Supersymmetry of the Dirac equation in a nonabelian chromomagnetic field

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It is shown that the Dirac equation is supersymmetric both in an abelian stationary and uniform magnetic field and in a nonabelian field of the magnetic type specified by constant potentials in the group $SU(2)$.

The study of different models in the supersymmetric quantum mechanics first proposed by Witten[1] (see also Ref. 2) can shed light on some of the rules of a future, consistent supersymmetric theory. It has been shown recently that the Pauli equation for the electron in an arbitrary two-dimensional abelian magnetic field, and also in a three-dimensional field with a definite parity, is supersymmetric.[3] Under certain conditions supersymmetry is also possessed by the Pauli equation for a particle with spin 1/2 and gyromagnetic ratio g equal to twice that for an ordinary Dirac particle.[4]

In the present paper we shall demonstrate that the property of supersymmetry is also intrinsic to the Dirac equation in a nonabelian chromomagnetic field specified by noncommuting constant potentials.

The Hamiltonian of the Dirac equation for a particle with mass m in an arbitrary nonabelian field of the magnetic type specified by constant potentials in the group $SU(2)$ has the form $(c = \hbar = 1)$

$$\hat{H}_D=\alpha\hat{\mathbf{P}}+\gamma_0 m, \tag{1}$$

where $\hat{\mathbf{P}} = \hat{\mathbf{p}} - e\mathbf{A}_a\tau^a/2$; α and γ_0 are the Dirac matrices; τ^a $(a = 1, 2, 3)$ are the Pauli matrices in the color (isotropic) group $SU(2)$. The components of the field $\mathbf{B}_a$ of the magnetic type in the general case are determined by the potentials $\mathbf{A}_a$ as follows:

$$B_i^a=-{}^1/_2\varepsilon_{ijk}F_{jk}{}^a,$$
$$F_{jk}{}^a=\partial_jA_k{}^a-\partial_kA_j{}^a-e\varepsilon_{abc}A_j{}^bA_k{}^c. \tag{2}$$

We shall consider a stationary and uniform field, oriented along the third axes of the configuration space and group space: $B_i^a = B\delta_{i3}\delta_3^a$. We shall confine ourselves first to the case of zero longitudinal momentum $p_3 = 0$. We introduce the operators

$$\hat{Q}_1={}^1/_2(\Sigma_1\hat{P}_1+\Sigma_2\hat{P}_2),\quad \hat{Q}_2={}^1/_2\gamma_0(\Sigma_2\hat{P}_1-\Sigma_1\hat{P}_2), \tag{3}$$

where Σ_i are the Dirac matrices. It is easily verified that these operators commute with the Dirac Hamiltonian (1), and this indicates the double (or, with allowance for the other degrees of freedom, even) degeneracy of the spectrum of the energy ε of the fermion. The conserved operators (3) also commute with the squared Dirac operator[1)]

$$\hat{H}=\hat{H}_D{}^2-m^2=\hat{P}_1{}^2+\hat{P}_2{}^2-{}^1/_2eB\Sigma_3\tau_3 \tag{4}$$

and form a supersymmetry algebra of the usual form

$$\{\hat{Q}_i\ \hat{Q}_j\}={}^1/_2\delta_{ij}\hat{H},\quad [\hat{Q}_i,\ \hat{H}]=0. \tag{5}$$

Thus, $\hat{Q}_1$ and $\hat{Q}_2$ can be called supercharges. The operators

$$\hat{Q}_\pm=\hat{Q}_1\pm i\hat{Q}_2={}^1/_2(\hat{P}_1\mp i\gamma_0\hat{P}_2)(\Sigma_1\pm i\gamma_0\Sigma_2) \tag{6}$$

are the generators of supersymmetry transformations between bosonic (spatial and color) and fermionic (spin) variables:

$$[\hat{Q}_\pm,\ \hat{x}_1]=-i\hat{\psi}_\pm,\quad [\hat{Q}_\pm,\ \hat{x}_2]=\mp\gamma_0\hat{\psi}_\pm,$$
$$\{\hat{Q}_\pm,\ \hat{\psi}_\mp\}=\hat{P}_1\mp i\gamma_0\hat{P}_2,\quad \{\hat{Q}_\pm,\ \hat{\psi}_\pm\}=0, \tag{7}$$

where

$$\hat{\psi}_\pm={}^1/_2(\Sigma_1\pm i\gamma_0\Sigma_2),\quad \{\hat{\psi}_+,\ \hat{\psi}_-\}=1. \tag{8}$$

The fermion-number operator

$$\hat{f}=[\hat{\psi}_+,\ \hat{\psi}_-]={}^1/_2\gamma_0\Sigma_3 \tag{9}$$

commutes with $\hat{H}_D$ and has eigenvalues $f = \pm 1/2$. Here,

$$[\hat{Q}_\pm,\ \hat{f}]=\mp\hat{Q}_\pm, \tag{10}$$

i.e., $\hat{Q}_+$ ($\hat{Q}_-$) increases (decreases) the fermion number by unity.

We obtain the general case $p_3\neq 0$ by applying the Lorentz-transformation operator

$$\exp(\alpha_3 \operatorname{arch}\gamma), \tag{11}$$

where $\gamma = (1 - v_3^2)^{-1/2}$, v_3 being the lognitudinal velocity of the particle. Then, in place of the operators (3) and (9), we find

$$\hat{Q}_1={}^1/_2\gamma(\Sigma_1\hat{P}_1+\Sigma_2\hat{P}_2)(1-\alpha_3v_3),\quad \hat{Q}_2={}^1/_2\gamma_0(\Sigma_2\hat{P}_1-\Sigma_1\hat{P}_2),$$
$$\hat{f}={}^1/_2(1-v_3{}^2)^{-1/2}\gamma_0(\Sigma_3+\gamma_5v_3), \tag{12}$$

in which the operator $\hat{f}$ is, to within a constant factor, the invariant spin $e\tilde{F}^{\mu\nu}\hat{P}_\mu\hat{S}_\nu$ of the Dirac equation in a magnetic field.[7]

In the case of potentials $\mathbf{A} = -1/2\mathbf{r}\times\mathbf{B}$, in abelian electrodynamics (the group $U(1)$) the ground state ("vacuum") has the following quantum numbers: Landau-level number $n = 0$, longitudinal momentum $p_3 = 0$, and fermion number $f = -1/2$. The vacuum vector Ψ_0 satisfies the conditions

$$\hat{Q}_+\Psi_0=0,\quad \hat{Q}_-\Psi_0=0, \tag{13}$$

whence, by virtue of (5), it follows that the vacuum energy (after subtraction of the rest mass) is equal to zero: $\varepsilon_0^2 = m^2$. This conclusion, according to Ref. 8, implies that the supersymmetry is not broken, and at the same time the ground state is found to be nondegenerate with respect to the fermion number (the degeneracy with respect to the centers of the orbits is preserved).

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We shall consider the case of nonabelian constant potentials in the group $SU(2)$ that specify the same stationary and uniform chromomagnetic field as do the linear potentials:

$$A_1^a=\lambda^{1/2}\delta_1^a, \quad A_2^a=\lambda^{1/2}\delta_2^a, \quad A_3^a=0, \qquad B_i^a=e\lambda\delta_3^a\delta_{i3}, \quad \lambda>0. \tag{14}$$

Besides $\hat{f}$, the operators $\hat{p}_i$ $(i=1, 2, 3)$ and

$$\hat{\tau}=(p_1\tau_1+p_2\tau_2+{}^1/_2e\lambda^{1/2}\tau_3\Sigma_3)\,(p_\perp^2+{}^1/_4e^2\lambda)^{-1/2}, \tag{15}$$

$$p_\perp^2=p_1^2+p_2^2,$$

also commute with $\hat{H}_D$. The energy spectrum has the form

$$\varepsilon^2=m^2+p_3^2+[\,(p_\perp^2+{}^1/_4e^2\lambda)^{1/2}-{}^1/_2e\lambda^{1/2}\tau]^2, \tag{16}$$

where $\tau=\pm 1$ are the eigenvalues of the operator (15). The ground state, with $\varepsilon_0^2=m^2$, has quantum numbers $\mathbf{p}=0$ and $\tau=1$, and the supersymmetry, as in the abelian case, is not broken. The essential difference from the above-considered case of an abelian field is that there now appears an additional, double degeneracy of the vacuum, since $\tau=1$ for states with the following eigenvalues of the matrices Σ_3 and τ_3:

$$\text{a) } \Sigma_3'=-1, \quad \tau_3'=-1 \text{ and b) } \Sigma_3'=1, \quad \tau_3'=1. \tag{17}$$

For $p_\perp=0$ the sign of τ_3' coincides with the sign of the particle charge $\tau_3'e$ and, therefore, the degeneracy can be lifted if we define the vacuum by requiring that the particle have a charge of definite sign. The choice of the vacuum obviously implies the spontaneous symmetry breaking $SU(2)\rightarrow U(1)$. A quark of a particular initial color changes it under the action of a nonabelian chromomagnetic field, but the solution of the Dirac equation remains in the group $U(1)$. Within the framework of the one-particle quantum-mechanical problem it is permissible to consider only those states which do not mix different initial particle charges, i.e., excited states of a vacuum that is defined in the manner indicated above. In this sense, the arguments of the author of Ref. 9, in which wave packets composed of vectors corresponding to different values of $\boldsymbol{\tau}\cdot\mathbf{n}(|\mathbf{n}|=1)$ were considered, are incorrect. In the abelian case, when the field is specified by potentials that are linear in the coordinates, to the operator $\hat{\tau}$ there corresponds a conserved operator of the third component of the total angular momentum:

$$\hat{J}_3=\hat{L}_3+{}^1/_2\Sigma_3=\hat{x}_1\hat{p}_2-\hat{x}_2\hat{p}_1+{}^1/_2\Sigma_3.$$

Here the sign of the charge in the group $U(1)$ has been fixed in advance, and, therefore, the vacuum is not degenerate (it corresponds to the eigenvalue $\Sigma_3'=-1$).

We note that the fermion vacuum in the nonabelian field (14), thanks to the positive-definiteness of the square (16) of the energy of the one-particle states, turns out to be stable, just like the vacuum of QED in a magnetic field.[10] The appearance of the third isotopic component A_μ^3, e.g., in a three-dimensional field of the magnetic type ($A_i^a=\lambda^{1/2}\delta_i^a$, $B_i^a=e\lambda\delta_i^a$), leads to breaking of the supersymmetry, in agreement with the conclusion of Ref. 3 that supersymmetry is absent in three-dimensional even potentials. The ground state in such a field, with energy $\varepsilon_0^2=m^2+e^2\lambda/4$, turns out to be degenerate. It can be seen that the breaking of the supersymmetry leads only to an increase of ε_0, and the fermion vacuum remains stable.

[1)]We note that the operator (4) is the Hamiltonian for the Schrödinger equation off the mass shell in the proper-time method of Schwinger[5] (see also the review Ref. 6).

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