Two-dimensional conformal symmetry and critical four-spin correlation functions in the Ashkin-Teller model

Al. B. Zamolodchikov

Scientific Council on General Cybernetics Problems, USSR Academy of Sciences
(Submitted 11 November 1985)
Zh. Eksp. Teor. Fiz. **90**, 1808–1818 (May 1986)

An explicit formula is obtained for the conformal-block four-point functions which arise in the theory of Virasoro algebras with central charge $C = 1$, when all four operators are assumed to have the same dimension $\delta = 1/16$. Using a particular block, we construct a one-parameter family of four-point correlation functions which satisfy crossing-symmetry relations. It is proposed that these functions describe critical correlations between four spins in the Ashkin-Teller model along the phase-transition line.

1. INTRODUCTION

The spatial symmetries of a two-dimensional conformally-invariant field theory are described by a Virasoro algebra (actually by the direct product of two such algebras—one "right-handed" and one "left-handed").[1,2] The state space can be classified according to representations of the Virasoro algebra with central charge C, which is a parameter of the theory. In Ref. 3, a way was suggested to construct a solution to the conformally-invariant field theory problem by combining the conformal invariance condition with the requirement that the operator algebra be associative (which is equivalent to crossing-symmetry conditions on the vacuum Green's functions). In order to carry out this "conformal bootstrap" program, four-point vacuum expectation values can be constructed out of "conformal blocks" (which represent the contribution of all states from a specific conformal class to one of the Green's function channels), combined in such a way as to satisfy the crossing symmetry of this function. This program can be explicitly carried out for the so-called "minimal models," [1] where the conformal blocks are solutions to linear differential equations and admit closed-form integral representations[4] while the set of conformal-invariant operators which enter into the algebra is finite.

In section 2 of this paper, we obtain explicit expressions for the conformal block function when the conformal theory has $C = 1$ and an "external" operator dimension of $\delta = 1/16$ for any value of the "intermediate" dimension Δ. It turns out that in the crossed channels this block does not exhibit a simple power-law asymptotic dependence, as is the case for minimal models, but also contains a logarithm which corresponds to a continuous spectrum of cross-dimensionality. Thus, the crossing-symmetric four-point Green's function cannot be built out of a finite number of such blocks; this implies that the algebra must contain an infinite set of invariant operators which combine in such a way that power-law asymptotic behavior in the crossed channels can be recovered after infinite summations.

In section 3, we derive a single-parameter family of such Green's functions constructed out of these blocks and satisfying the crossing-symmetry relations. When acting as intermediate states, these functions include a combination of an arbitrary number of conformal-invariant fermions from the massless Thirring model, so that the family can be parametrized by the coupling constant of this model. It is interesting that for irrational values of the coupling constant the "intermediate" dimensionality spectrum Δ ($\overline{\Delta}$) for each of the two Virasoro algebras consists of the everywhere dense interval $(0, \infty)$, while the spectrum of scaling dimensionality is discrete; in addition, there appear in the operator decomposition invariant fields with arbitrary integral (or half-integral) spin. For rational g, the spectra Δ and $\overline{\Delta}$ are discrete, and the Green's functions can be constructed from a finite set of blocks corresponding to contributions from operators with nonintegral dimensions. We note that the latter dimensions do not coincide with the conformal ones, and are apparently associated with the presence of higher symmetries of some sort in the rational-g theories. For example, the $g = 1/5$ case corresponds[5] to the Z_4-symmetric self-dual theory described in Ref. 5, where a symmetry generated by parafermion currents is present, while the case $g = -1/3$ corresponds to the conformal theory of the algebra investigated by Kac and Moody, which is invariant under $SU(2)$.

In the isotropic Ashkin-Teller model,[6] a phase transition line appears, along which the critical index varies continuously.[7,8] In this model one can introduce fermion variables whose critical correlations along the phase transition line are described by the massless Thirring model.[9] Apparently, the four-point functions obtained in section 3 describe correlations of four spin variables σ_1 or σ_2 (along with the corresponding dual variables μ_1 and μ_2) on this line. This hypothesis is confirmed by comparing with certain points along this line at which these correlation functions are well-known.

2. THE CONFORMAL BLOCK WITH $C=1$ AND $\delta=1/16$

The conformal block $F(\Delta,\delta_i,C,x)$ is the contribution of all states which belong to a particular continuous representation of the Virasoro algebra

$$[L_m L_n] = (m-n) L_{m+n} + C(m^3-m)\delta_{m+n}/12 \tag{2.1}$$

with principal weight Δ, to the intermediate channels of the

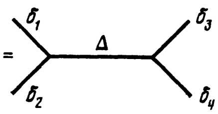
 0038-5646/86/051061-06$04.00

vacuum expectation value $\langle V_{\delta_4}(\infty) V_{\delta_3}(1) V_{\delta_2}(x) V_{\delta_1}(0)\rangle$ of four invariant operators possessing the following transformation properties relative to the generators (2.1):

$$[L_n,\ V_\delta(z)] = z^{n+1} dV_\delta(z)/dz + \delta(n+1) z^n V_\delta(z). \qquad (2.2)$$

By convention, this conformal block is represented by the diagram

$$F(\Delta, \delta_i, C, x) = \qquad (2.3)$$



The representation of the algebra (2.1) with principal weight Δ can be constructed out of differential operators in the space of functions of an infinite number of variables $\Psi(y_1, y_2, \ldots)$ in the following fashion:

$$L_k = \sum_{l=1}^{\infty} l y_l \frac{\partial}{\partial y_{l+k}} - \frac{1}{4}\sum_{l=1}^{k-1} \frac{\partial^2}{\partial y_l\, \partial y_{k-l}} + (i\lambda + \mu k)\frac{\partial}{\partial y_k}, \quad k>0,$$

$$L_0 = \Delta + \sum_{l=1}^{\infty} l y_l \frac{\partial}{\partial y_l}, \qquad (2.4)$$

$$L_k = \sum_{l=1}^{\infty} (l-k) y_{l-k} \frac{\partial}{\partial y_l} - \frac{1}{4}\sum_{l=1}^{k-1} y_l y_{k-l} + (i\lambda + \mu k) y_k, \quad k<0.$$

The parameters λ and μ are connected with C and Δ by the relations:

$$C = 1 + 24\mu^2, \quad \Delta = \lambda^2 + \mu^2.$$

The basis for this space which consists of finite monomial functions $\Psi_{n_1, n_2 \ldots} = y_1^{n_1} y_2^{n_2} \ldots y_l^{n_l}$ is the so called "oscillator" basis. A scalar product in this space is determined by the following functional integral:

$$\langle \Phi, \Psi \rangle = \int \Phi^*(\{y_l^*\}) \Psi(\{y_l\}) \times \exp\Big(-\sum_{l=1}^{\infty} 2l y_l y_l^*\Big) \prod_{l=1}^{\infty} \frac{dy_l\, dy_l^*}{2\pi}. \qquad (2.5)$$

Relation (2.2) leads to the following system of equations for the wave function of the state $V_{\delta_2}(x) V_{\delta_1}(0)|0\rangle = \Phi(y_1, y_2, \ldots | x)$:

$$x\partial\Phi/\partial x = (L_0 - \delta_1 - \delta_2)\Phi,$$
$$x^{k+1}\partial\Phi/\partial x = [L_k - (k+1)x^k \delta_2]\Phi, \quad k>0. \qquad (2.6)$$

(Out of this system, only the $k=1$ and $k=2$ equations are independent; the rest follow from them by virtue of (2.1).) The first equation implies that Φ actually is a function of the variables $\eta_l = x^l y_l$:

$$\Phi(y_1, y_2, \ldots | x) = x^{\Delta - \delta_1 - \delta_2} \Phi(\eta_1, \eta_2, \ldots).$$

The system of equations (2.6) for the wave function Φ with the representation of the operators L_k in the form (2.4) was first given and discussed by A. M. Polyakov.

Let us seek a solution to (2.6) in the form

$$\Phi = x^{\Delta - \delta_1 - \delta_2} \exp\Big\{\sum_{l=1}^{\infty} A_l \eta_l + \sum_{l,l'=1}^{\infty} B_{l,l'} \eta_l \eta_{l'}\Big\} \qquad (2.7)$$

for some A and B. Then (2.6) leads to a set of constraints on A and B:

$$l(B_{l+k,l'} - B_{l,l'}) + l'(B_{l,k+l'} - B_{l,l'}) = \sum_{p=1}^{k-1} B_{l,p} B_{k-p,l'},$$
$$l(A_{l+k} - A_l) - \sum_{p=1}^{k-1} A_{k-p} B_{p,l} + 2(i\lambda + \mu k) B_{k,l} = 0, \qquad (2.8)$$
$$(i\lambda + \mu k) A_k - \frac{1}{4}\sum_{p=1}^{k-1} (A_p A_{k-p} + 2B_{p,k-p}) = \Delta + k\delta_1 - \delta_2.$$

As shown by A. B. Zamolodchikov, this system has a solution only if $\mu = 0$, which corresponds to $C = 1$ and $\delta_1 = \delta_2 = 1/16$. In this case the solution takes the form

$$B(u,v) = -uv[(1-u)(1-v)]^{-1/2}[(1-u)^{1/2} + (1-v)^{1/2}]^{-2}/2,$$
$$A(u) = -2i\lambda[(1-u)^{-1/2} - 1]. \qquad (2.9)$$

Here we introduce the generating functions

$$B(u,v) = \sum_{l,l'=1}^{\infty} B_{l,l'} u^l v^{l'}, \quad A(u) = \sum_{l=1}^{\infty} A_l u^l. \qquad (2.10)$$

Taking (2.5) into account, we can write the conformal block $F(\Delta, 1/16, 1, x)$ in the form

$$F\Big(\Delta, \frac{1}{16}, 1, x\Big) = x^{\Delta - 1/8} \det{}^{-1/2}(1 - 4G^2) \exp(h(1-2G)^{-1} h), \qquad (2.11)$$

where

$$G_{l,l'} = x^{(l+l')/2} \frac{B_{ll'}}{2(ll')^{1/2}}, \quad h_l = x^{l/2} \frac{A_l}{(2l)^{1/2}}.$$

Using (2.9), it is not difficult to show that the operator $G_{l,l'}$ has a spectrum which coincides with that of the following eigenvalue problem with spectral parameter λ':

$$\lambda' W(t) = -\frac{xt}{2(1-xt)^{1/2}} \oint_C \frac{du\, W(u)}{2i\pi u[(1-xt)^{1/2} + (1-1/u)^{1/2}]}, \qquad (2.12)$$

where the contour C encloses the points 0 and 1, and

$$h(1-2G)^{-1} h = 2\Delta \oint_C \frac{du}{2i\pi u} Q(u) \ln\Big[4u \frac{1 - (1-1/u)^{1/2}}{1 + (1-1/u)^{1/2}}\Big], \qquad (2.13)$$

while $Q(t)$ satisfies the inhomogeneous equation

$$Q(t) - \frac{xt}{(1-xt)^{1/2}} \oint_C \frac{du\, Q(u)}{2i\pi u[(1-xt)^{1/2} + (1-1/u)^{1/2}]} = \frac{1}{(1-xt)^{1/2}} - 1. \qquad (2.14)$$

In these equations, it is convenient to make the elliptic-function substitution

$$\xi = \frac{1}{2}\int_0^t \frac{dt}{[t(1-t)(1-xt)]^{1/2}}, \quad t = \mathrm{sn}^2 \xi, \qquad (2.15)$$

with modulus $k = x^{1/2}$. Then

$$\lambda' W(\xi) = \frac{x\,\mathrm{sn}^2\xi}{\mathrm{dn}\,\xi}\int_0^{2K}\frac{\mathrm{cn}\,\eta\,\mathrm{dn}\,\eta W(\eta)\,d\eta}{2i\pi\,\mathrm{sn}\,\eta[\mathrm{dn}\,\xi-\mathrm{dn}(\eta+iK')]},$$

$$Q(\xi)+\frac{2x\,\mathrm{sn}^2\xi}{\mathrm{dn}\,\xi}\int_0^{2K}\frac{\mathrm{cn}\,\eta\,\mathrm{dn}\,\eta Q(\eta)\,d\eta}{2i\pi\,\mathrm{sn}\,\eta[\mathrm{dn}\,\xi-\mathrm{dn}(\eta+iK')]}$$

$$=\frac{1}{\mathrm{dn}\,\xi}-1, \tag{2.17}$$

where $K'(x) = K(1-x)$, while $K(x)$ is the complete elliptic integral of the first kind:

$$K(x)=\frac{1}{2}\int_0^1\frac{dt}{[t(1-t)(1-xt)]^{1/2}}. \tag{2.18}$$

From equation (2.16), it is clear that $W(\xi)$ is periodic with period $2K$, while under a shift of $2iK'$ $W(\xi)$ satisfies the relation

$$\lambda'[W(\xi+2iK')+W(\xi)]+W(\xi+iK')=0. \tag{2.19}$$

In addition, $W(\xi)$ has a double zero for $\xi = 0$, simple zeroes for $\xi = ink'$ (n an integer) and poles at $\xi = K + inK'$ ($n\neq 0$). Then we can write

$$W_n(\xi)=\frac{\mathrm{sn}\,\xi}{\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi}\sin\frac{\pi n\xi}{K},\qquad n=1,2,\ldots, \tag{2.20}$$

and it follows from (2.19) that

$$\lambda_n'=-1/2\cos\pi n\tau,\qquad \tau=iK'/K, \tag{2.21}$$

so that

$$\det{}^{-1/2}(1-4G^2)=\prod_{n=1}^{\infty}\frac{1+q^{2n}}{1-q^{2n}}. \tag{2.22}$$

The quantity $q = e^{i\pi\tau}$ is connected with x by the relation

$$x=\theta_2^4(q)/\theta_3^4(q). \tag{2.23}$$

Here we use the familiar θ-series

$$\theta_3(q)=\sum_{n=-\infty}^{\infty}q^{n^2},\qquad \theta_2(q)=\sum_{n=-\infty}^{\infty}q^{(n+1/2)^2}. \tag{2.24}$$

An analogous investigation of equation (2.27) leads to the solution

$$Q(\xi)=\mathrm{sn}\,\xi Z(\xi)/\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi, \tag{2.25}$$

where Z is the Jacobi zeta-function:

$$Z(\xi)=\frac{2\pi}{K}\sum_{n=1}^{\infty}\frac{q^{2n-1}\sin(\pi\xi/K)}{1-2q^{2n-1}\cos(\pi\xi/K)+q^{4n-2}}. \tag{2.26}$$

Evaluating the integral (2.13) gives

$$h(1-2G)^{-1}h=\Delta\ln(16q/x), \tag{2.27}$$

so that finally

$$F\left(\Delta,\frac{1}{16},1,x\right)=(16q)^{\Delta}[x(1-x)]^{-1/8}\theta_3^{-1}(q). \tag{2.28}$$

We should note the following features of the conformal block function so obtained:

a) The simple exponential dependence on the "intermediate" dimension Δ:

$$F\left(\Delta,\frac{1}{16},1,x\right)=(16q)^{\Delta}F_0(x). \tag{2.29}$$

It is well-known that in the general case the conformal block function has a pole in Δ at points corresponding to degeneracy of the conformal model. These values are determined by the Kac formula,[10] and for $C = 1$ they are given by the formula $\Delta_n = n^2/4$; $n = 0,1,2,\ldots$. However, for $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 1/16$ the residues at all these poles reduce to zero (the formula for these residues is given in Ref. 11) and the conformal block is an entire function of Δ.[(1)] We note that in this case the exponential dependence on Δ is dictated by the existence of the quasiclassical limit (see Ref. 1), while a basis for this can be calculated in the following way: in the quasiclassical limit, when the parameters C, Δ, and δ_i simultaneously go to infinity in such a way that their ratios are fixed, the conformal block function has an exponential asymptotic limit

$$F(\Delta,\delta_i,C,x)\sim\exp\left[-\frac{C}{24}S_{cl}\left(\frac{\Delta}{C},\frac{\delta_i}{C},x\right)\right], \tag{2.30}$$

while S_{cl} is connected with properties of the monodromic differential equation of the second kind:

$$\psi''(z)=\frac{1}{4}f(z)\psi(z),$$

$$f(z)=\frac{\lambda_1^2-1}{z^2}+\frac{\lambda_2^2-1}{(z-x)^2}+\frac{\lambda_3^2-1}{(1-z)^2}$$

$$-\frac{\lambda_4^2-\lambda_1^2-\lambda_2^2-\lambda_3^2}{z(1-z)}+\frac{x(1-x)\mathscr{C}(x)}{z(z-x)(1-z)}, \tag{2.31}$$

where $\delta_i = (C/24)(1-\lambda_i^2)$, $\Delta = (C/24)(1-\Lambda^2)$, and the dependence of the accessory coefficient on x is selected in such a way that the monodromic substitutional invariant corresponding to a circuit around the points 0 and x is constant and equals $2\cos 2\pi\Lambda$. In this case,

$$\mathscr{C}(x)=\partial S_{cl}(x)/\partial x. \tag{2.32}$$

Let $\Delta \gg C, \delta$. Then $\Lambda \gg 1$ and the accessory coefficient must be large compared with unity. In equation (2.31), we can use the short-wavelength approximation

$$\psi(z)\sim\exp\left\{\pm\frac{1}{2}\int^z\left[\frac{x(1-x)\mathscr{C}}{t(1-t)(1-xt)}\right]^{1/2}dt\right\}, \tag{2.33}$$

which at once leads to the asymptotic estimate (for $\Lambda\gg 1$):

$$S_{cl}=\Lambda^2(i\pi\tau+\mathrm{const}). \tag{2.34}$$

b) Modular transformations of the variable τ correspond to various anharmonic-group substitutions in the variable x. In particular, the transition to the crossed channel $x\to 1-x$ corresponds to the substitution $\tau\to\tau' = -\tau^{-1}$. It is not hard to see that

$$F_0(\tau)=(-i\tau')^{-1/2}F_0(\tau'), \tag{2.35}$$

i.e., for $x\to 1$,

$$F(\Delta,1/16,1,x)\sim(1-x)^{-1/8}\ln^{-1/2}(1-x),$$

so that near the singular points corresponding to 1 and ∞, and also near zeroes on other sheets of the Riemann surface of the function $F(x)$, the asymptotic limits contain logarithms.

3. CROSSING-SYMMETRIC GREEN'S FUNCTIONS

Let the operator expansion of the two fields of dimension 1/16, which we will denote by $\sigma(x,\bar{x})$ and assume are scalars, take the form

$$\sigma(x,\bar{x})\sigma(0,0)=\sum_i C_i x^{\Delta_i-1/8}\bar{x}^{\bar{\Delta}_i-1/8}[V_i(0,0)], \tag{3.1}$$

where Δ_i and $\bar{\Delta}_i$ correspond to the right- and left-dimensions of the invariant operator V_i while the symbol $[V_i]$ denotes the contribution of the conformal class[1] corresponding to V_i (because the spin of this operator is local, $\Delta_i - \bar{\Delta}_i$ must be an integer or half-integer). Then the simple exponential dependence (2.29) of the conformal block on the parameter Δ implies that the four-point Green's function can be written in the form

$$\langle\sigma(\infty,\infty)\sigma(1,1)\sigma(x,\bar{x})\sigma(0,0)\rangle=F_0\bar{F}_0\sum A_i q^{\Delta_i}\bar{q}^{\bar{\Delta}_i}, \tag{3.2}$$

where $A_i = 16^{\Delta_i+\bar{\Delta}_i}C_i^2$ are certain coefficients.

According to the conformal bootstrap program, we must select the dimensions $(\Delta_i, \bar{\Delta}_i)$ and constants A_i in such a way that they guarantee the necessary properties of crossing symmetry of the Green's function (3.2). In trying to obtain such a Green's function, we will rely on the properties of the four-spin correlation function for the isotropic Ashkin-Teller model along its second-order phase transition line (see Refs. 7, 8 and the Appendix). Therefore, in constructing the intermediate states in the Green's function, we are naturally led to the space of states of the massless Thirring model (see Ref. 9 and citations there). This model describes an interacting conformal-invariant two-component Fermi field $\psi(z,\bar{z}) = (\psi_1,\psi_2)$ which satisfies the equations of motion

$$\frac{\partial}{\partial\bar{z}}\psi_1=g:\psi_2^+\psi_2\psi_1:,\qquad \frac{\partial}{\partial z}\psi_2=g:\psi_1^+\psi_1\psi_2:, \tag{3.3}$$

where $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$ are coordinates in the two-dimensional space, and the symbol : : denotes normal ordering of products of fields taken at the same point. The fermion field in this model can be represented in the following way:

$$\begin{aligned}\psi_1(z,\bar{z})&=:\exp[-2i\alpha\phi(z)-2i\beta\bar{\phi}(\bar{z})]:,\\ \psi_2(z,\bar{z})&=:\exp[2i\beta\phi(z)+2i\alpha\bar{\phi}(\bar{z})]:,\\ \psi_1^+(z,\bar{z})&=:\exp[2i\alpha\phi(z)+2i\beta\bar{\phi}(\bar{z})]:,\\ \psi_2^+(z,\bar{z})&=:\exp[-2i\beta\phi(z)-2i\alpha\bar{\phi}(\bar{z})]:,\end{aligned} \tag{3.4}$$

where $\varphi(z,\bar{z}) = \phi(z) + \bar{\phi}(\bar{z})$ is a free massless Bose field normalized so that

$$\begin{aligned}\langle\phi(z)\phi(z')\rangle&=-{}^1/_2\ln(z-z'),\\ \langle\bar{\phi}(\bar{z})\bar{\phi}(\bar{z}')\rangle&=-{}^1/_2\ln(\bar{z}-\bar{z}'),\end{aligned} \tag{3.5}$$

while α and β are parameters which satisfy the relation

$$\alpha^2-\beta^2={}^1/_2 \tag{3.6}$$

and are connected with the coupling constant g of the Thirring model:

$$\alpha=[2(1-g^2)]^{-1/2},\quad \beta=g[2(1-g^2)]^{-1/2}. \tag{3.7}$$

Taking all this into account, let us write down the following expression:

$$G(x,\bar{x})=F_0\bar{F}_0\sum_{k,l} q^{(\alpha k+\beta l)^2}\bar{q}^{(\alpha l+\beta k)^2}, \tag{3.8}$$

where the pair of numbers k and 1 take on all integer values of the same parity (the latter condition corresponds to even numbers of fermions in the intermediate state). Using the parameter

$$\beta_\pm=\alpha\pm\beta, \tag{3.9}$$

we can write

$$G(x,\bar{x})=F_0\bar{F}_0\sum_{m,n} q^{(\beta_+m+\beta_-n)^2}\bar{q}^{(\beta_+m-\beta_-n)^2}, \tag{3.10}$$

where m and n now run over all integers. If we introduce the following function of two variables, Θ,

$$\Theta(v|\hat{t})=\sum_{(M)}\exp\{i\pi M^T\hat{t}M+2i\pi M^Tv\}, \tag{3.11}$$

where $M = (m_1,m_2)^T$ is an integer-valued column vector, $v = (v_1,v_2)^T$ are argument column vectors and $\hat{t}$ is a 2×2 matrix with positive definite imaginary part, then the function (3.10) can be expressed in the form

$$G(x,\bar{x})=F_0\bar{F}_0\Theta(0|\hat{t}), \tag{3.12}$$

$$\hat{t}=\begin{pmatrix}\beta_+^2(\tau-\bar{\tau}) & (\tau+\bar{\tau})/2\\ (\tau+\bar{\tau})/2 & \beta_-^2(\tau-\bar{\tau})\end{pmatrix}. \tag{3.13}$$

Using the Poisson sum formula

$$\Theta(v|\hat{t})=\det{}^{1/2}(-i\hat{t}')\exp(i\pi v^T\hat{t}'v)\Theta(\hat{t}'v|\hat{t}'), \tag{3.14}$$

where $\hat{t}' = -\hat{t}^{-1}$, we can show that $G(x,\bar{x})$ is fully crossing-symmetric.

It is natural to assume that this Green's function corresponds to the correlation function of four spins σ_1 (or σ_2) in the Ashkin-Teller model (see Appendix):

$$G(x,\bar{x})=\langle\sigma_1(\infty,\infty)\sigma_1(1,1)\sigma_1(x,\bar{x})\sigma_1(0,0)\rangle. \tag{3.15}$$

(Henceforth we will omit the operator arguments in the four-point functions and simply write $\langle\sigma_1\sigma_1\sigma_1\sigma_1\rangle$, having in mind the same sequence of arguments as (3.15).)

Analogously we can show that the set of functions

$$\begin{aligned}R_1(x,\bar{x})&=F_0\bar{F}_0\sum_{m,n} q^{(\beta_+m+\beta_-(n+1/2))^2}\bar{q}^{(\beta_+m-\beta_-(n+1/2))^2},\\ R_2(x,\bar{x})&=F_0\bar{F}_0\sum_{m,n}(-)^m q^{(\beta_+m+\beta_-n)^2}\bar{q}^{(\beta_+m-\beta_-n)^2},\\ R_3(x,\bar{x})&=F_0\bar{F}_0\sum_{m,n}(-)^m q^{(\beta_+m+\beta_-(n+1/2))^2}\bar{q}^{(\beta_+m-\beta_-(n+1/2))^2},\end{aligned} \tag{3.16}$$

satisfies the relations of crossing symmetry:

$$\begin{aligned}&R_1(1-x,1-\bar{x})=R_2(x,\bar{x}),\quad R_3(1-x,1-\bar{x})=R_3(x,\bar{x}),\\ &R_1\left(\frac{x}{x-1}\frac{\bar{x}}{\bar{x}-1}\right)=R_1(x,\bar{x}),\quad R_2\left(\frac{x}{x-1},\frac{\bar{x}}{\bar{x}-1}\right)=R_2(x,\bar{x}).\end{aligned} \tag{3.17}$$

This allows us to make the following correspondences:

$$R_1(x,\bar{x})=\langle\sigma_1\sigma_2\sigma_2\sigma_1\rangle,\quad R_2(x,\bar{x})=\langle\sigma_1\sigma_1\sigma_2\sigma_2\rangle,$$
$$R_3(x,\bar{x})=\langle\sigma_1\sigma_2\sigma_1\sigma_2\rangle. \qquad (3.18)$$

It is also possible to write out the functions with an odd number of fermions in the direct channel:

$$\Gamma_1(x,\bar{x})=F_0\bar{F}_0\sum_{m,n}q^{[\beta_+(m+1/2)+\beta_-(n+1/2)]^2}\bar{q}^{[\beta_+(m+1/2)-\beta_-(n+1/2)]^2}, \qquad (3.19)$$

$$\Gamma_2(x,\bar{x})$$
$$=iF_0\bar{F}_0\sum_{m,n}(-)^{m+n}q^{[\beta_+(m+1/2)+\beta_-(n+1/2)]^2}\bar{q}^{[\beta_+(m+1/2)-\beta_-(n+1/2)]^2}.$$

Here,

$$\Gamma_1(1-x,1-\bar{x})=\Gamma_3(x,\bar{x}),\quad \Gamma_2(1-x,1-\bar{x})=-\Gamma_2(x,\bar{x}), \qquad (3.20)$$

$$\Gamma_1\left(\frac{x}{x-1},\frac{\bar{x}}{\bar{x}-1}\right)=\Gamma_2(x,\bar{x}),\quad \Gamma_3\left(\frac{x}{x-1},\frac{\bar{x}}{\bar{x}-1}\right)=\Gamma_3(x,\bar{x}),$$

where

$$\Gamma_3(x,\bar{x})=F_0\bar{F}_0\sum_{m,n}(-)^{m+n}q^{(\beta_+m+\beta_-n)^2}\bar{q}^{(\beta_+m-\beta_-n)^2}. \qquad (3.21)$$

These functions can be related to correlations which include the dual variables:

$$\Gamma_1(x,\bar{x})=\langle\sigma_1\mu_1\mu_1\sigma_1\rangle,\quad \Gamma_2(x,\bar{x})=\langle\sigma_1\mu_1\sigma_1\mu_1\rangle,$$
$$\Gamma_3(x,\bar{x})=\langle\sigma_1\sigma_1\mu_1\mu_1\rangle. \qquad (3.22)$$

We note that the correlation functions R_1 and R_2 in formulae (3.16) correspond to the appearance in the operator expansion of $\sigma_1(x,\bar{x})\sigma_2(0,0)$ of a leading-term scalar operator with dimension $\beta_-^2/4=(1-g)/8(1+g)$; this operator corresponds to "paired spin" correlations $\sigma_1\sigma_2$. If we make the assumption that the duality relation applied to one of the variables $\sigma_{1,2}$ becomes an exact symmetry on the critical line, then we can suppose that the operator $\sigma_2\mu_2$ has dimension $\beta_+^2/4=(1+g)/8(1-g)$ and thus write another set of relations

$$H_1(x,\bar{x})=F_0\bar{F}_0\sum_{m,n}q^{(\beta_+(m+1/2)+\beta_-n)^2}\bar{q}^{(\beta_+(m+1/2)-\beta_-n)^2},$$

$$H_2(x,\bar{x})=F_0\bar{F}_0\sum_{m,n}(-)^{n}q^{(\beta_+m+\beta_-n)^2}\bar{q}^{(\beta_+m-\beta_-n)^2}, \qquad (3.23)$$

$$H_3(x,\bar{x})=F_0\bar{F}_0\sum_{m,n}(-)^{n}q^{(\beta_+(m+1/2)+\beta_-n)^2}\bar{q}^{(\beta_+(m+1/2)-\beta_-n)^2},$$

for the correlation function

$$H_1(x,\bar{x})=\langle\sigma_1\mu_2\mu_2\sigma_1\rangle,\quad H_2(x,\bar{x})=\langle\sigma_1\sigma_1\mu_2\mu_2\rangle,$$
$$H_3(x,\bar{x})=\langle\sigma_1\mu_2\sigma_1\mu_2\rangle. \qquad (3.24)$$

It remains to confirm the following properties of these infinite sums (3.8), (3.16), (3.19) and (3.23).

a) For irrational values of the ratio β_+/β_- (which correspond to irrational values of g) the spectra of right- and left-handed dimensions are everywhere dense on the interval $(0,\infty)$. At the same time, the scaling dimension spectrum is discrete, e.g., in the case of the functions G, R_2, $\bar{\Gamma}_3$ and H_2,

$$d_{mn}=\Delta_{mn}+\bar{\Delta}_{mn}=2\beta_+^2m^2+2\beta_-^2n^2,$$
$$s_{mn}=\Delta_{mn}-\bar{\Delta}_{mn}=2mn. \qquad (3.25)$$

b) For rational values of the ratio β_+/β_- the infinite sums can be expressed by using a finite number of blocks, corresponding to a summation of contributions whose dimensionality is nonintegral. For example, in the simplest case $\beta_+=\beta_-=2^{-1/2}$ (which corresponds to the case $g=0$) we can write

$$G(x,\bar{x})=F_0\bar{F}_0\sum_{m,n}q^{(m+n)^2/2}\bar{q}^{(m-n)^2/2}$$
$$=F_0\bar{F}_0\sum_{k,l}[q^{2k^2}\bar{q}^{2l^2}+q^{(2k+1)^2/2}\bar{q}^{(2l+1)^2/2}]$$
$$=[x\bar{x}(1-x)(1-\bar{x})^{-1/8}$$
$$\times[\theta_3(q^2)\theta_3(\bar{q}^2)+\theta_2(q^2)\theta_2(\bar{q}^2)]/2\theta_3(q)\theta_3(\bar{q}), \qquad (3.26)$$

which is in fact the four-spin correlation function of the Ising model.

The author would like to express his thanks to A. B. Zamolodchikov and V. A. Fateev for their stimulating discussions and comments.

APPENDIX

Here we describe briefly the isotropic Ashkin-Teller model (see Ref. 7), in order to fix notation and confirm certain duality properties of this model.

The Ashkin-Teller model describes the interaction of two spin variables, $\sigma_1=\pm1$ and $\sigma_2=\pm1$, on a planar lattice (which we here assume to be square) with an interaction energy

$$\varepsilon(\{\sigma_1,\sigma_2\})=\sum_{ij}[K(\sigma_1^i\sigma_1^j+\sigma_2^i\sigma_2^j)+L\sigma_1^i\sigma_1^j\sigma_2^i\sigma_2^j], \qquad (A1)$$

where the sum goes over all pairs of nearest neighbors.

The Boltzmann weights corresponding to each bond of the lattice can be conveniently parametrized in the following way:

$$W(\sigma,\sigma')=[1+\gamma_1(\sigma_1\sigma_1'+\sigma_2\sigma_2')+\gamma_2\sigma_1\sigma_1'\sigma_2\sigma_2']/(1+2\gamma_1+\gamma_2), \qquad (A2)$$

where σ and σ' are spins on neighboring sites, while γ_1 and γ_2 are parameters. Performing the standard duality transform on both spin variables (corresponding to the dual variables which we call μ_1 and μ_2), we can verify that the model is self-dual along the line

$$2\gamma_1+\gamma_2=1. \qquad (A3)$$

As is well-known (see Refs. 7, 8), the part of this line between $(\gamma_1,\gamma_2)=(1/3,1/3)$ and $(\gamma_1,\gamma_2)=(1/2,0)$ corresponds to a line of second-order phase transitions, while along this line the critical index varies continuously. Thus, the dimensionality of the fields σ_1 and σ_2 (and also that of the dual fields μ_1 and μ_2) are constant and equal to (1/16, 1/16), while correlations of the spinor fields $\sigma_1\mu_1$ and $\sigma_2\mu_2$

are described by the massless Thirring model.

The duality transformation on one of the variables σ, for example σ_1, leads to the following mapping of the line (A3):

$$(\gamma_1, \gamma_2) \to \left(\frac{1-\gamma_1}{1+\gamma_1}, \frac{\gamma_1-\gamma_2}{1+\gamma_1}\right) \tag{A4}$$

onto itself. Such a symmetry corresponds to the substitution $\mu_2 \leftrightarrow \sigma_2$ in the correlation functions. Thus, at the same time that the variables σ_1 and σ_2 in the original model (A1) are distributed on the exact same lattice sites, σ_1 and μ_2 occupy sites of the various mutually-dual lattices and the interaction between pairs of nearest neighbors $\sigma_1\sigma_1'$ and $\mu_2\mu_2'$ takes place at the intersecting edges of these lattices. One can, however, advance the supposition that at the critical point this difference in the details of the microscopic model no longer exists, so that the symmetry (A4) is an exact symmetry of the correlation function. Apparently, this symmetry corresponds to the exchange $g \to -g$ for the corresponding Thirring model. We note that the fixed point of the transformation (A4), i.e., $(\gamma_1, \gamma_2) = (2^{1/2} - 1, 3 - 2^{3/2})$, corresponds to the breakup of the Askhin-Teller model into two noninteracting Ising models. Thus, the Thirring coupling constant reduces to zero. We will assume the correctness of the symmetry (A4).

[1)]There is yet another case in which all the poles have zero residues: $C = 25$ and $\delta = 15/16$. In this case, the conformal block function has a form related to (2.28):

$$F(\Delta, 15/16, 25, x) = (16q)^{\Delta-1}[x(1-x)]^{-7/8}\theta_3^{-3}(q).$$

[1]A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Nucl. Phys. B241, 333 (1984).
[2]D. Friedan, Z. Qui, and S. H. Shenker, Phys. Rev. Lett. **52**, 1575 (1984).
[3]A. M. Polyakov, Zh. Eksp. Teor. Fiz. **66**, 23 (1974) [Sov. Phys. JETP **39**, 10 (1974)].
[4]V. S. Dotsenko and V. A. Fateev, Nucl. Phys. **B240**, 312 (1984); **B243**, 175 (1985).
[5]A. B. Zamolodchikov and V. A. Fateev, Zh. Eksp. Teor. Fiz. **89**, 320 (1985) [Sov. Phys. JETP **62**, 215 (1985)].
[6]J. Ashkin and E. Teller, Phys. Rev. **64**, 178 (1943).
[7]R. Baxter, Exactly Solved Models in Statistical Mechanics, Academic, London (1982).
[8]L. P. Kadanoff, Ann. Phys. (N.Y.) **120**, 39 (1979); L. P. Kadanoff and A. Brown, Ann. Phys. (N.Y.) **121**, 318 (1979).
[9]A. Wightman, Problemy v relyativistskoi dinamike kvantovannykh polei (Problems in Relativistic Quantum Dynamics), Moscow: Nauka (1968).
[10]V. G. Kac, Lecture Notes in Physics **94**, 441 (1979).
[11]Al. B. Zamolodchikov, Comm. Math. Phys. **96**, 419 (1984).

Translated by F. J. Crowne