Theory of quantum tunneling with linear dissipation

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The lifetime of a metastable state of a quantum system with linear dissipation is calculated for $T=0$ and arbitrary viscosity. The various types of potential that occur in superconducting weak links are considered. The dependence of the quantum-decay rate Γ on the mass for the potentials considered and in the limit of strong dissipation has the form $\Gamma \propto m^{-\delta}$ $(0<\delta<2)$, and, in the absence of Coulomb nonlinearity in the potential it satisfies $\Gamma \propto \ln(m^{-1})$.

1. INTRODUCTION

Recently investigations of various quantum-mechanical systems with dissipation have attracted considerable attention. As has been shown by Caldeira and Leggett,[1,2] dissipation can substantially increase the lifetime of a metastable state of a quantum system. In these papers the authors studied the quantum decay of a metastable state of a quasiclassical degree of freedom interacting with a thermostat that consists of a large number of quantum harmonic oscillators. The dissipation in such a system appears upon averaging over the variables of the thermostat.

The interest in the study of quantum phenomena in the presence of dissipation is due, in particular, to the real possibility of investigating such phenomena experimentally in macroscopic systems. Superconducting systems with weak coupling provide such a possibility. Quantum fluctuations of the difference in the phases of the order parameters are able to induce transitions between different current states of superconducting junctions. Transitions between the states of the electron subsystem lead to dissipation.

A microscopic expression for the effective action of tunnel junctions was obtained in Refs. 3 and 4, and, in the case of superconducting junctions with direct conduction, in Ref. 5. In a number of cases the effective action found in these papers in the adiabatic approximation has the same form as the effective action of the phenomenological theory of Refs. 1 and 2. The microscopic theory then makes it possible to determine the parameters appearing in the action of Refs. 1 and 2. For certain types of weak coupling (for more detail see Ref. 5) the effective action differs appreciably from that obtained in Refs. 1 and 2.

The case of linear dissipation is particularly important. The probability of the quantum decay of metastable states in the presence of linear dissipation was calculated with exponential accuracy in Refs. 1, 2, 4, and 5, and also in certain other papers. The expression for the coefficient of exponential in the case when the potential in the effective action has the form of a cubic parabola was obtained in the dissipationless limit by Likharev,[6] and in the limit of strong dissipation by Larkin and Ovchinnikov.[7] In the present paper the coefficient of the exponential in the expression for the rate of decay of metastable states will be calculated for arbitrary relationships between the mass and viscosity for the various potentials that describe practically all the types of superconducting weak coupling.

2. LIFETIME OF A METASTABLE STATE

The evolution of a quantum system can be described by means of the quantity

$$J(\varphi_{i+},\varphi_{i-};\varphi_{f+},\varphi_{f-})=\int D\varphi \exp\left\{i\int_{C_0} dt L_{\rm eff}\right\}, \qquad (1)$$

where C_0 is the Keldysh contour:

$$\int_{C_0} dt(\ldots)=\int_{t_i}^{t_f} dt_+(\ldots)-\int_{t_i}^{t_f} dt_-(\ldots)$$

and $L_{\rm eff}$ is the effective Lagrangian of the system, which depends on the variable $\varphi(t)$, and $\varphi_{i(f)\pm}=\varphi(t_\pm)|_{t=t_{i(f)}}$.

Let the state $\varphi=0$ of the system correspond to a local minimum of the potential $V(\varphi)$. In the case of superconducting junctions the quasiclassical variable φ describes fluctuations of the difference in the phases about the equilibrium value, and, in the adiabatic approximation, when a current I close to the critical current I_c is flowing across the junction, $V(\varphi)$ can be represented in the form

$$V(\varphi)=\frac{\varkappa}{2}\varphi^2-\frac{\lambda}{6}\varphi^3+\frac{\gamma}{2}(\chi-\varphi)\theta(\varphi-\chi). \qquad (2)$$

The constants $\varkappa$, λ, and γ for different types of junction will be determined below. Let the maximum of the potential $V(\varphi)$ be reached at the point $\varphi=\chi$. The probability that by the time t_f the system is in the region $\varphi>\chi$ (the metastable state has decayed) is equal to

$$W_{if}=\int_{\chi}^{\infty} d\varphi_f \iint_{-\infty}^{\infty} d\varphi_{i+}\, d\varphi_{i-} J(\varphi_{i+},\varphi_{i-};\varphi_f)\rho(\varphi_{i+},\varphi_{i-}). \quad (3)$$

Here $\varphi_f=\varphi_{f+}=\varphi_{f-}$, and ρ is the initial density matrix of the system. Henceforth we shall assume that

$$\rho(\varphi_{i+},\varphi_{i-})=\delta(\varphi_{i+})\times\delta(\varphi_{i+}-\varphi_{i-}).$$

As usual, we shall assume that the potential barrier is sufficiently high that the decay rate Γ of the metastable state is exponentially small, and to calculate it we can use the quasiclassical approximation. We shall consider first the case $t_f-t_i\ll\Gamma^{-1}$. In this case the quantity W_{if} is also exponentially small: $W_{if}\propto e^{-A}$, $A\gg 1$. To determine A we introduce a parametrization $\tau(t)$ (t, as before, varies on the contour C_0) such that

$$d\tau/dt=i. \qquad (4)$$

The quantity A is determined by the value of the action functional on the trajectory $\tilde\varphi(\tau(t))$, which contains the points

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$\varphi > \chi$ and satisfies the extremum (minimum) condition $\delta S/\delta\tilde{\varphi} = 0$, i.e.,[1)] $A = S[\tilde{\varphi}(\tau)]$.

Here we shall consider the region of temperatures much smaller than the characteristic fluctuation frequencies. With this condition the action $S[\varphi(\tau)]$ corresponding to the potential (2) has, in the case of linear dissipation, the form

$$S[\varphi]=\frac{1}{2}\int\frac{d\omega}{2\pi}\varphi_\omega\varphi_{-\omega}G_0^{-1}(\omega) -\int d\tau\left\{\frac{\lambda}{6}\varphi^3+\frac{\gamma}{2}(\chi-\varphi)\theta(\varphi-\chi)\right\},$$

$$G_0^{-1}(\omega)=m\omega^2+\alpha(|\omega|)+\varkappa. \qquad (5)$$

The function $\alpha(|\omega|)$ determines the dissipative contribution to the action of the system. For ohmic dissipation,

$$\alpha(|\omega|)=\eta|\omega|. \qquad (6)$$

In a number of cases $\alpha(|\omega|)$ has a more complicated form (see, e.g., Refs. 5 and 9). Here we shall be interested mainly in the case (6). For superconducting junctions the viscosity satisfies $\eta = 1/R_{\text{eff}}e^2$, where R_{eff} is the shunt resistance[4] or effective short-circuit resistance,[5] and the mass $m = C^*/e^2$, where C^* is the renormalized capacitance of the junction.

The coefficient of the exponential in the expression for W_{if} for $t_f - t_i \ll \Gamma^{-1}$ is determined by the deviations from the extremal trajectory $\tilde{\varphi}(\tau)$. We shall consider the parametrization $\tau(t-t_1)$, $t_i \leqslant t_1 \leqslant t_f$. For a given trajectory $\varphi(t)$ we define the quantity t_1 by means of the relation

$$a_1\equiv\int d\tau\,\varphi_1(\tau)\varphi(t(\tau)+t_1)=0, \qquad (7)$$

where

$$\varphi_1(\tau)=\frac{\partial\tilde{\varphi}}{\partial\tau}\left[\int d\tau\left(\frac{\partial\tilde{\varphi}}{\partial\tau}\right)^2\right]^{-1/2}. \qquad (8)$$

Substituting the identity

$$1=\frac{1}{2}\int_{t_i}^{t_f}dt_+\delta(t_+-t_1)+\frac{1}{2}\int_{t_i}^{t_f}dt_-\delta(t_--t_1)$$

into the integrands of (1) and (3), for $t_f - t_i \ll \Gamma^{-1}$ we find

$$W_{if}={}^1/_2e^{-A}\left[\int_{t_i}^{t_f}dt_+Y(t_+)+\int_{t_i}^{t_f}dt_-Y(t_-)\right],$$

$$Y(t)=\int D\delta\varphi(t)\delta(t-t_1) \times\exp\left\{-{}^1/_2\int d\tau\,d\tau'G^{-1}(\tau,\tau')\delta\varphi(t(\tau))\delta\varphi(t(\tau'))\right\}, \qquad (9)$$

$$G^{-1}=\delta^2S/\delta\tilde{\varphi}(\tau)\delta\tilde{\varphi}(\tau').$$

In formula (9) we have used the notation $\delta\varphi(t) = \varphi(t) - \tilde{\varphi}(\tau(t-t_1))$ and have expanded the action $S[\varphi]$ (5) to terms quadratic in $\delta\varphi$. Since the expression (9) for Y does not depend on t, we can set $t = 0$ in it. By making use of the relation

$$\delta(t_1)=-i\left|\int d\tau\,\varphi_1(\tau)\frac{\partial}{\partial\tau}\varphi(t(\tau)+t_1)\right|\delta(a_1),$$

in which in the leading approximation in $\delta\varphi$ we can assume that $\varphi(t) = \tilde{\varphi}(\tau(t-t_1))$, and integrating in (9) over $t_\pm$, we obtain for $t_f - t_i \ll \Gamma^{-1}$

$$W_{if}=(t_f-t_i)Be^{-A}, \qquad (10)$$

where

$$B=-iK^{-1}(0)\left[\int d\tau\left(\frac{\partial\tilde{\varphi}}{\partial\tau}\right)^2\right]^{1/2}\int\prod_{n=0}^{\infty}\frac{da_n}{(2\pi)^{1/2}}\delta(a_1) \times\exp\left[-\frac{1}{2}\sum_{n=0}^{\infty}\Lambda_na_n^2\right],\quad K(0)=(\det G_0^{-1})^{-1/2}, \qquad (11)$$

in which $K(0)$ is a normalization constant, and the a_n are the coefficients of the expansion in the eigenfunctions $\varphi_n(\tau)$ of the operator G^{-1}:

$$\delta\varphi(t)=\sum_{n=0}^{\infty}a_n\varphi_n(\tau(t-t_1)),\qquad\int d\tau\,\varphi_n(\tau)\varphi_{n'}(\tau)=\delta_{nn'},$$

and Λ_n are the eigenvalues corresponding to the $\varphi_n(\tau)$. For larger times $t_f - t_i \sim \Gamma^{-1}$ the probability W_{if} is of order unity. In this case it is necessary to take into account processes in which the system passes repeatedly through the classically forbidden region. To describe such processes we introduce a family of parametrizations $\tau(t-t_n)$, $n = 1,2,\ldots$, where $t_i \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n \leqslant t_f$. It is not difficult to see that the general expression for W_{if}, valid for all $t_f - t_i$, can be represented in the form of a sum of contributions of n-instanton trajectories describing n-fold intersections of the classically forbidden region by the system in the time $t_f - t_i$:

$$W_{if}=\sum_{n=1}^{\infty}(-1)^{n+1}\int_{t_i}^{t_f}dt_n\int_{t_i}^{t_n}dt_{n-1}\ldots\int_{t_i}^{t_3}dt_2\int_{t_i}^{t_2}dt_1B^ne^{-nA}, \qquad (12)$$

whence we easily find

$$W_{if}=1-\exp\{-(t_f-t_i)\Gamma\},\qquad\Gamma=Be^{-A}. \qquad (13)$$

The formula (11) for B is well known. To obtain it, a number of authors (see, e.g., Refs. 7 and 10) have used the imaginary-time technique. We note that the interpretation of the presence of the "zero mode" $\Lambda_1 = 0$ in the technique that we have used differs slightly from the usual interpretation in the imaginary-time technique (cf. Ref. 10).

As follows from (11), to determine B it is necessary to calculate the infinite product of eigenvalues Λ_n of the operator G^{-1}. In a number of cases (including those considered below), for various reasons this procedure turns out to be inconvenient, and to calculate the coefficient B of the exponential other methods must be invoked.

A. Nonquasiclassical potential, $\lambda = 0$

We shall consider first the potential (2) for $\lambda = 0$. Such a potential describes clean SNS bridges, if $\varkappa = 2I_c/e$ and $\gamma = 2\pi\varkappa$. The extremal trajectory for the action (5) is easily found[5]:

$$\tilde{\varphi}_\omega=\frac{\gamma}{\omega}G_0(\omega)\sin\omega\tau_0,\qquad\chi=\int\frac{d\omega}{2\pi}\tilde{\varphi}_\omega e^{-i\omega\tau_0}. \qquad (14)$$

When the relation (5) and the condition $\xi = 2\pi\varkappa\chi/\gamma \ll 1$ are fulfilled (for SNS bridges, $\chi = \pi(1 - I/I_c)/2$), we have[5]

$$A=\pi r\chi^2|m\varkappa-\eta^2/4|^{1/2},\qquad m\varkappa\gg\eta^2\xi,$$
$$A=(\pi/2)\eta\chi^2p(1-p/2),\qquad m\varkappa\ll\eta^2\xi, \qquad (15)$$

where

$$r^{-1}=\pi-2\,\mathrm{arctg}\frac{\eta}{(4m\varkappa-\eta^2)^{1/2}},\qquad \eta\leqslant 2(m\varkappa)^{1/2},$$
$$r^{-1}=\ln\left[\frac{\eta+(\eta^2-4m\varkappa)^{1/2}}{\eta-(\eta^2-4m\varkappa)^{1/2}}\right],\qquad \eta\geqslant 2(m\varkappa)^{1/2},$$

and p is found from the equation $p(1-C-\ln\xi p)=1$, where $C\cong 0.577$ is Euler's constant. The function G^{-1} in (9) for the potential (2) for $\lambda=0$ has the form

$$G_g^{-1}(\omega,\omega')=2\pi G_0^{-1}(\omega)\delta(\omega-\omega')-g[e^{i(\omega-\omega')\tau_0}+e^{-i(\omega-\omega')\tau_0}],\tag{16}$$

where τ_0 is defined in (14) and g is given by the relations

$$g=[f(0)-f(2\tau_0)]^{-1},\qquad f(\tau)=\int\frac{d\omega}{2\pi}G_0(\omega)\cos\omega\tau.\tag{17}$$

To calculate B we shall consider the functional integral

$$K(\tilde g)=\int D\delta\varphi\exp\left\{-\frac{1}{2}\int\frac{d\omega\,d\omega'}{(2\pi)^2}G_{\tilde g}^{-1}(\omega,\omega')\,\delta\varphi_\omega\delta\varphi_{\omega'}\right\},\tag{18}$$

where $\tilde g=g-\varepsilon$. Let $\mathrm{Re}\,\varepsilon$ be positive and large enough for the integral (18) to be well defined. We have the obvious equality

$$K(\tilde g)=\left[\prod_{n=0}^{\infty}\Lambda_n(\varepsilon)\right]^{-1/2},\tag{19}$$

where $\Lambda_n(\varepsilon)$ are the eigenvalues of the operator $G_{\tilde g}^{-1}$, which are found from the equation

$$1=\tilde g\int\frac{d\omega}{2\pi}\frac{1+(-1)^n\cos 2\omega\tau_0}{G_0^{-1}(\omega)-\Lambda_n(\varepsilon)}.\tag{20}$$

The integral $K(\tilde g)$ can also be calculated by another method. Differentiating $\ln K(\tilde g)$ with respect to $\tilde g$, we obtain

$$\frac{\partial\ln K(\tilde g)}{\partial\tilde g}=\frac{1}{2}\langle(\delta\varphi(\tau_0))^2+(\delta\varphi(-\tau_0))^2\rangle=G_{\tilde g}(\tau_0,\tau_0).\tag{21}$$

The function $G_{\tilde g}(\tau,\tau')$ is the inverse of the function (16). For $\tilde g<[f(0)+f(2\tau_0)]^{-1}$ it is found in explicit form:

$$G(\tau,\tau')=f(\tau-\tau')+\frac{g}{2}\left\{\frac{[f(\tau+\tau_0)-f(\tau-\tau_0)]}{1-g[f(0)-f(2\tau_0)]}\times[f(\tau'+\tau_0)-f(\tau'-\tau_0)]+\frac{[f(\tau+\tau_0)+f(\tau-\tau_0)][f(\tau'+\tau_0)+f(\tau'-\tau_0)]}{1-g[f(0)+f(2\tau_0)]}\right\}.\tag{22}$$

Substituting (22) into (21) and integrating over $\tilde g$, we obtain

$$K(\tilde g)=K(0)\{[1-\tilde g(f(0)+f(2\tau_0))][1-\tilde g(f(0)-f(2\tau_0))]\}^{-1/2}.\tag{23}$$

The quantity B in (11) is obviously a function of g, $B=B(g)$. From (11), using (19), we have

$$B(\tilde g)=-\frac{iK^{-1}(0)}{(2\pi)^{1/2}}\left[\int d\tau\left(\frac{\partial\tilde\varphi}{\partial\tau}\right)^2\right]^{1/2}K(\tilde g)\Lambda_1^{1/2}(\varepsilon).\tag{24}$$

Thus, to determine B there is no need to solve the transcendental equation (20) and determine all the $\Lambda_n(\varepsilon)$. As follows from (23) and (24), it is sufficient to determine only $\Lambda_1(\varepsilon)$, and then analytically continue the quantity $B(\tilde g)$ (24) in $\tilde g$ to $\tilde g=g$. For $\varepsilon\to 0$, with the aide of (20) we find

$$\Lambda_1(\varepsilon)=\varepsilon\gamma^2[f(0)-f(2\tau_0)]^2\left[2\int d\tau\left(\frac{\partial\tilde\varphi}{\partial\tau}\right)^2\right]^{-1}.\tag{25}$$

Using the relations (23)–(25), for $\tilde g\to g$ we obtain

$$B=\frac{\gamma}{(2\pi)^{1/2}}\frac{f(0)-f(2\tau_0)}{2[f(2\tau_0)]^{1/2}}.\tag{26}$$

For $\xi\ll 1$ in the case of ohmic dissipation (5), B is easily calculated:

$$B=\frac{\pi\chi r^{3/2}}{m}\left|\varkappa m-\frac{\eta^2}{4}\right|^{3/4},\qquad m\varkappa\gtrsim\eta^2\xi,$$
$$B=\frac{\gamma\eta}{2\pi}\left[\frac{p}{2(1-p)}\right]^{1/2}\left[\ln\left(\frac{2\pi p\chi\eta^2}{m\gamma}\right)+C\right],\qquad m\varkappa\lesssim\eta^2\xi,\tag{27}$$

where r and p are defined in Ref. 7.

We now indicate the range of applicability of the results obtained. We shall consider the extremal (one-instanton) trajectory $\tilde\varphi(\tau)$. In finding the expression (26) we neglected the contribution of trajectories intersecting the boundary $\varphi(\tau_0)=\chi$ more than twice, and also neglected the change in the time τ_0 of intersection of the boundary in higher orders in $\delta\varphi(\tau)$. A necessary condition for this is that the fluctuations $\delta\varphi(\tau)$ be small on the boundary $\tau=\tau_0$:

$$\zeta=\langle(\delta'\varphi(\tau_0))^2\rangle^{1/2}/\chi\ll 1.\tag{28}$$

For $\tau\neq\pm\tau_0$ the action is quadratic in $\varphi(\tau)$ and the contribution of fluctuations is taken into account exactly. The function $\delta'\varphi(\tau)$ in (28) is given by the expression

$$\delta'\varphi(\tau)=\sum_{n\neq 1}a_n\varphi_n(\tau).$$

We shall calculate the correlator

$$\langle\delta'\varphi(\tau)\delta'\varphi(\tau')\rangle=\sum_{n\neq 1}\varphi_n(\tau)\varphi_n(\tau')\Lambda_n^{-1}.\tag{29}$$

We rewrite the relation (29) in the form

$$\langle\delta'\varphi(\tau)\delta'\varphi(\tau')\rangle=\lim_{\varepsilon\to 0}[G(\tau,\tau')-\varphi_1(\tau)\varphi_1(\tau')\Lambda_1^{-1}(\varepsilon)].\tag{30}$$

With allowance for (12), (18), and (21) we obtain

$$\langle\delta'\varphi(\tau)\delta'\varphi(\tau')\rangle=f(\tau-\tau')-[f(\tau+\tau_0)+f(\tau-\tau_0)][f(\tau'+\tau_0)+f(\tau'-\tau_0)][4f(2\tau_0)]^{-1}.\tag{31}$$

Setting $\tau=\tau'=\tau_0$ in (31), we find the condition for the fluctuations of $\varphi(\tau)$ to be small in the form

$$\zeta=\frac{f(0)-f(2\tau_0)}{2\chi[f(2\tau_0)]^{1/2}}=\frac{(2\pi)^{1/2}}{\chi\gamma}B\ll 1.\tag{32}$$

The applicability of the noninteracting-instanton approximation used is assessed in the usual way. The restriction thereby obtained on the parameters of the system turns out to be less stringent than the inequality (32). Thus, the above calculations of the quantity Γ are valid under the condition (32). In the case of superconducting junctions this inequality is fulfilled by a large margin.

B. The potential $V(\varphi)=\varkappa\varphi^2/2-\lambda\varphi^3/6$

We now consider another extremely important case, viz., $\gamma=0$. In this case the function $V(\varphi)$ is a cubic parabola. A potential of this form, with the current I close to I_c, describes different types of superconducting weak coupling—tunnel junctions with short circuits, and short superconducting bridges in the case of very dirty superconductors (for more detail, see Ref. 5). For such systems,

$$\varkappa=\frac{1}{e}[2e\lambda(I_c-I)]^{1/2},\quad \lambda=-\left.\frac{\partial^2 I}{e\partial\varphi^2}\right|_{I=I_c},\quad \chi=\frac{2\varkappa}{\lambda}\ll 1.$$

In the presence of ohmic dissipation (6),

$$A=9M\frac{m^{1/2}\varkappa^{5/2}}{\lambda^2}\left[\frac{y}{2}+F(y)\right][1-F^2(y)]^2,$$
$$F(y)=[(y^2+5)^{1/2}+y]^{-1},\quad y=N\eta/(m\varkappa)^{1/2},\tag{33}$$

where M and N, generally speaking, are functionals of $\tilde{\varphi}(\tau)$ and, consequently, functions of y, although the change of M and N when y goes from 0 to ∞ is small. In the dissipationless limit, $M=5^{3/2}/6\cong 1.863$ and $N=9\sqrt{5}\zeta(3)/\pi^3\cong 0.780$, while in the strong-dissipation limit, $M=2^{5/2}\pi/9\cong 1.975$ and $N=1/\sqrt{2}\cong 0.707$, and the expression (33) in this case coincides with the results of Refs. 2 and 4. There are also numerical calculations of the quantity A (Ref. 11) and an interpolation formula for A (Ref. 12), describing the case of intermediate dissipation. Thus, the dependence $A(y)$ has been sufficiently well studied for arbitrary y.

The expression for the coefficient B of the exponential, as already noted, has been obtained in the absence of dissipation[6] and in the limit of strong dissipation.[7] The calculation of B for arbitrary dissipation is complicated by the fact that the analytical expression for the extremal trajectory in this case is unknown. To determine B in the case of an arbitrary relationship between the mass and the viscosity we shall make use of the effective-action formalism of Ref. 13. It will be convenient to go over in (1), (4) to integration over the quantity $q(\tau)=\lambda\varphi(\tau)$. Then the effective action $S(-\lambda^2, \bar{q}(\tau))$ is a functional of the classical field $\bar{q}(\tau)$ and, under the condition (5), is given by the relations

$$\exp\{-S(-\lambda^2,\bar{q})\}$$
$$=\int Dq\exp\left\{-\frac{\tilde{A}(\varkappa,q)}{\lambda^2}+\int d\tau h(\tau)[q(\tau)-\bar{q}(\tau)]\right\},\tag{34}$$

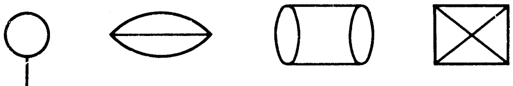
$$\tilde{A}(\varkappa,q(\tau))=\frac{1}{2}\int\frac{d\omega}{2\pi}q_\omega q_{-\omega}G_0^{-1}(\omega)-\frac{1}{6}\int d\tau q^3(\tau),\tag{35}$$

where the field $h(\tau)$ is found from the condition for the maximum of the right-hand side of (34) for a given $\bar{q}(\tau)$. The functional (34) can be represented in the form

$$S(-\lambda^2,\bar{q})=\tilde{A}(\varkappa,\bar{q})/\lambda^2-Q(-\lambda^2,\bar{q}),$$
$$Q(-\lambda^2,\bar{q})=\sum_{r\geqslant 1}Q_r(\bar{q})\lambda^{2r-2},\quad \frac{\delta S(-\lambda^2,\bar{q})}{\delta\bar{q}}=-h(\tau).\tag{36}$$

The quantity Q describes the contribution of quantum fluctuations to the effective action and is the result of summing the irreducible r-loop diagrams of the φ^3 field theory. The extremum of $S(-\lambda^2, q)$ is reached at $h=0$, and the extremal quantity $\bar{q}(\tau)$ gives the average value of $q(\tau)$. As follows from (34), (35), the contribution of a diagram having k loops appears with a factor λ^{2k}, so that the effective action $S(-\lambda^2,\bar{q})$ is a function of the variable $z=-\lambda^2$.

We shall calculate the functional integral (34), (35) by the method of steepest descent. We first find the uniform



a b c

FIG. 1. Irreducible diagrams determining the effective action S in the one-loop (a), two-loop (b), and three-loop (c) approximations.

solution $\bar{q}_0$ that produces an extremum of the action $S(-\lambda^2, \bar{q})$. In the one-loop approximation the quantity Q is determined by the simple diagram a (see the figure), the contribution from which is easily calculated. For $T\to 0$ we have

$$Q_1(\bar{q})=-\int_0^\infty\frac{d\omega}{2\pi}\ln[1-\bar{q}G_0(\omega)].\tag{37}$$

Minimizing $S(-\lambda^2,\bar{q})$ (36) with respect to $\bar{q}$, taking (37) into account we find that the quantum fluctuations give rise to a nonzero average value $\bar{q}=\bar{q}_0$, which is determined by the relations

$$\tilde{\varkappa}\equiv\varkappa-\bar{q}_0=[\varkappa^2-\lambda^2\beta(\varkappa)]^{1/2},$$

$$\beta(\varkappa)=\frac{2}{\pi}\frac{1}{(\eta^2-4m\varkappa)^{1/2}}\ln\frac{\eta+(\eta^2-4m\varkappa)^{1/2}}{2(m\varkappa)^{1/2}},\quad \eta>2(m\varkappa)^{1/2},$$
$$\beta(\varkappa)=\frac{2}{\pi}\frac{1}{(4m\varkappa-\eta^2)^{1/2}}\operatorname{arctg}\left(\frac{4m\varkappa}{\eta^2}-1\right)^{1/2},\quad \eta<2(m\varkappa)^{1/2}.\tag{38}$$

In the calculation of the quantity $\beta(\varkappa)=\partial Q/\partial\varkappa$ we have assumed, as before, that the relation (6) is fulfilled.

As in the previously considered case of the potential (2) with $\lambda=0$, one can show that the action $S(-\lambda^2, q)$ possesses an imaginary part that determines the decay rate Γ of the metastable state. We have

$$\Gamma=-\frac{\operatorname{Im}S(-\lambda^2,\bar{q}_0)}{t_f-t_i}=Be^{-A},\quad B=\frac{\tilde{B}(\varkappa)}{\lambda},\quad A=\frac{\tilde{A}(\varkappa)}{\lambda^2},\tag{39}$$

where $\tilde{A}$ and $\tilde{B}$ do not depend on λ, and the function $A(\varkappa)$ is determined by the expression (33). To calculate B for arbitrary relative magnitudes of the parameters η^2 and $m\varkappa$ we shall use dispersion relations for the effective action $S(z,\bar{q})$. In accordance with (39) the function of the complex variable $z=-\lambda^2$ has a discontinuity on the negative semiaxis $z<0$:

$$S(z+i0,\bar{q})-S(z-i0,\bar{q})=2i\operatorname{Im}S(z,\bar{q}),$$

while for $z>0$ we have $\operatorname{Im}S(z,\bar{q})=0$. With the assumption that the function $S(z,\bar{q})$ is analytic in the plane of the complex variable z with a cut along the real semiaxis $z<0$ we write the dispersion relation for $S(z,\bar{q})$ with two subtractions, which for $x>0$ has the form

$$S(x,\bar{q})=-\frac{\tilde{A}(\varkappa)}{x}-Q_1(\bar{q})+\frac{x}{\pi}\int_{-\infty}^{0}\frac{dz}{z-x}\frac{\operatorname{Im}S(z,q)}{z}.\tag{40}$$

We note that dispersion relations in the coupling constant for a stable field theory have been used in a number of papers.[14-17] In contrast to these papers, we have used the method of dispersion relations directly for the effective-action functional (35). From (40) and (36) there follows the relation

$$Q_r(\bar{q})=\frac{1}{\pi}\int_0^\infty \frac{dz}{z^r}\,\mathrm{Im}\,S(-z,\bar{q}). \tag{41}$$

Substitution of the imaginary part (39) of the action into (41) makes it possible to determine the coefficient of the exponential:

$$\tilde{B}(\tilde{\varkappa})=\{2\pi[\tilde{A}(\tilde{\varkappa})]^{r-1/2}/\Gamma(r-1/2)\}Q_r(\bar{q}), \tag{42}$$

where $\Gamma(x)$ is the Euler gamma function.

As follows from (41), the accuracy of the relation (42) increases with the number r. Expanding the root in (38), we find

$$\bar{q}_0=\varkappa-\tilde{\varkappa}=\lambda^2\beta(\varkappa)/2\varkappa. \tag{43}$$

In the coefficient (39) of the exponential the difference between $\tilde{\varkappa}$ and $\varkappa$ can be neglected, and the expansion (43) of the argument of the exponential gives an additional contribution to the coefficient of the exponential:

$$B=\tilde{B}(\varkappa)\theta/\lambda,\quad \theta=\exp\left[\frac{1}{2\varkappa}\frac{\partial\tilde{A}(\varkappa)}{\partial\varkappa}\beta(\varkappa)\right]. \tag{44}$$

Thus, the fluctuational shift (43) of the position of the ground state of the system leads to the renormalization (44) of the coefficient of the exponential.

It is interesting to compare our result (44), (42) with the known expressions[6,7] for B, found in the limiting cases of weak and strong dissipation. The parameter θ is easily calculated, so that in the limiting cases of interest to us we have

$$\theta=e^3,\quad \eta\ll(m\varkappa)^{1/2};\qquad \theta=\eta^4/(m\varkappa)^2,\quad \eta\gg(m\varkappa)^{1/2}. \tag{45}$$

For $\eta\gg(m\varkappa)^{1/2}$ both the exponent A (33) and the quantity B (42) are independent of m. Consequently, in this case the quantity θ completely determines the dependence of Γ on m. In the simplest two-loop approximation the quantity Q_2 is determined by diagram b (see the figure). Taking (42) and (44) into account, in this approximation we obtain

$$B=\frac{\sqrt{\pi}}{3\lambda}\tilde{A}^{3/2}X,\quad X=\iint\frac{d\omega\,d\omega'}{(2\pi)^2}G_0(\omega)G_0(\omega')G_0(\omega+\omega'). \tag{46}$$

Calculation of the function X for arbitrary η yields extremely cumbersome expressions, which will not be given here. In the limiting cases, from (46) we have

$$B=\left(\frac{32\pi}{375}\right)^{1/2}e^3\frac{\varkappa^{7/4}}{\lambda m^{1/4}},\quad \eta\ll(m\varkappa)^{1/2};$$

$$B=\frac{\pi^2\sqrt{2}}{9}\frac{\eta^{7/2}}{\lambda m^2},\quad \eta\gg(m\varkappa)^{1/2}. \tag{47}$$

It can be seen that in the limit of strong dissipation the expression (47) differs from the exact formula for B in Ref. 7 only by a factor $\pi^2/9$. In the dissipationless limit the agreement of (47) with the exact expression for B in Ref. 6 is somewhat worse: To obtain the result of Ref. 6 we must replace the numerical coefficient $(32\pi/375)^{1/2}e^3\cong 10.40$ in (47) by the quantity $12/\sqrt{\pi}\cong 6.77$. It should be stressed, however, that the proposed method for $r\to\infty$ makes it possible to calculate the quantity Γ to any specified accuracy, and, for finite r, determines an upper bound for this quantity. In other words, the parameter that regulates the accuracy of the proposed method is $1/r$. Thus, calculations in the next order ($r=3$) give better agreement (in comparison with those with $r=2$) with the exact formulas of Refs. 6 and 7 for B. In the three-loop approximation the coefficient of the exponential is determined by the diagrams c (see the figure), calculation of which in the dissipationless limit gives for the numerical factor in the expression for B the value $e^3 266\sqrt{2\pi}/375\sqrt{15}\cong 9.21$.

3. DISCUSSION OF THE RESULTS

Thus, we have shown that the lifetime of a metastable state of a quantum system with linear dissipation can be calculated even in those cases in which the application of standard methods encounters considerable difficulties. Thus, in the case of the nonquasiclassical potential (2) with $\lambda=0$ the determination of the coefficient B of the exponential in the expression for Γ by means of the usual method of calculating the product of the eigenvalues Λ_n of the operator (16) is difficult, since the analytical determination of Λ_n involves solving the transcendental equation (20). This difficulty can be circumvented by expressing B in terms of the Green's function G, which can be calculated exactly. In this case Eq. (20) is used only for the correct isolation of the zero eigenvalue $\Lambda_1=0$.

In the case of a potential of the cubic-parabola type, with an arbitrary relationship between the mass and the viscosity, the problem of determining Γ becomes still more complicated, since we do not know the analytical expression even for the extremal (for the action) trajectory that determines the exponent A. Nevertheless, the quantity A is calculated almost exactly for arbitrary relative magnitudes of the parameters η and $(m\varkappa)^{1/2}$ (Refs. 2, 5, 11, and 12), and for the determination of B, as shown in the present paper, one can use dispersion relations for the effective action, which is regarded as a function of the complex variable $z=-\lambda^2$. By means of such relations one can find the imaginary part of the effective action—the part which determines the decay rate Γ.

The proposed method for calculating Γ is applicable for a wide class of potentials $V(\varphi)$. We note also that this method can be used to find the imaginary part of the free energy of the system for a finite temperature T. For this, in the diagrams determining the quantity Q_r, one must, as usual, replace the integration over ω by summation over the Matsubara frequencies $\omega_n=2\pi nT$.

The potential considered, as already noted, describe different types of superconducting junctions. By means of the methods we have used one can also investigate the more general case of the potential (2) when neither λ nor γ is equal to zero. For $\varkappa=I_c\chi/e$, $\lambda=3I_c/2e$, $\gamma=2I_c/e$, and $\chi=\pi(1-I/I_c)/2$, such a potential, with the current close to I_c, describes superconductor-constriction-superconductor (SCS) junctions with a small quantity of impurities. Here we shall discuss only the question of how the quantity Γ depends on the mass m in the strong dissipation limit. For $\lambda=0$, as follows from (27), the quantity Γ in this limit depends logarithmically weakly on m, while for $\gamma=0$ we have

$\gamma \propto m^{-2}$ (Ref. 7). This difference is easily understandable, since the dependence of Γ on m is determined by the average value $\langle\varphi\rangle$ calculated with allowance for quantum fluctuations. The potential of the cubic-parabola type is "less symmetric" about the point $\varphi = 0$ than the potential (2) with $\lambda = 0$; i.e., the quantity $\langle\varphi\rangle$ when $\gamma = 0$ should be greater than $\lambda = 0$. In the limit of strong dissipation with $\gamma = 0$ the average $\langle\varphi\rangle = \bar{q}/\lambda \propto \ln(m^{-1})$, whereas for $\lambda = 0$ the average satisfies $\langle\varphi\rangle \propto \ln\ln(m^{-1})$.

Now let $\lambda \neq 0$ and $\gamma \neq 0$. In this case, to determine A we can use a method analogous to that used in Ref. 5 for the case of a potential of the cubic-parabola type. The quantity A in this case is expressed in terms of the invariants of the two-parameter group of scale transformations with respect to the coordinate φ and time τ. The values of these invariants depend weakly on the relationship between the mass and the viscosity. The dependence of Γ on the mass m in the limit of strong dissipation, as in the case $\gamma = 0$ [see (44)], is determined by the factor

$$\theta = \exp\left\{\frac{\lambda^2}{2\varkappa}\frac{\partial A}{\partial\varkappa}\beta(\varkappa)\right\}.$$

When we neglect the logarithmic dependence on m in comparison with the power dependence, $\beta(\varkappa)$ is determined by the expression (38). The quantity A is easily found in the limiting cases of small γ or small λ. As a result we have $\Gamma \propto m^{-\delta}$, where

$$\delta = (1-p)\left(\frac{\lambda\chi p}{2\varkappa}\right)^2 + \frac{\pi}{9}\left(\frac{\lambda\chi p}{2\varkappa}\right)^3$$

for small λ, and

$$\delta = 2 - \frac{\gamma\lambda^2\chi}{2\pi\varkappa^2}\left(\frac{4\varkappa}{\lambda\chi} - 1\right)^{1/2}$$

for small γ. We note also that the formal divergence of Γ as $m \to 0$ disappears if we express Γ in terms of the quantity $\bar{\varkappa}$ renormalized with allowance for the quantum fluctuations.

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[1)] The way in which the factor e^{-A} is separated from the expression for W_{if} is obviously somewhat arbitrary in character, and, in general, nonunique. Our parametrization $\tau(t)$ (4) is convenient in that in the dissipationless limit it gives for A the same result as the usual WKB approximation (cf. Ref. 8).

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