{\mathbf{k}} + i\gamma_{\mathbf{k}} \operatorname{sign}\nu)^{-1}$ where $\gamma_{\mathbf{k}}^{\nu}$ is the viscous damping. The mass operators $\Sigma_{q_1q_2}^{\nu_1\nu_2}$ and $\Phi_{q_1q_2}^{\nu_1\nu_2}$ can be written

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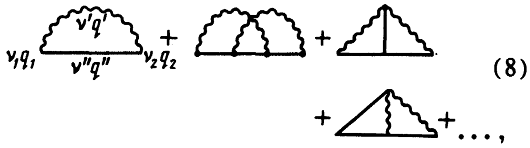
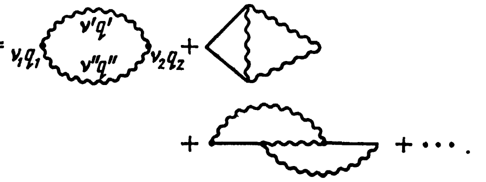
in the form of the following diagram series

$\Sigma^{\nu_1\nu_2}_{q_1q_2} =$ 

(8)

$\Phi^{\nu_1\nu_2}_{q_1q_2} =$ 

(9)

The wavy lines depict $n^{\nu_i\nu_j}_{q_iq_j}$ and the straight lines $G^{\nu_i\nu_j}_{q_iq_j}$.

Assuming that there is a low level of nonlinearity and specifically requiring that the nonlinear frequency shift $\Delta\omega^{\nu}_{k}$ be much smaller than any of the frequencies, the Dyson equations can in the usual way be reduced to a set of KE for the quantities $n^{\nu}_{\mathbf{k}} = \int n^{\nu_i - \nu}_{qq} d\omega$—the single-time correlator of the complex amplitude (see Refs. 4, 5).[2)] In the stationary and spatially homogeneous case the set of KE has the following form:

$$I_{st}^{\nu}\{n\} = \sum_{\nu_1,\nu_2=\pm1,2} \int d\mathbf{k}_1\, d\mathbf{k}_2 \{D^{\nu\nu_1\nu_2}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}(n^{\nu_1}_{\mathbf{k}_1} n^{\nu_2}_{\mathbf{k}_2} - n^{\nu}_{\mathbf{k}} n^{\nu_1}_{\mathbf{k}_1} - n^{\nu}_{\mathbf{k}} n^{\nu_2}_{\mathbf{k}_2})$$

$$- D^{\nu_1\nu_2\nu}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}}(n^{\nu_2}_{\mathbf{k}_2} n^{\nu}_{\mathbf{k}} - n^{\nu_1}_{\mathbf{k}_1} n^{\nu}_{\mathbf{k}} - n^{\nu_1}_{\mathbf{k}_1} n^{\nu_2}_{\mathbf{k}_2}) \quad (10)$$

$$- D^{\nu_2\nu\nu_1}_{\mathbf{k}_2\mathbf{k}\mathbf{k}_1}(n^{\nu}_{\mathbf{k}} n^{\nu_1}_{\mathbf{k}_1} - n^{\nu_2}_{\mathbf{k}_2} n^{\nu}_{\mathbf{k}} - n^{\nu_2}_{\mathbf{k}_2} n^{\nu_1}_{\mathbf{k}_1})\} = 0, \quad \nu = 1, 2.$$

Here

$$D^{\nu\nu_1\nu_2}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} = \frac{\pi}{2} |V^{\nu\nu_1\nu_2}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}|^2 \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)\, \delta(\omega^{\nu}_{\mathbf{k}} - \omega^{\nu_1}_{\mathbf{k}_1} - \omega^{\nu_2}_{\mathbf{k}_2}). \quad (11)$$

Equations (10) are the same as the KE used to describe phonon systems (see Refs. 5, 6) with the difference that we have dropped spontaneous processes in them (formally because $n^{\nu}_{\mathbf{k}} \gg \hbar$).

One can check in the usual way (see, e.g., Ref. 7) that Eqs. (10) have solutions of the form

$$n_{\mathbf{k}}^{1} = \tilde{T}/c_1|\mathbf{k}|, \quad n_{\mathbf{k}}^{2} = \tilde{T}/c_2|\mathbf{k}|, \quad (12)$$

where $\tilde{T}$ is a constant which has the meaning of a temperature. The solution (12) is an equilibrium Rayleigh distribution and is characterized by the absence of any fluxes (in **k**-space); it is accordingly not suited for a statement of the AT problem.

1. NON-EQUILIBRIUM SOLUTION OF THE KE

In a non-equilibrium situation the statement of the problem of finding the spectra $n^{\nu}_{\mathbf{k}}$ assumes, apart from an equation, the presence of a source and a sink for waves. In general, the solution $n^{\nu}_{\mathbf{k}}$ depends on the actual choice for the form of the source (and the sink). However, as often happens, the regions where the source and sink are important are widely separated in **k**-space, i.e., $k_+ \ll k_-$. In that case a distribution $n^{\nu}_{\mathbf{k}}$ which is independent of the shape of the source (and the sink) can be established in a range of wave numbers which is far from both k_+ and from k_- (i.e., $k_+ \ll k \ll k_-$), i.e., in the so-called inertial range (IR). The role of the source and the sink is then reduced to some kind of boundary conditions which select from the ensemble of solutions those which guarantee that the flux of some physical quantities—in our case, the energy—is constant.

The problem of finding the spectra is thus reduced to finding such solutions of the KE (10) which guarantee that the energy flux is constant. This problem is very complicated, as it is connected with solving a set of nonlinear integral equations. Even in the simplest case of a single wave mode one can find an exact solution only in the isotropic situation and under very rigid restrictions on the form of the functions $\omega^{\nu}_{\mathbf{k}}$, $V^{\nu_1\nu_2\nu_3}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}$, namely, under the requirement that these quantities are homogeneous functions of their arguments. This requirement, as well as the condition $k_+ \ll k_-$ by virtue of which one can put $k_+ = 0$ and $k_- = \infty$, leads to the assumption of a seale-invariant problem, i.e., to the absence of characteristic scales for k. This enables us to assume that the solution $n_{\mathbf{k}}$ has a power-law form: $n_{\mathbf{k}} = Ak^s$. The power index s is then evaluated using the so-called Zakharov-Kats-Kontorovich transformations (for details see Ref. 7). Using these transformations the two last terms in the braces in Eq. (10) can be reduced to the form of the first term with a factor which is a function of $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, and the parameter s, i.e., the integrand can be factorized. One of the factors leads to the Rayleigh distribution $n_{\mathbf{k}} \propto k^{-1}$, the other leads to a distribution $n_{\mathbf{k}} \propto k^s$ which is characterized by a nonvanishing energy flux. In particular, Zakharov and Sagdeev[1] found in this way the AT spectrum in a classical fluid. They found that $s = -9/2$ and evaluated the connection between the amplitude of the spectrum A and the power P of the source of the wave energy.

Because of the presence of several kinds of nonlinear interactions in He II one cannot factorize the collision integral directly. Nonetheless, we shall now show that the set of KE (10) has an isotropic scale-invariant solution of the form

$$n_{\mathbf{k}}^{1} = Ak^s, \quad n_{\mathbf{k}}^{2} = Bk^s \quad (13)$$

with the same power index s.

To prove that statement we can proceed as follows. If we substitute the spectra (13) into the KE (10) and evaluate all integrals occurring in it the following important fact emerges. By virtue of the identical degree of homogeneity of all vertices $V^{\nu_1\nu_2\nu_3}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}$ and also of the linearity of the dispersion $\omega^{\nu}_{\mathbf{k}}$, the external argument $\mathbf{k}$ occurs in all terms in the form of a factor k^{5+2s}. As a result, after canceling k^{5+2s}, the KE reduce to a set of algebraic equations for the quantities A and B:

$$X_{AA}A^2 + X_{AB}AB + X_{BB}B^2 = 0, \quad (14)$$

$$Y_{AA}A^2 + Y_{AB}AB + Y_{BB}B^2 = 0. \quad (15)$$

The quantities X and Y are evaluated from the corresponding terms occurring in the KE; they are functions of the parameter s. As the set of Eqs. (14), (15) is homogeneous (in A, B) it has a solution only for some well defined values of s which play the role of eigenvalues. We show below that the value $s = -9/2$ is an eigenvalue and give reasons that other values of s lead to spectra which do not satisfy the requirement of scale invariance.

We write down the terms in the set of KE (10) corresponding to decay processes (from our exposition it will be clear that for the Cherenkov processes a similar situation is realized). Denoting them by J^{122}, J^{212} where the first upper index denotes the number of equations and the two others indicate the form of the process, we get

$$J^{122}=\int D^{122}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}(n_{\mathbf{k}_1}{}^2 n_{\mathbf{k}_2}{}^2-n_{\mathbf{k}}{}^1 n_{\mathbf{k}_1}{}^2-n_{\mathbf{k}}{}^1 n_{\mathbf{k}_2}{}^2)\,d\mathbf{k}_1\,d\mathbf{k}_2, \qquad 16)$$

$$J^{212}=-\int D^{122}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}}(n_{\mathbf{k}_2}{}^2 n_{\mathbf{k}}{}^2-n_{\mathbf{k}_1}{}^1 n_{\mathbf{k}_2}{}^2-n_{\mathbf{k}_1}{}^1 n_{\mathbf{k}}{}^2)\,d\mathbf{k}_1\,d\mathbf{k}_2 \qquad (17)$$
$$-\int D^{122}_{\mathbf{q}_2\mathbf{k}\mathbf{q}_1}(n_{\mathbf{k}}{}^2 n_{\mathbf{q}_1}{}^2-n_{\mathbf{q}_2}{}^1 n_{\mathbf{k}}{}^2-n_{\mathbf{q}_2}{}^1 n_{\mathbf{q}_1}{}^2)\,d\mathbf{q}_1\,d\mathbf{q}_2.$$

We consider the second term in Eq. (17). The conservation laws occurring in the factor $D^{122}_{\mathbf{q}_2\mathbf{k}\mathbf{q}_1}$ require that the following conditions hold:

$$c_1|\mathbf{q}_2|=c_2|\mathbf{k}|+c_2|\mathbf{q}_1|, \quad \mathbf{q}_2=\mathbf{k}+\mathbf{q}_1. \qquad (18)$$

Similarly the conservation laws in the integral J^{122} have the following form:

$$c_1|\mathbf{k}|=c_2|\mathbf{k}_1|+c_2|\mathbf{k}_2|, \quad \mathbf{k}=\mathbf{k}_1+\mathbf{k}_2. \qquad (19)$$

In Figs. 1 a,b we draw triads of vectors satisfying (18) and (19) and we choose the triangles to be similar ones. We turn the triangle $\mathbf{q}_2\mathbf{k}\mathbf{q}_1$ so that the direction of $\mathbf{k}$ is the same as the direction of $\mathbf{k}_1$ of the first triangle $\mathbf{k}\mathbf{k}_1\mathbf{k}_2$ (Fig. 2) and we extend it by a factor k/k_1 after which the two triangles coincide. These operations are equivalent to a formal change in variables of the following form:

$$|\mathbf{q}_1|=\frac{k}{k_1}|\mathbf{k}_2|, \quad |\mathbf{q}_2|=\left(\frac{k}{k_1}\right)^2|\mathbf{k}_1|. \qquad (20)$$

Using the substitutions (20) and the homogeneity property of the quantities $V^{\nu_1\nu_2\nu_3}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}$ and $\omega^{\nu}_{\mathbf{k}}n^{\nu}_{\mathbf{k}}$ we reduce the term we study to the following form:

$$\int (k/k_1)^{8+2s} D^{122}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}(n_{\mathbf{k}_1}{}^2 n_{\mathbf{k}_2}{}^2-n_{\mathbf{k}}{}^1 n_{\mathbf{k}_1}{}^2-n_{\mathbf{k}}{}^1 n_{\mathbf{k}_2}{}^2)\,d\mathbf{k}_1\,d\mathbf{k}_2. \qquad (21)$$

The second integral term in Eq. (17) thus differs from J^{122} in (16) by a factor $(k/k_1)^{8+2s}$ in the integrand. Similar calculations lead to the result that the first term in (17) takes the form of J^{122} with a factor $(k/k_2)^{8+2s}$ under the integal. Then multiplying (16) by c_1 and (17) by c_2 and adding them we get

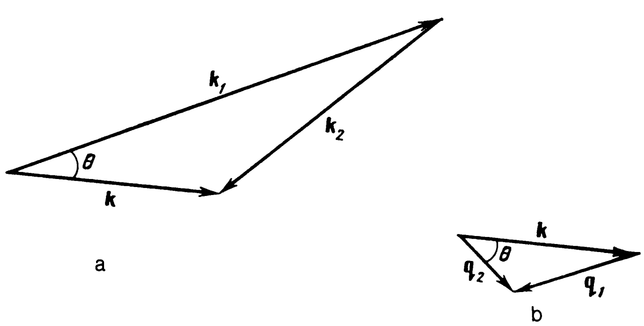


FIG. 1

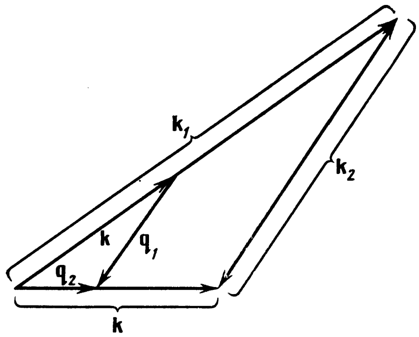


FIG. 2

$$c_1J^{122}+c_2J^{212}=\int D^{122}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}[c_1-c_2(k/k_1)^{8+2s}-c_2(k/k_2)^{8+2s}]$$
$$\times[n_{\mathbf{k}_1}{}^2 n_{\mathbf{k}_2}{}^2-n_{\mathbf{k}}{}^1 n_{\mathbf{k}_1}{}^2-n_{\mathbf{k}}{}^1 n_{\mathbf{k}_2}{}^2]\,d\mathbf{k}_1\,d\mathbf{k}_2. \qquad (22)$$

One sees easily that if $8+2s=-1$ (i.e., if $s=-9/2$) the first square bracket in (22) is the same as the argument of the frequency δ-function occurring in $D^{122}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$ so that the integral vanishes. Thus, *irrespective* of their dependence on the amplitudes A and B of the spectra, when s equals $-9/2$ the quantities J^{122} and J^{212} are connected by the following relation:

$$c_1J^{122}+c_2J^{212}=0. \qquad (23)$$

In particular, if we choose the relation between A and B such that J^{122} vanishes, J^{212} must also vanish. As a result the contribution to I^{ν}_{st} from the decay processes in both Eqs. (10) vanishes. Contributions to the collision integral from nonlinear processes "inside" each of the wave modes vanish automatically when $s=-9/2$, as the situation is completely analogous to AT in classical fluids described in Ref. 1. Therefore, by a choice of the relation between A and B (and putting $s=-9/2$) we can make both collision integrals I^{ν}_{st} vanish, in other words, obtain a solution of the form (13) for the KE. The connection between the amplitudes A and B of the spectra can be established using either of the Eqs. (14), (15). The results of the calculations are shown in Fig. 3 where we plot the ratio A/B as function of the temperature T. We draw attention to the fact that the ratio A/B is close to the value $\frac{1}{2}(2c_2/c_1)^{9/2}$ (the latter is shown in Fig. 3 by the dashed curve). One can give this fact the following physical explanation. Calculations show that the contribution to the quantities X, Y [see (14), (15)] from the decay processes is much larger than that from the Cherenkov processes. However, it was shown in Ref. 2 that in the decay processes a

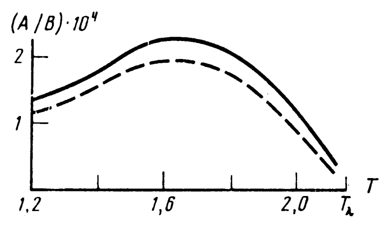


FIG. 3

single first sound quantum with momentum $\mathbf{k}$ decays into two second sound quanta with momenta $\mathbf{k}_1$, $\mathbf{k}_2$ while k_1, $k_2 \approx (c_1/2c_2)k$. As to order of magnitude the following relation holds: $n^1(k) \approx \frac{1}{2} n^2(c_1 k/2c_2)$, whence we get $2A/B \approx (2c_2/c_1)^{9/2}$, if we take into account that $n_\mathbf{k}^1$, $n_\mathbf{k}^2 \propto k^{9/2}$.

We show schematically the first and second sound spectra as functions of the quantity k (Fig. 4). Region I is the region where the source has an effect, and in region III the effects of the viscous damping of the waves are important. Region II is the inertial range in which the spectra found above exist.

We estimate that energies ε^1, ε^2 stored in the wave fields:

$$\varepsilon^1 = \int \omega_\mathbf{k}^1 n_\mathbf{k}^1 \, d\mathbf{k} = 4\pi c_1 A \int k^{-3/2} \, dk \approx 8\pi c_1 A / k_+^{1/2}. \tag{24}$$

The cutoff of the integral is at the size k_+ ($\approx L^{-1}$) where L, we reiterate, is the size of the system. Similarly, we have for second sound $\varepsilon^2 = 8\pi c_2 B / k_+^{1/2}$. The ratio of the energies ε^1 and ε^2 is equal to $\varepsilon^1/\varepsilon^2 \approx 4(2c_2/c_1)^{7/2}$, i.e., the energy stored in the second, softer mode is appreciably larger than the energy of the first, more rigid sound. This is a peculiarity of the non-equilibrium distribution as these energies are equal for the Rayleigh solution (12), in accordance with the energy equipartition law.

We consider the problem of energy fluxes in $\mathbf{k}$-space. The set of Eqs. (10) conserves the total energy $\varepsilon = \varepsilon^1 + \varepsilon^2$ and as a consequence can be written in the form of a continuity equation in $\mathbf{k}$-space for the spectral density $\varepsilon_\mathbf{k}$. In the stationary case which is of interest to us this equation has the following form:

$$\operatorname{div}_\mathbf{k} \mathbf{P}_\mathbf{k} = P\delta(\mathbf{k}). \tag{25}$$

Here $\mathbf{P}_\mathbf{k}$ is the energy flux vector. On the right-hand side of Eq. (25) we introduced the source of the wave energy concentrated at the origin $\mathbf{k} = 0$. By analogy with electrostatics one finds easily that $\mathbf{P}_\mathbf{k} = (P/4\pi k^2)\mathbf{k}/k$. The absolute magnitude of the spherically normalized energy flux $\widetilde{P}_\mathbf{k} = 4\pi k^2 |\mathbf{P}_\mathbf{k}|$ is then independent of the vector $\mathbf{k}$ which means that the total flux through any surface surrounding the origin is constant. To obtain the connection between the amplitudes A and B of the spectra and the power of the source P it is necessary to express $\widetilde{P}_\mathbf{k}$ in terms of the collision integrals. To do this we multiply the first of Eqs. (10) by $4\pi c_1 k^3$ and the second one by $4\pi c_2 k^3$ and add. Changing to the isotropic case and writing the sum in divergent form we get

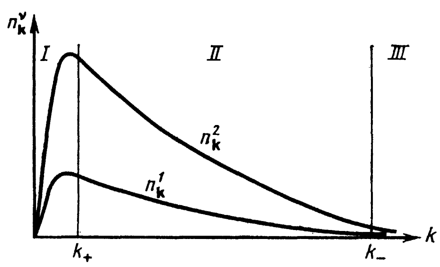


FIG. 4

$$\widetilde{P}_\mathbf{k} = \frac{4\pi k^{9+2s}}{9+2s} \{ c_1 \tilde{I}_{st}^1 + c_2 \tilde{I}_{st}^2 \} = P. \tag{26}$$

Here $\tilde{I}_{st}^\nu$ are the collision integrals taken without the factor k^{5+2s}. It is clear that when $s = -9/2$ the quantity $\widetilde{P}_\mathbf{k}$ is independent of k, i.e., the solutions found for the spectra $n_\mathbf{k}^\nu$ guarantee that the energy flux is constant. To obtain the connection between the quantities A, B, and P in which we are interested we must resolve the undetermined expression $\{\}/(9+2s)$, where $\{\}$ is the expression in the braces. This uncertainty can be resolved by putting s different from $-9/2$ and then letting the expression $9 + 2s$ tend to zero. Separating the contributions from the different forms of nonlinear processes we get

$$\widetilde{P}_\mathbf{k} = P = \lim_{s \to -9/2} \left\{ \frac{4\pi k^{9+2s}}{9+2s} [c_1 J^{111} + c_2 J^{222}] \right.$$
$$+ \frac{4\pi k^3}{9+2s} \int D_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}^{122} \left[c_1 - c_2 \left(\frac{k}{k_1} \right)^{8+2s} - c_2 \left(\frac{k}{k_2} \right)^{8+2s} \right] \tag{27}$$
$$\left. \times [B^2 k_1^s k_2^s - ABk^s k_1^s - ABk^s k_2^s] \, d\mathbf{k}_1 \, d\mathbf{k}_2 \right\}.$$

For the sake of simplicity we have here again restricted ourselves to the decay processes. We consider the integral term in (27). The integrand contains the product of the two square brackets. The first of these vanishes (when $s = -9/2$) on the resonance surface given by condition (19). The second bracket vanishes by virtue of the fact that the connection between A and B was chosen such that the collision integral I_{st}^1 vanished (*vide supra*). Therefore, if the quantity s differs little from $-9/2$ the integral expression is proportional to $(9+2s)^2$ whereas the denominator contains $(9+2s)$ to the first power. As a result the contribution to the total energy flux from the cross interaction vanishes, i.e., notwithstanding the general state of non-equilibrium of the system, the "gases" of the first and second quanta are mutually in equilibrium. As to the contributions to the energy flux from the nonlinear eigenprocesses, for them calculations show that the following relations hold:

$$\widetilde{P}_\mathbf{k}^1 \approx A^2 (6\alpha_1^2 c_1/\rho), \quad \widetilde{P}_\mathbf{k}^2 \approx B^2 (6\alpha_2^2 c_2 \rho_s/\rho_n). \tag{28}$$

Here α_1, α_2 are the coefficients of the first- and second-sound non-linearities (see, e.g., Ref. 8) which are defined by the following relations:

$$\alpha_1 = 1 + \frac{\partial \ln c_1}{\partial \ln \rho}, \quad \alpha_2 = \frac{\sigma T}{C} \frac{\partial}{\partial T} \ln \left(c_2^3 \frac{\partial \sigma}{\partial T} \right). \tag{29}$$

The sum of the expressions $\widetilde{P}_\mathbf{k}^1$, $\widetilde{P}_\mathbf{k}^2$ is the total energy flux along the spectrum. Knowing the ratio A/B we can thus express the spectral amplitude A (or B) in terms of the power P of the source.

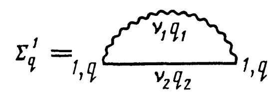
Concluding this section we discuss the problem of the uniqueness of the solutions found by us for the spectra $n_\mathbf{k} \propto k^{-9/2}$. Generally speaking, by virtue of the non-linearity Eqs. (14), (15) can have other eigenvalues of s which differ from $-9/2$. However, one sees easily that in that case the energy flux $\widetilde{P}_\mathbf{k} \propto k^{9+2s}$ depends in an essential way on k. As a result when investigating the connection between the spectral amplitudes A,B and the power P of the source we must introduce the external turbulence scale k_+ which contradicts the original assumption of scale invariance of the turbulence.

2. ACOUSTIC PROPERTIES OF TURBULENT He II

He II in which AT has been excited possesses acoustic properties which are different from those of the unperturbed liquid. Indeed, any sound wave propagating in turbulent helium will interact with the developed wave fields. The result of this interaction is an additional damping Γ and dispersion Δ. We evaluate these quantities.

Assuming that the wave vector $\mathbf{k}$ of the external wave belongs to the inertial range ($k_+ \ll k \ll k_-$), the required quantities are equal to the imaginary and real parts of the mass operator $\Sigma_q^\nu = \Sigma_{qq}^{\nu,-\nu}$ evaluated on the mass surface $\omega = \omega_k^\nu$ [see (6), (8)]. The evaluation of the quantities Σ_q^ν will be performed in first order of perturbation theory, i.e., we restrict ourselves to the first loops in the expansion (8). Taking higher orders into account will be done at the end of the section.

We give first of all the calculations for first sound. The mass operator Σ_q^1 is represented by the following loop:



The analytical expression for this diagram has the following form:

$$\Sigma_q^1 = \sum_{\nu_1,\nu_2=\pm 1,2} \int |V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}^{1\nu_1\nu_2}|^2 n_{q_1}^{\nu_1} G_{q_2}^{\nu_2} d^4q_1\, d^4q_2. \tag{30}$$

Here $n_q^\nu = n_{qq}^{\nu,-\nu}$, $G_q^\nu = G_{qq}^{\nu,-\nu}$. We consider the contribution to Σ_q^1 due to the interaction of the external wave with the wave field of a first sound wave. By analogy with the collision integrals we denote this contribution by Σ_q^{111}. When we attempt to evaluate $\Sigma_{\mathbf{k},\omega_\mathbf{k}}^{11,1}$ using the "bare" Green function G_{0q}^ν there arises the "standard" difficulty which appears, e.g., when one evaluates the damping of sound in a system of phonons with a linear dispersion relation (see Refs. 6, 9). This difficulty is related to the fact that for such systems resonance conditions are fulfilled for collinear vectors due to which the argument of the Green function $\omega_\mathbf{k}^1 - \omega_{\mathbf{k}_1}^1 - \omega_{\mathbf{k}-\mathbf{k}_1}^1$ vanishes identically, whereas at the same time the phase volume of the integration also vanishes. The uncertainty is resolved when one takes interactions into account due to which the δ-function (the imaginary part of the bare Green function) is replaced by a narrow Lorents profile with width $\approx |\Sigma_\mathbf{k}^1|$. Referring to Ref. 9 for details of the calculations we give the final result:

$$\Gamma_\mathbf{k}^{111} = \frac{\alpha_1^2 A}{2\pi\rho} \frac{k}{k_+^{1/2}}, \tag{31}$$

$$\Delta_\mathbf{k}^{111} = \frac{\alpha_1^2 A}{\pi^2\rho} \frac{k}{k_+^{1/2}} \ln \frac{c_1 k}{|\Sigma_k^1|}. \tag{32}$$

Here $\Gamma_\mathbf{k}^{111}$, $\Delta_\mathbf{k}^{111}$ are the damping and dispersion of the first sound caused by the interaction with the wave field of the first mode. In evaluating (31) and (32) one has used the fact that the spectra $n_\mathbf{k}^1 = Ak^{-9/2}$ and introduced a cutoff on the integrals at the lower limit k_+ ($\approx L^{-1}$).

We find thus that a wave propagating in turbulent helium undergoes additional damping, proportional to k, and a dispersion the role of reduces by virtue of the linear dependence on k to a renormalized velocity, i.e., $\Delta c_1^{111} = \Delta_\mathbf{k}^{111}/k$.

We now consider the contribution to $\Sigma_\mathbf{k}^1$ due to the decay interaction of the external wave, i.e., we put $\nu_1 = 2$, $\nu_2 = 2$. The values $\nu_1 = -2$ or $\nu_2 = -2$ are forbidden by the conservation laws. In this case without fear of contradictions we can for the evaluation of $\Sigma_\mathbf{k}^{122}$ substitute the bare Green function. The damping $\Gamma_\mathbf{k}^{122}$ is expressed by the following relation:

$$\Gamma_\mathbf{k}^{122} = \operatorname{Im} \Sigma_\mathbf{k}^{122} = \pi |V_{\mathbf{k}\mathbf{k}_1\mathbf{k}-\mathbf{k}_1}^{122}|^2 n_{\mathbf{k}_1}^2 \delta(\omega_\mathbf{k}^1 - \omega_{\mathbf{k}_1}^2 - \omega_{\mathbf{k}-\mathbf{k}_1}^2)\, d\mathbf{k}_1. \tag{33}$$

The calculation of the integral reduces to integration over the resonance surface which is the ellipsoid $c_1|\mathbf{k}| = c_2|\mathbf{k}_1| - c_2|\mathbf{k}-\mathbf{k}_1|$. The quantities $|\mathbf{k}_1|$, $|\mathbf{k}-\mathbf{k}_1|$ are here close to the value $c_1k/2c_2$. Integrating and neglecting terms of order $(c_2/c_1)^2$ in relation to the others we find that the damping $\Gamma_\mathbf{k}^{122}$ is equal to

$$\Gamma_\mathbf{k}^{122} = \frac{\langle\alpha_p\rangle}{16\pi\rho} \left(\frac{2c_2}{c_1}\right)^{1/2} Bk^{1/2}. \tag{34}$$

Here

$$\langle\alpha_p\rangle = \int_0^\pi (\alpha_p - \cos^2\theta)^2\, d\cos\theta,$$

where $\alpha_p = -\dfrac{\rho_n}{2\rho_s} - \dfrac{\rho_n\rho}{2\rho_s}\left(\dfrac{\partial\rho_n^{-1}}{\partial p}\right)_\sigma + \dfrac{1}{2}\dfrac{(\partial^2 p/\partial\sigma^2)_\rho}{(\partial T/\partial\sigma)_\rho}$.

We turn to the evaluation of the contribution of the decay processes to the first-sound dispersion $\Delta_\mathbf{k}^{122} = \operatorname{Re} \Sigma_\mathbf{k}^{122}$. The analytical expression for $\Delta_\mathbf{k}^{122}$ has the following form:

$$\Delta_\mathbf{k}^{122} = \int |V_{\mathbf{k}\mathbf{k}_1\mathbf{k}-\mathbf{k}_1}^{122}|^2 \frac{n_{\mathbf{k}_1}^2\, d\mathbf{k}_1}{\omega_\mathbf{k}^1 - \omega_{\mathbf{k}_1}^2 - \omega_{\mathbf{k}-\mathbf{k}_1}^2}. \tag{35}$$

The integral in (35) must be understood in the sense of a principal value. Close to the resonance surface the integrand has a singularity of the $(k - k_{\rm res})^{-1}$ type with different signs on both sides of the surface of the resonance ellipsoid. Nonetheless the integral does not vanish, owing to the presence of the fast decreasing function $n_{\mathbf{k}_1}^2 \propto k_1^{-9/2}$. We estimate approximately the dispersion $\Delta_\mathbf{k}^{122}$, neglecting the dependence of the matrix element $V_{\mathbf{k}\mathbf{k}_1\mathbf{k}-\mathbf{k}_1}^{122}$ on direction, i.e., putting $V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}^{122} = \text{const}(k_1k_2k_3)^{1/2}$. In that case one can perform the integration analytically over the angle between $\mathbf{k}$ and $\mathbf{k}_1$. As a result of the integration the $(k - k_{\rm res})^{-1}$ type singularity splits into two logarithmic type singularities; the integrand as function of $|\mathbf{k}_1|$ takes the form shown schematically in Fig. 5. The main contribution to the integral comes

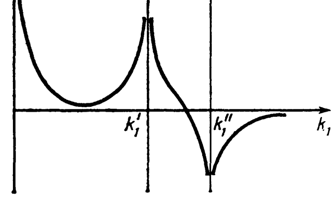


FIG. 5

from the infrared region $k_1 \to 0$ and also the regions near k_1', k_1''. The first region leads to the following (order of magnitude) expression for ${}^1\Delta_{\mathbf{k}}^{122}$:

$$ {}^1\Delta_{\mathbf{k}}^{122} \approx \frac{B}{\pi^2\rho}\frac{k}{k_+^{1/2}}. \tag{36} $$

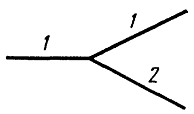
The role of that part of the dispersion reduces to a renormalization of the sound speed Δc^{122}, where $\Delta c^{122} \gg c^{111}$.

Estimates made for the regions close to k_1', k_1'' lead to the following result:

$$ {}^2\Delta_{\mathbf{k}}^{122} \approx \frac{\langle \alpha_p \rangle}{\pi^2\rho}\left(\frac{c_2}{c_1}\right) B k^{1/2}. \tag{37} $$

The non-linear decay processes thus lead to additional damping and dispersion which has a square-root dependence on the wave number k.

The contributions from the Cherenkov processes to the quantities $\Gamma_{\mathbf{k}}^1$, $\Delta_{\mathbf{k}}^1$ are evaluated in a similar way. Qualitatively the results are similar, i.e., additional $\Gamma_{\mathbf{k}}^{112}$, $\Delta_{\mathbf{k}}^{112}$ proportional to $k^{1/2}$. However, quantitatively these corrections are much smaller than the decay ones, formally because the Cherenkov vertices are small and also the phase volume for integration is small in the processes

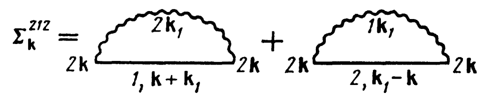


We turn to the problem of evaluating the correlation characteristics of second sound. By analogy with first sound there are here contributions to the damping $\Gamma_{\mathbf{k}}^{222}$ and to the dispersion $\Delta_{\mathbf{k}}^{222}$; these contributions are connected with nonlinear processes "inside" the second sound mode:

$$ \Gamma_{\mathbf{k}}^{222} = \frac{\alpha_2{}^2\rho_s B}{2\pi\rho\rho_n}\frac{k}{k_+^{1/2}}, \qquad \Delta_{\mathbf{k}}^{222} = \frac{\alpha_2{}^2\rho_s B}{\pi^2\rho\rho_n}\frac{k}{k_+^{1/2}}\ln\frac{c_2 k}{|\Sigma_{\mathbf{k}}{}^2|}. \tag{38} $$

Therefore, as in first sound, there is linear damping and dispersion, the role of which reduces to a renormalization of the sound speed.

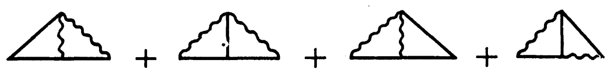
Similarly to first sound, the cross-term nonlinear processes contribute to the damping and dispersion proportional to $k^{1/2}$. However, these contributions are small compared to $\Gamma_{\mathbf{k}}^{222}$, $\Delta_{\mathbf{k}}^{222}$. We consider, e.g., decay processes. The mass operator $\Sigma_{\mathbf{k}}^{212}$ consists of the following diagrams:



The first graph contains the Green function $G_{0\mathbf{k}+\mathbf{k}_1}^2 = (\omega_{\mathbf{k}}^2 + \omega_{\mathbf{k}_1}^2 - \omega_{\mathbf{k}+\mathbf{k}_1}^1)^{-1}$. The corresponding resonance surface is almost a sphere with cross section $\propto (c_2 k / c_1)^2$ so that the integral is small because that factor is small. A similar situation occurs for the second diagram and also for the Cherenkov processes.

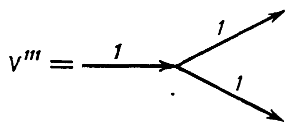
We briefly summarize the results. A first sound propagating in turbulent helium undergoes damping and dispersion caused by the interaction with the wave fields. The largest contribution comes here from the decay processes due to which the quantities $\Gamma_{\mathbf{k}}^1$ and $\Delta_{\mathbf{k}}^1$ have a square-root dependence on the wave number k and can easily be observed experimentally. The damping and dispersion of second sound are first and foremost due to nonlinear eigenprocesses in the second mode. The quantities $\Gamma_{\mathbf{k}}^2$ and $\Delta_{\mathbf{k}}^2$ are then linear in k and can also easily be distinguished from the normal viscous damping $\gamma_{\mathbf{k}}^2 \propto \gamma^2 \mathbf{k}^2$.

We now consider the omitted higher order diagrams in the series (8),(9) for Σ_q^ν, Φ_q^ν. We write, e.g., down the diagrams of second order in the square of the interaction $|V|^2$ for the operator Σ_q^ν:

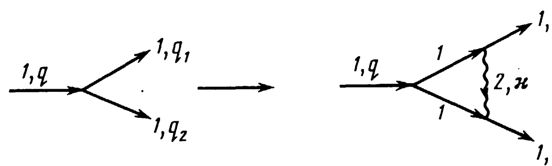


(39)

It is clear from (39) that taking higher order diagrams into account reduces normally to a greater complication of one of the vertices. We give estimates for the more complicated vertices. We consider as example the case when the vertex



is made more complicated in the following way:



(40)

The analytical expression for the more complicated vertex ΔV^{111} has the following form:

$$ \Delta V_{qq_1q_2}^{111} = \delta(q - q_1 - q_2) \tag{41} $$
$$ \times \int \frac{V^{111}|V^{112}|^2 n_\varkappa{}^2\, d\varkappa}{(\omega_{\mathbf{k}_1}{}^1 + \omega_\varkappa{}^2 - \omega^1_{\mathbf{k}_1+\varkappa} + i\Gamma)(\omega_{\mathbf{k}_2}{}^1 - \omega_\varkappa{}^2 - \omega^1_{\mathbf{k}_2-\varkappa} + i\Gamma)}. $$

To estimate the integral we use the fact that due to the presence of the δ-function $\delta(q - q_1 - q_2)$ the momenta $\mathbf{k}_1$ and $\mathbf{k}_2$ are "clamped" to the direction of the vector $\mathbf{k}$ (Fig. 6). The main contribution to the integral comes from the resonance surfaces—regions where each of the factors in the denominator vanish. These regions are Cherenkov spheres (see Ref. 2) which touch one another. The integral over each of such surfaces is a quantity of the order of the Cherenkov damping $\Gamma_{\mathbf{k}}^{122}$, while at the same time the other bracket in the denominator gives a factor of the order $1/\omega_{\mathbf{k}}^1$. We have thus for the correction $\Delta V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}^{111}$ to the vertex $V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}^{111}$

$$ \Delta V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}^{111} \approx V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}^{111}\Gamma_{\mathbf{k}}^{112}/\omega_{\mathbf{k}}{}^1. \tag{42} $$

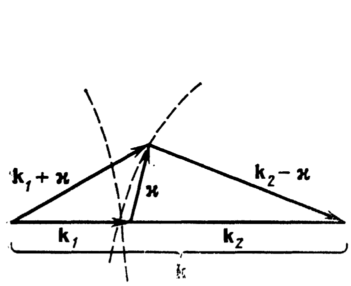


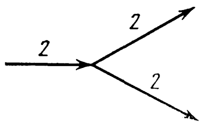
FIG. 6

The situation described here is typical for the cases when either the vertex V^{111} (V^{222}) is made more complicated by another sound mode taking part, or when the decay and Cherenkov vertices are made more complicated. As a result the relative correction to the listed vertices does not exceed the quantity $\xi = \Gamma_{\mathbf{k}}^{\nu}/\omega_{\mathbf{k}}^{\nu} \ll 1$ where $\Gamma_{\mathbf{k}}^{\nu}$ is the total nonlinear damping while the series (8),(9) are expansions in the small parameter ξ.

We now consider the case when the vertex V^{111} is made more complicated by the n_q^1, G_q^1 lines of first sound. In that case the calculations made to estimate the quantity $\Delta V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}^{111}$ are invalid as because of the linearity of the dispersion law the resonance surfaces are the same and merge into a narrow "tube" close to the vector **k** with a width which is determined by the damping Γ. An estimate of the integrals such as (41), where we make the substitutions $\omega^2 \to \omega^1$, $\mathbf{V}^{112} \to \mathbf{V}^{111}$ performed in the spirit of Ref. 9 shows that the relative correction $\Delta V^{111}/V^{111}$ to the vertex V^{111} equals

$$\Delta V^{111}/V^{111} \approx \Gamma_{\mathbf{k}}^{111}/\Gamma_{\mathbf{k}}^{1}. \qquad (43)$$

Here $\Gamma_{\mathbf{k}}^{111}$ is the first sound damping due to the non-linear eigenprocesses in the first mode, $\Gamma_{\mathbf{k}}^{1}$ the total nonlinear first sound damping. As we have already mentioned the main contribution to $\Gamma_{\mathbf{k}}^{1}$ comes from the decay processes and hence the ratio $\Gamma_{\mathbf{k}}^{111}/\Gamma_{\mathbf{k}}^{1}$ is small and with it also the corrections to the vertex. An exception in that sense is the vertex

$V^{222} =$ 

the relative correction $\Delta V^{222}/V^{222}$ to which is a quantity of order unity. Formally this follows from the fact that the main contribution to the damping comes from the eigenprocesses in the second-sound mode. This correction is, however, small at temperatures T_α where the second sound nonlinearity coefficient α_2 [see (29)] is small, i.e., near $T_\alpha = 0.95$ K, $T_\alpha = 1.88$ K. Thus, in the vicinity of these temperatures the terms dropped in the expansions (8),(9) when the KE were derived contain small corrections which justifies the application of the KE method.

It seems that in the other temperature regions, where α_2 is not a small quantity, there are also the grounds for using the kinetic equations to study stochastic wave fields. We expound our considerations basing ourselves upon a qualitative theory of turbulence of waves with a linear dispersion law (in ordinary liquids), which was developed in Refs. 10, 11 where it was shown that stochastic waves have a tendency to form separate narrow tubes ("jets") in **k**-space. A spectral distribution for the energy density $\varepsilon_{\mathbf{k}}$ of the form $\varepsilon_{\mathbf{k}} \propto k^{-2}$ is established inside each jet. The difference from the Zakharov-Sagdeev spectrum ($\varepsilon_{\mathbf{k}} \propto k^{-3/2}$) arises because in each jet there is established a rigid correlation between the phases of the different harmonics and leads in the **r**-representation to a steepening of the wave profile and the formation of shock fronts.[3] Quantitative estimates given in Ref. 11 show, however, that the characteristic time for the formation of discontinuities $\tau_{\rm disc}$, which gives a criterion for the time to establish correlations, is comparable to the characteristic time $\tau_{\rm kin}$ following from the KE and serving as a criterion for the time in which the phases are randomized. The problem of three-dimensional turbulence of acoustic waves in classical liquids remains thus open.[11]

The situation is changed in He II where there is an additional wave mode. Coherent wave processes of the cross interaction between the sounds are characterized by times $\tau_{\rm coh}$ which are considerably shorter than the kinetic time $\tau_{\rm kin}$ (this is a consequence of $\Delta\omega_{\mathbf{k}}^{\nu} \ll \omega_{\mathbf{k}}^{\nu}$) and, hence than the time to form discontinuities $\tau_{\rm disc}$ in the separate jets. As a result the nonlinear interaction "inside" the wave modes leading to the occurrence of discontinuities proceeds more slowly than, say, the decay of a wave or Cherenkov emission. The cross-term processes themselves do not prevent the randomization of the phases as the interaction of the waves proceeds at an angle. Formally this is reflected in the fact that the further complication of the vertices with the participation of both modes changes the bare vertex weakly. As a result the processes leading to the phase randomization dominate and this justifies the application of the KE method.

3. CRITERIA, NUMERICAL ESTIMATES

In the calculations performed above we used a number of parameters the relation between which is important for the operation of the calculations; we enumerate them.

1. The nonlinear frequency shift $\Delta\omega_{\mathbf{k}}^{\nu}$ (equal to the nonlinear damping $\Gamma_{\mathbf{k}}^{\nu}$ and also to the reciprocal of the time for the kinetic processes $\tau_{\rm kin}^{-1}$) is much smaller than the frequency of the sounds[4)] $\omega_{\mathbf{k}}^{\nu}$.

2. The viscous boundary of the inertial range (IR) k_- is much larger than the reciprocal of the size of the system L^{-1} ($\approx k_+$) (the region where the source has an effect), $k_+ \ll k_-$.

3. The viscous damping $\gamma_{\mathbf{k}}^{\nu}$ is much smaller than the nonlinear damping $\Gamma_{\mathbf{k}}^{\nu}$.

4. The characteristic time for the coherent processes $\tau_{\rm coh}$ is much smaller than the kinetic times $\tau_{\rm kin}$.

Using simple estimates one can show that the last two conditions are a consequence of the first two. For instance, the viscous boundary of the IR can be estimated from the condition that the energy flux along the spectra equals the viscous dissipation. Bearing in mind that the wave energy is concentrated mainly in the second mode we have

$$\tilde{P}_{\mathbf{k}}^2 = \int \gamma_{\mathbf{k}}^2 4\pi k^2 \omega_{\mathbf{k}}^2 n_{\mathbf{k}}^2 \, dk \approx B c_2 \gamma^2 k_-^{3/2}. \qquad (44)$$

Comparing this expression with (28) we find that as to order of magnitude the viscous boundary k_- equals

$$k_- \approx (B/\rho\gamma^{(2)})^{2/3}. \qquad (45)$$

Further expressing the second sound damping $\Gamma_{\mathbf{k}}^2$ [see (38)] in terms of k_- we find that the condition $\gamma_{\mathbf{k}} \ll \Gamma_{\mathbf{k}}$ is equivalent to the relation $(k/k_-)(k_+/k_-)^{1/2} \ll 1$, i.e., as was to be proved, condition 3 follows from condition 2.

We prove the equivalence of conditions 1 and 4. The characteristic time for the coherent interaction $\tau_{\rm coh}$ is evaluated from the formula $\tau_{\rm coh}^{-1} = |V|a$, where a is the amplitude of the monochromatic wave $a_{\mathbf{k}} = a\delta(\mathbf{k} - \mathbf{k}_0)$. From the formal relation $n_{\mathbf{k}} = a^2\delta(\mathbf{k} - \mathbf{k}_0)$ we have $a^2 = \varepsilon/\omega_{\mathbf{k}}$, where ε is the energy density of the wave and, hence,

$$1/\tau_{\rm coh}^2 = |V|^2\varepsilon/\omega_{\mathbf{k}}. \tag{46}$$

The reciprocal of the time for the kinetic processes $\tau_{\rm kin}$ is as to order of magnitude equal to

$$1/\tau_{\rm kin} = \Gamma_{\mathbf{k}} = \int \frac{|V|^2 n_{\mathbf{k}}\, d\mathbf{k}}{\omega_{\mathbf{k}}} \approx \frac{|V|^2\varepsilon}{\omega_{\mathbf{k}}}\frac{1}{\omega_{\mathbf{k}}}. \tag{47}$$

Comparing (46) and (47) we find the following equation

$$\tau_{\rm kin}^{-1}/\omega_{\mathbf{k}} = (\tau_{\rm coh}^{-1}/\omega_{\mathbf{k}})^2, \qquad \tau_{\rm kin}^{-1}/\tau_{\rm coh}^{-1} = \tau_{\rm coh}^{-1}/\omega_{\mathbf{k}}. \tag{48}$$

The requirement $\omega_{\mathbf{k}} \gg \tau_{\rm kin}^{-1}$ follows thus from the conditions $\tau_{\rm coh}^{-1} \ll \omega_{\mathbf{k}}$, $\tau_{\rm kin}^{-1} \ll \tau_{\rm coh}^{-1}$. The last inequality shows the equivalence of conditions 1 and 4.

One can easily explain the physical meaning of the criteria 1 and 4 using a quantum-mechanical analogy, i.e., the representation of wave fields by a system of quasiparticles. The requirement $\tau_{\rm coh} \ll \tau_{\rm kin}$ is then equivalent to the fact that the time for interaction of the particles is much shorter than the free flight time which is important for the derivation of the KE.

It has been often shown that the possibility of the KE method is connected with the randomization of phases of separate waves. Phase relations are conserved in coherent interaction processes so that in order that they are destroyed it is necessary that quasiparticles experience successive collisions with other sound quanta which are random with respect to them. In other words, it is necessary that there be sufficiently many other quasiparticles along a mean free path $l_{mfp} = c\tau_{\rm kin}$. Using the fact that the number of quasiparticles per unit volume is proportional to the phase-space volume $(\Delta k)^3 \approx k^3$ we get the necessary condition

$$(ck/\tau_{\rm kin}^{-1}) = \omega_{\mathbf{k}}/\Gamma_{\mathbf{k}} \gg 1. \tag{49}$$

The condition $\Gamma_{\mathbf{k}} \ll \omega_{\mathbf{k}}$ is thus necessary from the physical point of view for the randomization of the phases.

If condition 1 requires for its fulfillment that there is a low level of nonlinearity (i.e., small A and B), in contrast to this, condition 2 requires large amplitudes as is clear from Eq. (45). To elucidate the simultaneous fulfillment of these conditions we express the damping Γ and the quantity k_- in terms of the external parameters of the problem. The simultaneous fulfillment of criteria 1 and 2 leads to the following chain of inequalities:

$$\gamma/k_+^{3/2} \ll B \ll \omega_{\mathbf{k}} k_+^{1/2}/k. \tag{50}$$

Substituting into (50) $\gamma^2 = 10^{-4}$ cm^2/s, $k_+ = L^{-1} = 10^{-1}$ cm^{-1}, $c = c_2 = 2\times 10^3$ cm/s, we see that the right-hand side is larger than the left-hand side by a factor $c_2 L/\gamma^2 \approx 10^8$. This estimate verifies the existence of a sufficiently wide margin for the inertial range.

In reality there is, apart from the requirement $\Gamma_{\mathbf{k}} \ll \omega_{\mathbf{k}}$, yet another, often technical, restriction on the increase in the AI intensity. Indeed, the wave energy is dissipated in the volume and the possibility to support a stationary situation is limited by the technical possibilities to remove the heat from the system. For instance, when the heat is removed by pumping away the vapor, realistic fluxes are of the order of magnitude of 1 W/cm^2. For a volume $L^3 \approx 10^3$ cm^3 we see thus that $\widetilde{P}_{\mathbf{k}} \sim 0.1$ W/cm^3. In accordance with Eq. (28) we find for the amplitude B the value $B = (1 \text{ to } 10)$ g/cm$^{5/2}\cdot$s. Substituting this value into (45) we find that the viscous limit k_- of the IR is of the order of magnitude 10^4 cm^{-1}, i.e., from this point of view there is a sufficiently large margin for the inertial range.

In conclusion the author expresses his gratitude to the participants in the Bakuriani colloquium for discussions of this paper and also to V. V. Lebedev who made a number of comments on reading a preliminary variant of this paper.

[1)]The notation used in this paper corresponds to Ref. 2.

[2)]Generally speaking, the derivation of the KE used in the cited papers refers to the case of a single wave mode ($\nu = \pm 1$). However, when $\Delta\omega_{\mathbf{k}}^{\nu} \ll \omega_{\mathbf{k}}^{\nu}$ this derivation applies to our case. Indeed, the terms in the tensors n, G which are off-diagonal in the indices ν are proportional to the quantity $\delta^{\Gamma}(\omega_{\mathbf{k}_1}^{\nu_1} - \omega_{\mathbf{k}_2}^{\nu_2})$, where δ^{Γ} is a δ-function which is smeared out by allowance for interactions. Therefore, under the conditions $\Delta\omega_{\mathbf{k}}^{\nu} \ll \omega_{\mathbf{k}}^{\nu}$ one can neglect the off-diagonal terms after which the derivation of the KE is the same as the one expounded in Refs. 4, 5 with only that difference that there are added terms corresponding to cross interaction between the sounds.

[3)]It is well known (see, e.g., Ref. 6) that one of the main requirements which allows the possibility to use the KE method is that there be no correlations between the phases of separate waves.

[3)]Generally speaking, there are several different times $\tau_{\rm kin}$ (and also coherent times $\tau_{\rm coh}$) which depend on the actual form of the nonlinear processes. However, they are all, on the one hand, described by similar formulae and, on the other hand, they are comparable as to order of magnitude. Therefore, in the following calculations we shall not actually define these parameters.

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