Group resonance, universal mapping, and stochastic particle dynamics

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(Submitted 2 April 1985)
Zh. Eksp. Teor. Fiz. **89**, 1632–1647 (November 1985)

Under certain restrictions on the structure of the wave packet, the equations of motion of a particle in the field of a wave packet are exactly reduced to a universal mapping, regardless of the relation between the particle velocity and the group velocity of the packet. It thus becomes possible to find the region of stochastic dynamics, a kinetic equation, and the steady-state distribution of the particles. Some consequences for the basic types of waves in plasma are discussed. The phenomenon of a group resonance between particles and a wave packet is described and analyzed.

1. INTRODUCTION

A description of the dynamics of charged particles in the field of wave packets is one of the central problems in the theory of low-density plasmas. Its formal content can be expressed by an equation, seemingly quite simple, which would be written in the one-dimensional case as

$$\ddot{x}=\frac{e}{m}\sum_n E_n\cos(k_nx-\omega_nt). \tag{1.1}$$

The basic properties of Eq. (1.1) depend on the structure of the wave packet, which is determined by the parameters E_n, k_n, and ω_n. In turn, this structure reflects the particular physical situation.

Is is possible to simplify Eq. (1.1) and draw general conclusions regarding the dynamics of particles under rather general assumptions regarding the right-hand side of this equation? A simplification of this sort was first proposed by Vedenov *et al.*[1] and Drummond and Pines,[2] who thereby laid the foundation for the quasilinear theory of plasmas. That simplification consisted of a transformation from the description of the particle motion by means of the ordinary differential equation (1.1) to a kinetic equation (also called a quasilinear equation)

$$\frac{\partial F(v,t)}{\partial t}=\frac{\partial}{\partial v}D_0(v)\frac{\partial F(v,t)}{\partial v}, \tag{1.2}$$

where the diffusion coefficient $D_0(v)$, found by an averaging procedure, is

$$D_0(v)=\pi\frac{e^2}{m^2}\sum_n E_n{}^2\delta(\omega_n-k_nv). \tag{1.3}$$

If the fields E_n had random phases, or if Eq. (1.1) contained some other random element, it would be possible to make the transformation from (1.1) to (1.2) through some suitable technique for deriving a Fokker-Planck equation. In reality, a low-density plasma has the distinctive feature that the statistical element of the dynamics must appear as a result of nonlinear processes, not as a result of collisions, which are infrequent and have a long scale time.

The problem of the transformation from (1.1) to a kinetic equation thus basically reduces to one of determining (in the present terminology) the conditions for the onset of chaos in the dynamic system (1.1). The first such qualitative condition was formulated by Vedenov *et al.*[1] That condition is that there are no particles which are trapped by any of the waves of the packet on the right side of (1.1) (or, more precisely, the number of such particles is quite small). Although this condition appears to have an unexceptional physical content, it has turned out to be quite difficult to verify explicitly. The great difficulites which arise here lead to an equally great latitude in the choice of other physical hypotheses for introducing a statistical description.[3–5] At this point we can assert that the attempts which have been made over the course of more than 20 years to revise the quasilinear theory by one approach or another have been caused primarily by the absence of any reliable conditions of a general nature for the onset of chaos in system (1.1) (see also the criticism of renormalization ideas by Galeev *et al.*[6]).

Our purpose in the present paper is to present some exact results on the appearance of chaos in system (1.1) and the exact form of the corresponding equation under certain restrictions on the shape of the wave packet (these restrictions, however, are physically quite general). Some qualitative ideas on the subject were published previously in Ref. 7 and, in more general form, in Ref. 8. Exact results for the two limiting cases of time-like and space-like wave packets were first derived in Ref. 9. These cases correspond to the inequalities $v\ll v_g$ and $v\gg v_g$, where v is the velocity of a particle, and v_g is the group velocity of the wave packet. We will pursue the development of the method described in Ref. 9, which leads to a universal description of the onset of chaos for arbitrary relations between v and v_g (including the region $v\sim v_g$). This more general result has several important physical consequences, which we believe will force changes in certain *a priori* ideas regarding the range of applicability of the quasilinear equation.

Furthermore, there are two other physical situations which should be approached in a way different from that which is often taken. The first of these situations involves determining the region in which particles interact strongly with the wave packet. We will show below that the interaction is strongest at velocities v close to v_g, where chaos sets in. We thus see the importance of studying the region $v\sim v_g$.

The second physical situation in which we need to take a more accurate look at earlier representations involves the criterion for an overlap of resonances. We will show that the condition for the onset of chaos in the motion of a particle in the field of the wave packet is not the same as the condition

 0038-5646/85/110945-08$04.00

for the resonances of two neighboring wave packets to overlap (this is the reason for the difficulty of actually ensuring the Vedenov-Velikhov-Sagdeev condition of the "absence of trapped particles"[1]).

2. UNIVERSAL MAPPING

We write the equation of motion of a particle as a pair of Hamilton's equations for the variables p, x:

$$\dot{p}=e\sum_{n=-N}^{N} E_n \cos(k_n x-\omega_n t), \qquad \dot{x}=p/m=v, \tag{2.1}$$

where $2N+1$ is the number of harmonics in the wave packet. We make the following simplifying assumptions regarding the structure of the packet:

$$k_n=k_0+n\Delta k, \quad \omega_n=\omega_0+n\Delta\omega, \quad E_n=E_0, \tag{2.2}$$

where n is an integer. Expressions (2.2) mean that dispersive effects are weak and that the spectral characteristics of the wave packet are uniform and symmetric. Conditions (2.2) are discussed in detail in Ref. 10.

We introduce

$$\theta=k_0x-\omega_0 t, \qquad \xi=\Delta kx-\Delta\omega t. \tag{2.3}$$

Substitution of (2.2) and (2.3) into (2.1) yields

$$\dot{p}=eE_0\cos\theta\sum_{n=-N}^{N}\cos n\xi, \qquad \dot{\xi}=\Delta kp/m-\Delta\omega. \tag{2.4}$$

The change in the energy $\mathscr{E}=mv^2/2$ of the particle is determined by the equation

$$\dot{\mathscr{E}}=eE_0v\cos\theta\sum_{n=-N}^{N}\cos n\xi. \tag{2.5}$$

We now assume $\Delta k\neq 0$ (the case $\Delta k=0$ was discussed in Ref. 8), and we introduce the new variable

$$w=m(v-v_g)|v-v_g|/2, \tag{2.6}$$

where $v_g=\Delta\omega/\Delta k$ is the group velocity of the wave packet. The quantity $|w|$ represents the energy of the particle in a coordinate system moving at the velocity v_g. The relationship between v and w is

$$v=v_g+\operatorname{sign} w(2|w|/m)^{1/2}. \tag{2.7}$$

Differentiating (2.6) with respect to t, we find

$$\dot{w}=m|v-v_g|\dot{v}. \tag{2.8}$$

From (2.4) and (2.7) we find

$$\dot{\xi}=\Delta k(v-v_g)=\Delta k\operatorname{sign} w(2|w|/m)^{1/2}. \tag{2.9}$$

Combining (2.8) and (2.9), we find

$$\dot{w}=(m/\Delta k)|\dot{\xi}|\dot{v}=(m/\Delta k)\dot{\xi}\dot{v}\operatorname{sign} w. \tag{2.10}$$

Using (2.8)–(2.10), we can rewrite the system (2.4) as

$$\frac{dw}{d\xi}=\frac{eE_0}{\Delta k}\cos\theta\operatorname{sign} wF_N(\xi), \qquad \frac{d\theta}{d\xi}=\frac{k_0}{\Delta k}\frac{v(w)-v_0}{v(w)-v_g}, \tag{2.11}$$

where

$$F_N(\xi)=\sum_{n=-N}^{N}\cos n\xi, \qquad v_0=\omega_0/k_0$$

and the functional dependence $v=v(w)$ is determined with the help of (2.7). The quantity v_0 is the phase velocity of the central mode of the wave packet. It is not difficult to see that system (2.11) is a Hamilton's system with the Hamiltonian

$$H(w,\theta,\xi)=H_0(w)-\operatorname{sign} w\frac{eE_0}{\Delta k}\sin\theta F_N(\xi),$$
$$H_0(w)=\frac{k_0}{\Delta k}[w+(v_g-v_0)(2m|w|)^{1/2}] \qquad (w\neq 0), \tag{2.12}$$

where the variable ξ serves as the time; i.e.,

$$\frac{dw}{d\xi}=-\frac{\partial H(w,\theta,\xi)}{\partial\theta}, \qquad \frac{d\theta}{d\xi}=\frac{\partial H(w,\theta,\xi)}{\partial w}.$$

The last simplification, which corresponds to the physical situation and which will be discussed in more detail below, is the assumption that N is large. We can therefore replace the function $F_N(\xi)$ in (2.11) and (2.12) by

$$F(\xi)=\lim_{N\to\infty}F_N(\xi)=2\pi\sum_{n=-\infty}^{+\infty}\delta(\xi-2\pi n). \tag{2.13}$$

The system (2.11) becomes

$$\frac{dw}{d\xi}=eE_0L\cos\theta\operatorname{sign} w\sum_{n=-\infty}^{+\infty}\delta(\xi-2\pi n),$$
$$\frac{d\theta}{d\xi}=\frac{k_0L}{2\pi}\left[1+\operatorname{sign} w\left(\frac{m}{2|w|}\right)^{1/2}(v_g-v_0)\right]. \tag{2.14}$$

The quantity $L=2\pi/\Delta k$ is the length scale of the system. We can transform from Eqs. (2.14) to finite-difference equations. For this purpose we introduce a sequence of values of the independent variable $\xi_n=2\pi n$ and a sequence of functions

$$w_n=w(\xi_n-0), \quad \theta_n=\theta(\xi_n-0).$$

Integration of (2.14) over the interval $(\xi_n-0, \xi_{n+1}-0)$ yields

$$w_{n+1}=w_n+eE_0L\hat{s}_{n,n+1}\cos\theta_n, \qquad \theta_{n+1}=\theta_n+\omega(w_{n+1}), \tag{2.15}$$

where the quantity

$$\omega(w)=k_0L[\operatorname{sign} w+(m/2|w|)^{1/2}(v_g-v_0)] \tag{2.16}$$

is a nonlinear oscillation frequency of the particle in the field of the wave packet. In deriving mapping (2.15) we took into account the circumstance that the system is in free motion on the interval $\xi_n+0, \xi_{n+1}-0$, and we introduce the operator $s_{n,n+1}$. To explain the meaning of this operator, we note that on the right side of the first equation in (2.14) a product of two generalized functions appears. This product, analyzed in Ref. 11, is determined unambiguously only for some fixed class of functions in whose space the generalized functions operate. The reader is referred to Ref. 9 for a discussion of this question for equations of the type in (2.14). In our symmetric case we can assume $\hat{s}_{n,n+1}=1$, which is clearly valid under the condition $eE_0L\ll|w|$. The same condition $s_{n,n+1}=1$, can be extended to the case of arbitrary values of the quantity eE_0L, by treating it as a definition of the limit $N\to\infty$.

To determine the time interval of the sequence of δ-function pulses by means by the relation $\Delta\xi$

$=\xi_{n+1}-\xi_n=2\pi$, we start with the equations

$$\Delta kx(t_n)-\Delta\omega t_n=2\pi n,\qquad \Delta kx(t_{n+1})-\Delta\omega t_{n+1}=2\pi(n+1).$$

From (2.1) we find

$$x(t_{n+1})-x(t_n)=v_{n+1}(t_{n+1}-t_n).$$

From the last two equations we find

$$\Delta t=|t_{n+1}-t_n|=2\pi/\Delta k|v_{n+1}-v_g|=L(m/2|w_{n+1}|)^{1/2}. \qquad (2.17)$$

We will call the mapping (2.15) a "universal" mapping. It preserves the measure in the (w,θ) phase space. Using the definition $\hat{s}_{n,n+1}=1$, we can put the mapping (2.15) in the form

$$w_{n+1}=w_n+eE_0L\cos\theta_n,$$

$$\theta_{n+1}=\theta_n+k_0L[\operatorname{sign} w_{n+1}+(m/2|w_{n+1}|)^{1/2}(v_g-v_0)]. \qquad (2.18)$$

With $\Delta\omega=0$ we have, according to (2.6) and (2.16),

$$w=\tilde{\mathscr{E}}=\mathscr{E}\operatorname{sign} v,\quad \omega(\tilde{\mathscr{E}})=L[k_0\operatorname{sign}\tilde{\mathscr{E}}-\omega_0(m/2|\tilde{\mathscr{E}}|)^{1/2}]. \qquad (2.19)$$

After substitution into (2.18), expressions (2.19) lead to a so-called L-mapping:

$$\tilde{\mathscr{E}}_{n+1}=\tilde{\mathscr{E}}_n+eE_0L\cos\theta_n,\ \theta_{n+1}=\theta_n+k_0L\operatorname{sign}\tilde{\mathscr{E}}_{n+1}-\omega_0L(m/2|\tilde{\mathscr{E}}_{n+1}|)^{1/2}. \qquad (2.20)$$

Relations (2.19) and (2.20) describe the dynamics of a particle in the field of a wave in x-space packet,[9] which arises under the inequality $\Delta kv\gg\Delta\omega$.

Under the condition $v_g=v_0$, which is characteristic of oscillations of the acoustic type, we find $\omega=k_0L$ from (2.16). The associated mapping reduces to a simple phase rotation, i.e., a linear oscillator. In this case the dynamics of the particle is trivially stable.

3. REGION OF CHAOS

Let us examine the mapping (2.18). We can analyze the boundary of the onset of chaos in this mapping by a fairly standard approach if we impose the restriction $eE_0L\ll|w|$, which implies that the energy perturbation is small. For this purpose we need to determine the quantity

$$K=|d\theta_{n+1}/d\theta_n-1|=eE_0k_0L^2|v_g-v_0|(m/8|w|^3)^{1/2}|\sin\theta|. \qquad (3.1)$$

The condition for chaos is $K\gtrsim 1$ or, in a more convenient form,

$$\Omega_0^2\Omega L^2|\sin\theta|/k_0|v-v_g|^3\gtrsim 1, \qquad (3.2)$$

where

$$\Omega_0^2=eE_0k_0/m,\qquad \Omega=k_0|v_g-v_0|. \qquad (3.3)$$

Except in that region of phases θ in which chaos sets in, condition (3.2) leads to the following velocity interval for the chaos region:

$$|v-v_g|\leqslant(\Omega_0^2\Omega L^2/k_0)^{1/3}. \qquad (3.4)$$

Under the conditions $v\gg v_g$ and $v_0\gg v_g$, expression (3.4) becomes the expression derived in Ref. 9.

For expression (3.4) we can draw some conclusions which are important for applications: 1) the velocity interval of the chaos region is symmetric with respect to the packet group velocity v_g. 2) This interval is bounded, and its boundary is proportional to $E_0^{1/3}$. 3) At $v_g=v_0$, there is no region of chaos.

Let us examine some limiting cases of condition (3.4). When we have $v\ll v_g$ and $v_0\ll v_g$, that condition becomes the inequality

$$K_1=\Omega_0^2T^2\gtrsim 1,\qquad T=2\pi/\Delta\omega,$$

which was derived in Ref. 7. If $v\sim v_0$, we find from (3.4) and (3.3)

$$K_2=\Omega_0^2/(\Delta k)^2(v_g-v_0)^2\sim\Omega_0^2/(\Delta k)^2(v_g-v)^2\gtrsim 1.$$

This condition was presented in Ref. 8 and, in a form closer to that given here, in Ref. 12. In the same studies,[8,12] the existence of a restriction on the velocity in the stochastic hearing of particles was first mentioned. An analogous result for a similar model with $v_g=0$ was recently published by Fuchs *et al.*[13]

Let us assume $v_0\gg v_g$, as is frequently done. We then see easily from (3.4) that chaos always sets in for velocities v near v_g, while it may not set in for v near v_0. In connection with this comment, let us consider the expression for the phase velocity of the nth harmonic of the packet. From (2.2) we have

$$v_{\rm ph}(n)=\frac{\omega_n}{k_n}=\frac{\omega_0+n\Delta\omega}{k_0+n\Delta k}.$$

At small values of n $(n\Delta\omega<\omega_0,\ n\Delta k<k_0)$ we have $v_{\rm ph}(n)\sim v_0$, so that the typical wave phase velocities are comparable in magnitude to the phase velocity of the central mode of the packet, v_0. At large values of n $(n\Delta\omega>\omega_0,\ n\Delta k>k_0)$ we have $v_{\rm ph}(n)\sim v_g$. We denote by N_0 a characteristic number which distinguishes between the cases of large and small values of n. In the case discussed above, the number of harmonics in the packet is $N\to\infty$; i.e., we have $N\gg N_0$. This result means that for essentially all the waves of the packet we have $v_{\rm ph}(n)\approx v_g$, provided that $\Delta k\neq 0$.

Real packets, however, have a finite N, so that the validity of condition (3.4) requires further analysis for real packets. This analysis will also tell us whether chaos sets in at $v_{\rm ph}(n)>v_g$.

Finally, we note that the ordinary condition for the overlap of resonances, $K_1\gtrsim 1$, is valid only if these resonances are sufficiently "immobile" with respect to each other in phase space. The condition for this can easily be found from our initial equations, (2.4) and (2.3): $\Delta kx\ll\Delta\omega$, i.e., $v\ll v_g$.

For simplicity we consider the case

$$K_0=\max K=\Omega_0^2\Omega L^2/k_0|v-v_g|^3\gg 1. \qquad (3.5)$$

We can then find the following estimate[12] for the phase correlation:

$$R(t)=\frac{1}{2\pi}\int_0^{2\pi}\exp\{i[\theta(t)-\theta(0)]\}d\theta(0)\sim\exp[ik_0(v-v_0)t]\exp(-t/\tau_c), \qquad (3.6)$$

where the time scale for the loss of correlation is

$$\tau_c=2\Delta t/\ln K_0, \qquad (3.7)$$

and Δt is the time between two successive steps of the mapping. Substituting expression (2.17) for Δt into (3.7), we find

$$\tau_c=2L(|v-v_g|\ln K_0)^{-1}=2L\left(|v-v_g|\ln\frac{\Omega_0^2\Omega L^2}{k_0|v-v_g|^3}\right)^{-1},$$
$$K_0\gg 1. \qquad (3.8)$$

It can be seen from (3.8) that in the limit $v\to v_g$ the time scale for the loss of phase correlation tends toward infinity, although chaos condition (3.5) [or (3.1)] holds better as the quantity $|v - v_g|$ becomes smaller. The reason is the strong interaction between the particle and the wave packet in the region $v\sim v_g$. We will examine the dynamics near v_g in more detail below. We will also mention the obvious fact that an increase in L (i.e., in the dimensions of the region of the motion) is also accompanied by an increase in the time τ_c.

4. KINETIC EQUATION

Let us assume that condition (3.5), for chaos in the phases θ, holds. The loss of phase correlation in (3.6) allows us to transform to a kinetic description of the motion of the particle, by means of a distribution function $F(w,t)$.

We consider the velocity region in which the perturbation is small, i.e., $eE_0L\ll w$ or, equivalently,

$$\Omega_0(k_0L)^{1/2}\ll k_0|v-v_g|. \qquad (4.1)$$

This condition, combined with (3.4), yields the following inequality for the field:

$$\Omega_0\ll\Omega(k_0L)^{1/2}=|v_g-v_0|(k_0^3L)^{1/2}. \qquad (4.2)$$

This inequality is expressed exclusively in terms of the parameters of the wave packet.

By virtue of condition (4.1), we can then use a Fokker-Planck-Kolmogorov equation for the function $F(w,t)$:

$$\frac{\partial F}{\partial t}=-\frac{\partial}{\partial w}\left(\left\langle\!\left\langle\frac{\Delta w}{\Delta t}\right\rangle\!\right\rangle F\right)+\frac{1}{2}\frac{\partial^2}{\partial w^2}\left(\left\langle\!\left\langle\frac{(\Delta w)^2}{\Delta t}\right\rangle\!\right\rangle F\right), \qquad (4.3)$$

where $\langle\langle...\rangle\rangle$ means an average over the phase θ, Δw is the change in w in one step of the mapping ($\Delta w = w_{n+1} - w_n$), and Δt is the time interval between two successive steps of the mapping. In this case, the quantity Δt depends on the variable w in accordance with (2.17). To find the correct expressions we should thus put mapping (2.18) in a symmetric form (see §3.1 in Ref. 12).

We denote by x_n and x_{n+1} two successive points in the cross sections of a trajectory at which two successive δ-function perturbation pulses act. We replace these points by

$$\bar{x}_n=x_n+L/2, \qquad \bar{x}_{n+1}=x_{n+1}+L/2.$$

On the interval $(x_n + L/2, x_{n+1})$ the particle moves at a velocity v_n, while on $(x_{n+1}, x_{n+1} + L/2)$ it moves at v_{n+1}. We can thus write

$$\Delta t=L/2|v_n-v_g|+L/2|v_{n+1}-v_g|. \qquad (4.4)$$

Using (2.15) and (4.4), we find the following result, which holds to within small terms of second order in the parameter eE_0L/w:

$$\left\langle\!\left\langle\frac{\Delta w}{\Delta t}\right\rangle\!\right\rangle=\frac{1}{8}\frac{e^2E_0^2L}{(m|w|/2)^{1/2}},$$
$$\left\langle\!\left\langle\frac{(\Delta w)^2}{\Delta t}\right\rangle\!\right\rangle=\frac{1}{2}e^2E_0^2L\left(\frac{2w}{m}\right)^{1/2}. \qquad (4.5)$$

From (4.5) we find

$$\frac{1}{2}\frac{\partial}{\partial w}\left\langle\!\left\langle\frac{(\Delta w)^2}{\Delta t}\right\rangle\!\right\rangle=\left\langle\!\left\langle\frac{\Delta w}{\Delta t}\right\rangle\!\right\rangle, \qquad (4.6)$$

which is in general a consequence of the principle of detailed balance.[14]

Substitution of (4.5) and (4.6) into (4.3) yields the kinetic equation

$$\frac{\partial F(w,t)}{\partial t}=\frac{1}{2}\frac{\partial}{\partial w}D(w)\frac{\partial F(w,t)}{\partial w},$$
$$D(w)=e^2E_0^2L(|w|/2m)^{1/2}, \qquad (4.7)$$

which is meaningful in the w region determined by condition (3.4) and definition (2.6):

$$|w|\leqslant w_0=(m/2)(\Omega_0^2\Omega L^2/k_0)^{2/3}. \qquad (4.8)$$

The boundary condition should have the form of the condition that no particles stream across the boundary w_0 (Ref. 15); i.e.,

$$D(w)\frac{\partial F(w,t)}{\partial w}\bigg|_{w=w_0}=0. \qquad (4.9)$$

A steady-state solution of Eq. (4.7) is thus

$$F(w)=\text{const}=\rho/2w_0, \qquad |w|\leqslant w_0, \qquad (4.10)$$

where ρ is the number of particles in region (4.8).

It should be stipulated that the boundary w_0 is "impenetrable" for particles in a somewhat crude sense, since condition (3.4) establishes the boundary only when chaos is lightly developed. The possible existence of weak and slow processes which cause diffusion of particles into the region $|w| > w_0$ is not ruled out. We will not discuss such processes here. There is also some "fine structure" in the distribution function F near the point $w = 0$, which we will discuss below in connection with the phenomenon of group resonance.

We also consider the velocity distribution function $F(v)$. From (2.6) and (4.10) we have

$$F(v)=F(w)|dw/dv|=(m\rho/2w_0)|v-v_g|, \qquad (4.11)$$
$$|v-v_g|\leqslant(2w_0/m)^{1/2}.$$

Distributions (4.10) and (4.11) mean that there is an energy "plateau" in the coordinate system moving with the wave packet; equivalently, there is a velocity "cone" with respect to the point $v = v_g$. Such a distribution is unstable, so that the label "steady-state" applied to it should be understood in the narrow sense that we are considering only the interaction of the particles with the given wave packet.

Finally, we consider a generalization of the equation of motion (1.1):

$$\ddot{x}=\frac{e}{m}\sum_k E_k\cos(kx-\omega_kt), \qquad (4.12)$$

where the quantities E_k, k, and ω vary slowly along the packet and are slightly different from (2.2). At the same level of accuracy, we can thus retain the analysis in Sec. 3 of the onset of chaos. Condition (3.4) now becomes

$$|v-d\omega_k/dk|\leqslant(\Omega_k^2\Omega L^2/k)^{1/3}, \qquad (4.13)$$

where the quantities

$$\Omega_k^2=eE_kk/m, \qquad \Omega=|\omega-kd\omega_k/dk| \qquad (4.14)$$

refer to certain characteristic values of k for the wave packet.

As we mentioned back in Sec. 2, the variables w, θ serve as the action-angle canonical pair of variables.

Let us ignore the k dependence of v_g. The definition of the variable w in (2.6) thus remains in force, and an equation of $\dot{w}$ can be found from (2.8) and (4.12):

$$\dot{w}=e|v(w)-v_g|\sum_k E_k\cos(kx-\omega_k t).$$

The corresponding structure of the kinetic equation was found in Ref. 12 (§6.3):

$$\frac{\partial F(w,t)}{\partial t}=\frac{1}{2}\frac{\partial}{\partial w}D(w)\frac{\partial F(w,t)}{\partial w},$$

$$D(w)=\frac{4\pi e^2}{m}\sum_k E_k^2|w|\Delta\left(\frac{1}{\tau_c},\omega-kv\right), \tag{4.15}$$

$$\Delta\left(\frac{1}{\tau_c},z\right)=\frac{1}{\pi}\frac{1/\tau_c}{(1/\tau_c)^2+z^2}\xrightarrow[1/\tau_c\to 0]{}\delta(z),$$

where we have used expression (3.6) for the phase correlation. The correlation time (3.8) can be written as follows, where we are using the notation in (4.14):

$$\frac{1}{\tau_c}=\frac{1}{2L}\left|v(w)-\frac{d\omega_k}{dk}\right|\ln\frac{\Omega_k^2\Omega L^2}{k|v(w)-d\omega_k/dk|^3}. \tag{4.16}$$

It is not difficult to see that expression (4.15) for $D(w)$ becomes (4.7) if we set $E_k^2=\text{const}=E_0^2/2$ and make the transformation

$$\sum_k\Delta\left(\frac{1}{\tau_c},\omega_k-kv\right)\to\frac{1}{\Delta k}\int dk\,\delta(\omega_k-kv)=\frac{L}{2\pi|v-v_g|}.$$

Equation (4.15) is the most complete form of the quasilinear equation, written in the space of the generalized canonical variable w and incorporating the finite time for the onset of phase chaos. To show how the original and ususal form of this equation[1] can be found from (4.15), we consider, say, the limiting case $v\ll v_g$. In this cas we have

$$w\approx-mv_g^2/2+mvv_g,\quad dw=mv_g dv.$$

Substituting these expressions into (4.15), we find

$$\frac{\partial F(v,t)}{\partial t}=\frac{\partial}{\partial v}D(v)\frac{\partial F(v,t)}{\partial v},$$

$$D(v)=\pi\frac{e^2}{m^2}\sum_k E_k^2\Delta\left(\frac{1}{\tau_c},\omega_k-kv\right),\qquad \tau_c=\frac{2T}{\ln(\Omega_k^2T^2)}, \tag{4.17}$$

where $T=2\pi/\Delta\omega$. Expressions (4.17) are the same as those found in Ref. 8, while $D(v)$ becomes the same as $D_0(v)$ in (1.2) and (1.3) in the limit $1/\tau_c\to 0$.

5. STOCHASTIC DYNAMICS OF PLASMA PARTICLES

Let us consider some applications of the results of the preceding sections when wave packets interact with particles in a homogeneous and isotropic plasma. We know that there are three types of waves in such a plasma: longitudinal electrostatic plasma (Langmuir) and ion acoustic waves, and transverse electromagnetic waves.

For plasma waves the dispersion relation is (Ref. 14, for example)

$$\omega=\omega_{L_e}\left(1+\frac{3}{2}k^2r_d^2\right), \tag{5.1}$$

where $\omega_{L_e}=(4\pi ne^2/m)^{1/2}$ is the plasma frequency, and $r_d=(T_e/4\pi n_e{}^2)^{1/2}$ the Debye length of the electrons. The group velocity of these waves in the long-wave region, $kr_d\ll 1$, is substantially less than the electron thermal velocity, $v_g\sim 3v_T(kr_d)\ll v_T$, while the phase velocity of the waves is substantially greater than the thermal velocity, $v_{\text{ph}}\sim v_T/(kr_d)\gg v_T$. The Landau damping of long plasma waves is thus exponentially weak, so that a packet of plasma waves can certainly cause chaos in particles at velocities $v\sim v_{\text{ph}}\gg v_T$ if the inequality

$$K=eE_0L^2\omega_{L_e}/mv^3\gtrsim 1 \tag{5.2}$$

holds. This inequality determines the existence of an upper bound[9] on the energy of a particle in the region of its stochastic dynamics:

$$\mathscr{E}_{max}=\frac{m}{2}\left(\frac{e}{m}E_0L^2\omega_{L_e}\right)^{2/3}. \tag{5.3}$$

Expression (5.3) also determines the boundary for stochastic heating of particles. In this case the correlation time τ_c is

$$\frac{1}{\tau_c}=\frac{1}{2Lm^{1/2}}(2\mathscr{E})^{1/2}\ln\frac{\Omega_k^2\omega_{L_e}L^2m^{3/2}}{k(2\mathscr{E})^{3/2}}. \tag{5.4}$$

A question which remains unanswered is whether there is a region of stochastic dynamics of the particles for velocities $v\sim v_g\ll v_T$ if condition (5.2) does not hold. As we mentioned back in Section 3, there is no region of chaos if $v_g=v_0$. A similar situation can arise in a plasma when particles interact with a very broad packet of long ion acoustic waves, for which we have

$$\omega^2=k^2v_s^2/(1+k^2r_d^2)\to k^2v_s^2, \tag{5.5}$$

where $v_s=\omega_{L_i}r_d$ is the ion acoustic velocity. According to (4.13) and (5.5), the region of the stochastic instability is the velocity interval

$$|v-v_g|\lesssim(\Omega_0^2L^2v_s)^{1/3}(kr_d)^{2/3}/(1+k^2r_d^2)^{1/2}. \tag{5.6}$$

In the long-wave limit ($kr_d\ll 1$), interval (5.6) is very narrow, and the system of dynamic equations of motion in (1.1) becomes approximately integrable.

Let us examine in more detail the interaction of electrons with a packet of electromagnetic waves. For transverse waves in an isotropic plasma the dispersion relation is

$$\omega^2=\omega_{L_e}^2+k^2c^2, \tag{5.7}$$

and the phase velocity is greater than the velocity of light. Consequently, there are no phase resonance effects (Čerenkov radiation or Landau damping).[14] Since the group velocity of the transverse waves is less than the velocity of light, $v_g=c^2/v_{\text{ph}}<c$, we can expect chaos for the particles at velocities $v\sim v_g$.

For definiteness, we consider the motion of an electron in the field of a packet of circularly polarized transverse waves:

$$\mathbf{A}=\sum_n A_n\{\mathbf{e}_y\cos(\omega_nt-k_nx)+\mathbf{e}_z\sin(\omega_nt-k_nx)\}, \tag{5.8}$$

where $\mathbf{A}$ is the vector potential of the field in the Coulomb gauge. Corresponding to the motion of an electron in field (5.8) is the Hamiltonian

$$H=[m^2c^4+(c\mathbf{P}-e\mathbf{A})^2]^{1/2}, \qquad (5.9)$$

where $\mathbf{P}$ is the generalized momentum. The Hamiltonian (5.9) is a function of the time:

$$\dot{H}=-\frac{ec}{H}\frac{\partial}{\partial t}\mathbf{PA}+\frac{e^2}{2H}\frac{\partial}{\partial t}\mathbf{A}^2. \qquad (5.10)$$

According to (5.9), the canonical equations of motion are

$$\dot{\mathbf{r}}=\frac{c}{H}(c\mathbf{P}-e\mathbf{A}),\qquad \dot{\mathbf{P}}=\frac{ec}{H}\nabla(\mathbf{PA})-\frac{e^2}{2H}\nabla\mathbf{A}^2. \qquad (5.11)$$

Since Hamiltonian (5.9) and the vector potential (5.8) are independent of the coordinates y and z, the transverse component $P_\perp$ of the generalized momentum is a constant of motion.

We now assume that the amplitude of one of the harmonics, with index $n=i$, is large in comparison with the amplitudes of the other waves ($A_i \gg A_n$), and we retain in the Hamiltonian only the bilinear terms A_iA_n which correspond to two-photon Compton scattering. This approximation is legitimate if the particle oscillation velocity in field (5.8) is large in comparison with the transverse thermal velocity, i.e., if $e|\mathbf{A}|\gg cP_\perp$. In this approximation, the Hamiltonian (5.9) becomes

$$H=\Big[\mathscr{E}^2+2e^2A_i\sum_n A_n\cos(\Omega_nt-\varkappa_nx)\Big]^{1/2},\qquad \mathscr{E}=(m^2c^4+c^2p_x^2)^{1/2}, \qquad (5.12)$$

where $\mathscr{E}$ is the sum of the rest energy and the kinetic energy of the particle, and $\Omega_n=\omega_i-\omega_n$ and $\varkappa_n=k_i-k_n$ are the frequency and wave number of the beats of the electromagnetic waves. Equations of motion (5.11) also simplify:

$$\dot{x}=\frac{c^2p_x}{H},\qquad \dot{p}_x=\frac{e^2}{H}A_i\sum_n B_n\sin(\Omega_nt-\varkappa_nx), \qquad (5.13)$$

where $B_n=\varkappa_nA_n$ is the effective magnetic field of the wave. Correspondingly, the change in the energy is

$$\dot{\mathscr{E}}=\dot{x}\frac{e^2A_i}{\mathscr{E}}\sum_n B_n\sin(\Omega_nt-\varkappa_nx). \qquad (5.14)$$

Let us examine the motion of a particle in the field of a packet of long electromagnetic waves, with $k_nc\ll\omega_{L_e}$, in the case in which the beat phase velocity $\Omega_n/\varkappa_n$ is close to the packet group velocity, $\Omega_n/\varkappa_n\sim v_g\approx(c^2k/\omega_{L_e}\ll c$; in the region $v\sim v_g$, this is a nonrelativistic motion: $|p|\ll mc$. Furthermore, proceeding as in Section 2, we adopt the simplifying assumption (2.2) regarding the frequencies Ω_n and the wave numbers $\varkappa_n$. The equations of motion of the particle then remain the same as in (2.14), aside from the replacement

$$E_0\to\frac{eA_iB_\perp}{mc^2}.$$

Hence, there is a change in the condition for chaos (3.4):

$$|v-v_g|\lesssim(e^2A_iB_\perp L^2\omega_{Le}/m^2c^2)^{1/4}. \qquad (5.15)$$

The stochastic dynamics of the particles in region (5.15) will be determined by kinetic equation (4.15), in which we should make the replacement

$$E_k\to eA_iB_k/mc^2. \qquad (5.16)$$

Clearly, the structure of a packet of electromagnetic waves differs from the structure considered in Section 2, since the phase velocities of all the waves in the packet satisfy $v_{\rm ph}(n)>c$, and the velocities of all the particles satisfy $v>c$. Nevertheless, a packet of electromagnetic waves gives rise to chaos of those particles whose velocities lie outside the phase velocities of the waves of the packet.

If condition (5.15) holds, the particles undergo stochastic heating. To determine this heating rate, we multiply (4.7) by $w^{3/2}$ and integrate over w. We find

$$\frac{d}{dt}\langle w^{3/2}\rangle=\frac{3}{2^{5/2}}\frac{e^4B_\perp^2A_i^2L}{m^{5/2}c^4}\equiv Q, \qquad (5.17)$$

where we have used replacement (5.16), and $B_\perp$ is a characteristic value of B_k for the packet. From (5.17) we see that the energy of the particles in the coordinate system of the wave packet initially increases in accordance with

$$\langle w^{3/2}\rangle=\mathrm{const}+Qt.$$

Similar increases in the other moments, e.g., $\langle w\rangle$, also occur.

Since the energy balance must hold, the latter result means that we have

$$\rho_0\frac{d}{dt}\langle w\rangle=-\frac{d\mathscr{E}_{wp}}{dt},$$

where $\mathscr{E}_{wp}$ is the energy density for the wave packet. The energy density $\mathscr{E}_{wp}$ thus decreases until a steady state is reached in the energy exchange between the particles and the waves.

For transverse wave packets, a question remains unanswered: Under what conditions do the single-photon processes for which the $\mathbf{P}\cdot\mathbf{A}$ terms in equations of motion (5.11) are responsible lead to chaos of the particles?

6. GROUP RESONANCE

The phenomenon which is the subject of this section of the paper is one of the principal physical consequences of this study. The essence of this subject is the group resonance, i.e., a resonance between a particle and a wave packet. Such a resonance, if it occurs at all, should occur at particle velocities near v_g.

However, the point $v=v_g$ is singular, as can be seen from (2.6)–(2.11). Near this point, various (even small) deviations from conditions (2.2) on the structure of the wave packet may have a major effect on the nature of the particle motion. We will thus assume that the wave packet is finite, and in the n dependence of the frequency ω_n we take the dispersion into account by introducing a term $\sim n^2$.

The equation of motion becomes

$$\ddot{x}=\frac{e}{m}E_0\sum_{n=-N}^{N}\cos[k_0x-\omega_0t+n(\Delta kx-\Delta\omega t)-\alpha n^2t], \qquad (6.1)$$

where α is small. The last term in the phase comes into play only at times $t\gtrsim t_0$, where

$$t_0=1/\alpha N^2. \qquad (6.2)$$

To analyze the motion near the point $v = v_g$ we set

$$x=v_g t+\delta x, \quad \dot{x}=v_g+\delta v=v_g+\delta\dot{x}. \tag{6.3}$$

Using (6.3), we can put Eq. (6.1) in the form

$$\delta\ddot{x}=\frac{e}{m}E_0\sum_{n=-N}^{N}\cos\{\Omega t+k_0\delta x+\alpha t n_0^2(t)-\alpha t[n-n_0(t)]^2\}, \tag{6.4}$$

where $n_0(t) = \Delta k\delta x/2\alpha t$ and, as before, $\Omega = k_0(v_g - v_0)$. The quantity $n_0(t)$ reaches its maximum value at the minimum value of t, at which the terms caused by the frequency dispersion are important. This time is determined by t_0. Using (6.2), we introduce

$$n_0\equiv n_0(t_0)=\Delta k\delta x N^2/2. \tag{6.5}$$

It is now obvious that the limiting structure of the right side of (6.4) is determined by the relation between n_0 and N.

As a first case we assume $n_0 \ll N$ or, according to (6.5),

$$N\Delta k\delta x\ll 1. \tag{6.6}$$

Ignoring the terms in the phase which contain n_0 in (6.4), we write

$$\delta\ddot{x}=\frac{e}{m}E_0\sum_{n=-N}^{N}\cos(\Omega t+k_0\delta x-\alpha n^2 t). \tag{6.7}$$

Except at very early times we can replace the summation in (6.7) by an integral over n and push the limits to infinity. We then find

$$\delta\ddot{x}=\frac{e}{m}E_0\left(\frac{\pi}{|\alpha|t}\right)^{1/2}\cos\left(\Omega t+k_0\delta x-\frac{\pi}{4}\operatorname{sign}\alpha\right). \tag{6.8}$$

Introducing the new variable

$$y=\Omega t+k_0\delta x-\pi/2-(\pi/4)\operatorname{sign}\alpha,$$

we finally find

$$\ddot{y}+\Omega_0^2(\pi/|\alpha|t)^{1/2}\sin y=0. \tag{6.9}$$

The solution of Eq. (6.9) is obvious. It describes oscillations with a characteristic "frequency"

$$\Omega_g(t)=\Omega_0(\pi/|\alpha|t)^{1/4}, \tag{6.10}$$

which decreases as a function of time.

The physical meaning of this result is easy to see. It can be shown that the maximum value of $\Omega_g(t)$ in (6.10) is

$$\Omega_g=\Omega_g(t_0)\sim\Omega_0 N^{1/2} \tag{6.11}$$

[here it is necessary to use our original equation, (6.8), at small t]. Substituting t_0 and (6.11) into (6.9), we note that we have

$$\delta\ddot{x}\sim(e/m)NE_0;$$

i.e., the perturbation increases by a factor of about N. This increase is a direct consequence of the fact that the particle is in resonance not with a single wave but with N waves, whose influence is effectively summed.

The quantity Ω_g given by (6.11) may be called the "frequency of the group resonance." The decay of the effective frequency $\Omega_g(t)$ over time [see (6.10)] results from a spreading of the wave packet.

Let us summarize the characteristics of a group resonance:

1. The resonance occurs when the particle velocity equals group velocity of the wave packet.[1)]

2. Near the resonance, the dynamics corresponds to the oscillations of a nonlinear pendulum [see (6.9)] with a slowly decreasing frequency.

3. The characteristic frequency Ω_g of the group resonance is equal to the frequency Ω_0 of a nonlinear resonance in the field of a single wave, multiplied by the square root of the number of waves in the packet.

4. The oscillation frequency at a group resonance decays because the wave packet spreads (or it increases if the packet condenses).

Near a group resonance, the dynamics of a particle is regular. The boundary of the group resonance can be estimated roughly from

$$\delta\dot{v}_g\sim\max\delta\dot{x}\sim\Omega_g/k_0. \tag{6.12}$$

For which values of N will a group resonance occur inside the region of chaos? For this to occur, the quantity δv_g defined in (6.12) obviously must be below the chaos boundary set by (3.4). We thus find

$$N<(\Omega k_0^2L^2/\Omega_0)^{2/3}. \tag{6.13}$$

Inequality (6.6) also means that the perturbation is smooth, since $1/N\Delta k$ is the characteristic width of the perturbation pulse for the particle, while δx is the displacement of the particle caused by this perturbation.

We turn now to a second limiting case, $n_0 \gg N$ or

$$N\Delta k\delta x\gg 1. \tag{6.14}$$

In Eq. (6.4) we need retain only the term of first order in n; we then write

$$\delta\ddot{x}=\frac{e}{m}E_0\sum_{n=-N}^{N}\cos(\Omega t+k_0\delta x+n\Delta k\delta x)$$

$$=\frac{e}{m}E_0\cos(\Omega t+k_0\delta x)\sum_{n=-N}^{N}\cos n\Delta k\delta x. \tag{6.15}$$

Aside from some changes in notation, this equation is the same as the first equation in (2.4). For this equation we can immediately write an expression for the region of stochastic dynamics, (3.4):

$$|\delta v|=|v-v_g|\lesssim(\Omega_0^2\Omega L^2/k_0)^{2/3}, \tag{6.16}$$

or

$$|\delta x|\sim\frac{|\delta v|}{\Omega}\lesssim\left(\frac{\Omega_0}{\Omega}\right)^{2/3}\left(\frac{L^2}{k_0}\right)^{1/3}, \quad \Omega>\Omega_0,$$

$$|\delta x|\sim\frac{|\delta v|}{\Omega_0}\lesssim\left(\frac{\Omega}{\Omega_0}\right)^{1/3}\left(\frac{L^2}{k_0}\right)^{1/3}, \quad \Omega<\Omega_0. \tag{6.17}$$

Conditions (6.16) and (6.17) thus imply a stochastic disruption of a group resonance. Such a disruption occurs if the number of waves in the packet (N) is greater than some critical value N_0. From (6.14) and (6.17) we have

$$N\gg N_0=\frac{1}{\Delta k\,\delta x}=\left(\frac{\Omega}{\Omega_0}\right)^{2/3}\left(\frac{k_0}{\Delta k}\right)^{1/3}, \quad \Omega>\Omega_0,$$

$$N\gg N_0=\frac{1}{\Delta k\,\delta x}=\left(\frac{\Omega_0}{\Omega}\right)^{1/3}\left(\frac{k_0}{\Delta k}\right)^{1/3}, \quad \Omega<\Omega_0. \tag{6.18}$$

Clearly, in this case we can use the approximation of an infinitely wide wave packet ($N \to \infty$).

To put expression (6.18) for N_0 in a different form, we introduce the oscillation amplitude of a particle in the field E_0:

$$r_E = eE_0/m\omega_0^2.$$

We then write

$$N_0 = \left|1 - \frac{v_g}{v_0}\right|^{2/3} (\Delta k r_E)^{-1/3}, \qquad \Omega > \Omega_0,$$

$$N_0 = (k_0 r_E)^{1/6}\left(\frac{k_0}{\Delta k}\right)^{1/3} |1 - v_g/v_0|^{-1/6}, \qquad \Omega < \Omega_0. \tag{6.19}$$

The second expression in (6.19) shows, among other things, that for waves which are almost purely acoustic waves, with $|v_g - v_0| \to 0$, the boundary value becomes infinite ($N_0 \to \infty$), and there can be no stochastic disruption of the group resonance.

7. CONCLUSION

Let us summarize the phenomenon. The standard analysis of the interaction of waves with a particle in a plasma is based on two limiting situations, involving either one wave or a wave packet. In the latter case, the interaction with the group of waves is treated in a quasilinear theory. The heuristic value of this analysis is beyond question. However, exactly when one approximation or the other is valid is not settled. In reality, we are always dealing with wave packets, and to what extent they can be regarded as narrow or broad, and just how these assumptions actually affect the dynamics of the particle, remain open questions.

It was shown above that the phenomenon of a group resonance, analogous in many ways to an ordinary nonlinear resonance (i.e., analogous to the dynamics of particles trapped by a single wave), occurs. However, a group resonance occurs for particles which are moving at a velocity near $v_g \sim v_0$. A question which has not been answered is whether a resonance exists at $v \sim v_g$ if all the phase velocities of the waves in the packet are far from v. If such a situation were possible, it would become necessary to reexamine several physical processes, e.g., the propagation and damping of electromagnetic waves in a plasma.

Under certain conditions, which were established in this paper, the group resonance is disrupted, and the particle dynamics becomes stochastic. In this case, we can use a kinetic equation to describe the evolution. This equation has different structures, depending on the relation between the particle velocity and v_g. The general structure of the kinetic equation makes it possible to find the limiting particle energy distribution, which in turn creates a nonequilibrium situation. These secondary effects will be discussed in a separate paper.

We also note that there have been several studies, beginning in Ref. 3, in which some type of renormalization of the quasilinear theory has been introduced (these studies are reviewed by Krommes[16]). One purpose of the renormalization is to take into account the broadening of resonances between a particle and a wave. Postponing to another paper a critical analysis of the extreme liberties taken with the derivation of renormalized kinetic equations in Refs. 3 and 5, we simply note that under the restrictions which we have invoked in the present paper the kinetic equation which arises has a unique structure. This structure, also called a "generalized quasilinear equation," contains no renormalization of any sort. Furthermore, it is an exact consequence of the initial equations of motion.

We are sincerely indebted to S. S. Moiseev and R. Z. Sagdeev for constant interest in this study and for discussions. We are also indebted to B. V. Chirikov for useful comments.

[1)]In the case under consideration here, the center of the resonance corresponds to $\dot{y} = 0$, i.e., $v = v_0$. The reason is that the condition that the correction $\delta\dot{x}$ be small leads to the inequality $|v_0 - v_g| \ll v_g$. The wave-particle resonance, however, occurs in a way quite different from that of an ordinary phase resonance, since the resonance frequency Ω_g decays in time because of the spreading of the wave packet.

[1]A. A. Vedenov, E. P. Velikhov, and R. Z. Sagdeev, Nucl. Fusion **1**, 82 (1961).
[2]W. E. Drummond and D. Pines, Nucl. Fusion Suppl. **3**, 1049 (1962).
[3]T. Dupree, Phys. Fluids **9**, 1773 (1966); **10**, 1049 (1967).
[4]A. S. Bakaĭ, Dokl. Akad. Nauk SSSR **237**, 1069 (1977) [Sov. Phys. Dokl. **22**, 753 (1977)].
[5]J. C. Adam, G. Laval, and D. Pesme, Phys. Rev. Lett. **43**, 1671 (1979).
[6]A. A. Galeev, R. Z. Sagdeev, V. D. Shapiro, and V. I. Shevchenko, Zh. Eksp. Teor. Fiz. **79**, 2167 (1980) [Sov. Phys. JETP **52**, 1095 (1980)].
[7]G. M. Zaslavskiĭ and N. N. Filonenko, Zh. Eksp. Teor. Fiz. **54**, 1590 (1968) [Sov. Phys. JETP **27**, 851 (1968)].
[8]G. M. Zaslavskiĭ, S. S. Moiseev, R. Z. Sagdeev, *et al.*, Fiz. Plazmy **9**, 62 (1983) [Sov. J. Plasma Phys. **9**, 37 (1983)].
[9]G. M. Zaslavskiĭ, Zh. Eksp. Teor. Fiz. **88**, 1984 (1985) [Sov. Phys. JETP **61**, 1176 (1985)].
[10]G. M. Zaslavskiĭ and R. Z. Sagdeev, Zh. Eksp. Teor. Fiz. **52**, 1081 (1967) [Sov. Phys. JETP **25**, 718 (1967)].
[11]N. N. Bogolyubov, A. A. Logunov, and I. T. Todorov, Osnovy aksimaticheskogo podkhoda v kvantovoĭ teorii polya (Elements of the Axiomatic Approach in Quantum Field Theory), Nauka, Moscow, 1969.
[12]G. M. Zaslavskiĭ, Stokhastichnost' dinamicheskikh sistem (Stochastic Nature of Dynamic Systems), Nauka, Moscow, 1984.
[13]V. Fuchs, V. Krapchev, A. Ram, and A. Bers, Physica **D14**, 141 (1985).
[14]E. M. Lifshitz and L. P. Pitaevskiĭ, Fizicheskaya kinetika, Nauka, Moscow, 1979 (Physical Kinetics, Pergamon Press, Oxford, 1981).
[15]G. M. Zaslavskiĭ and B. V. Chirikov, Dokl. Akad. Nauk SSSR **159**, 306 (1964) [Sov. Phys. Dokl. **9**, 989 (1965)].
[16]J. A. Krommes, A. A. Galeev, and R. N. Sudan, Eds., in: Basic Plasma Physics, vol. 2, North-Holland, Amsterdam (1984), p. 183.

Translated by Dave Parsons