Vacuum birefringence in an intense laser radiation field

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(Submitted 23 May 1985)
Zh. Eksp. Teor. Fiz. **89**, 1181–1189 (October 1985)

Polarization phenomena accompanying the interaction of laser beams and due to scattering of light by light are considered. It is shown that a linearly polarized wave becomes elliptically polarized after interaction, and that the ellipse axis is rotated through a certain angle relative to the direction of the initial polarization. The possibilities of observing these effects in experiment are discussed.

One of the most beautiful effects predicted by quantum electrodynamics is photon-photon scattering due to polarization of vacuum. Although this phenomenon was investigated back in the Thirties,[1–3] it has not yet been directly confirmed by experiment.[1)] Of course, one can hardly doubt now that this phenomenon indeed exists, but this does not detract from the importance of searching for it in experiment.

From the viewpoint of the magnitude of the effects, the most convenient for the study of $\gamma\gamma$ scattering is the region of x-ray photons of frequency $\omega \sim m$ (m is the electron mass), where the cross section for $\gamma\gamma$ scattering has a maximum. One cannot exclude, however, the possibility that interactions between photons will be first observed in the optical band. This is due both to the increasing intensity of lasers capable of yielding light waves with more intense electric fields, and to the high measurement accuracy attainable in optical experiments.

We analyze here certain phenomena connected with the interaction of intense polarized light beams. The analysis is classical and is based on the Heisenberg-Euler Lagrangian.[6]

In Sec. 1 we discuss the interaction of head-on linearly polarized plane waves. We show that coherent interaction of photons causes each of these waves to acquire an effective refractive index. The value of this index depends on the field intensity of the head-on wave and on the relative orientation of the polarization vectors. After traversing a certain distance in the interaction region, a linearly polarized wave is generally speaking transformed into an elliptically polarized one.

In Sec. 2 are considered effects of the interaction of narrow light beams, for which the transverse dimensions of the interaction region are of substantial significance. It is shown that in this case the wave polarization not only changes from linear to elliptic, but the ellipse is furthermore rotated relative to the initial polarization plane. The two effects, i.e., the degree of ellipticity and the rotation angle, are found to be of the same order of magnitude for narrow beams. In the same section, we consider briefly effects connected with the appearance of combination frequencies in interaction of light waves.

Numerical estimates given in Sec. 3 show that these effects can already be observed with the aid of existing ultrapowerful pulsed lasers.

1. INTERACTION OF PLANE MONOCHROMATIC WAVES

Interactions of electromagnetic waves at optical frequencies ($\omega \ll m$) and at realistic electromagnetic field strengths ($\alpha^2 E^2/m^4 \ll 1$) can be described with the aid of the Heisenberg-Euler Lagrangian[6]

$$\begin{aligned} L &= \frac{1}{2}(E^2-H^2) + \frac{2}{45}\frac{\alpha^2}{m^4}\{(E^2-H^2)^2+7(\mathbf{EH})^2\} \\ &= -\frac{1}{4}F_{ik}F_{ik} + \frac{1}{180}\frac{\alpha^2}{m^4}\{14F_{ik}F_{kl}F_{lm}F_{mi} - 5(F_{ik}F_{ik})^2\}, \end{aligned} \tag{1}$$

where F_{ik} is the electromagnetic field intensity 4-tensor, **E** and **H** are the electric and magnetic field strengths, m is the electron mass, and $\alpha = \frac{1}{137}$ is the fine-structure constant. We use the Heaviside system of units and let, in addition, $h = c = 1$. The Lagrangian (1) leads to the field equations

$$\begin{aligned} \nabla_i F_{ik} = \frac{2}{45}\frac{\alpha^2}{m^4}\{&14F_{lm}F_{mi}\nabla_i F_{kl} + 14F_{kl}F_{lm}\nabla_i F_{mi} \\ &-5F_{lm}^2\nabla_i F_{ik} - 3F_{ik}F_{lm}\nabla_i F_{lm}\}, \end{aligned} \tag{2}$$

which can be solved by successive approximations, by putting

$$\begin{aligned} &F_{ik} = F_{ik}^{(0)} + F_{ik}^{(1)}, \quad \nabla_i F_{ik}^{(0)} = 0, \quad \nabla_i F_{ik}^{(1)} = j_k, \\ &j_k = \frac{2}{45}\frac{\alpha^2}{m^4}\{14F_{lm}^{(0)}F_{mi}^{(0)}\nabla_i F_{kl}^{(0)} - 3F_{ik}^{(0)}F_{lm}^{(0)}\nabla_i F_{lm}^{(0)}\}, \end{aligned} \tag{3}$$

where j_k is the effective 4-current density that results from the photon interaction.

We analyze with the aid of these equations the interaction of two plane monochromatic waves that approach each other head-on[2)] and have frequencies ω_1 and ω_2 and wave vectors k_1 and k_2,

$$k_1 = \omega_1\{1, \mathbf{n}\}, \quad k_2 = \omega_2\{1, -\mathbf{n}\}, \quad \mathbf{n}^2 = 1. \tag{4}$$

Let waves 1 and 2 be linearly polarized (circularly polarized waves will be considered below). In the zeroth approximation the intensity of the electric field can then be written in the form

$$\begin{aligned} &\mathbf{E}^{(0)} = \mathbf{E}_1 \sin k_1 x + \mathbf{E}_2 \sin k_2 x, \\ &k_1 x = \omega_1(t - \mathbf{nr}), \quad k_2 x = \omega_2(t + \mathbf{nr}), \\ &\mathbf{E}_n = E_n \mathbf{e}_n, \quad \mathbf{e}_n^2 = 1, \quad \mathbf{n}\mathbf{e}_n = 0 \ (n = 1, 2), \end{aligned} \tag{5}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are the wave polarization vectors. Substituting (5) in expression (3) for the effective current j_k we obtain

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$$j_0=0,$$

$$\mathbf{j}=\frac{16}{45}\frac{\alpha^2}{m^4}\{\omega_1[7\mathbf{E}_1E_2^2-3(\mathbf{E}_1\mathbf{E}_2)\mathbf{E}_2]\sin^2 k_2x\cos k_1x+\omega_2[7\mathbf{E}_2E_1^2-3(\mathbf{E}_1\mathbf{E}_2)\mathbf{E}_1]\sin^2 k_1x\cos k_2x\}. \qquad (6)$$

Introducing the magnetic-field intensities

$$\mathbf{H}_1=[\mathbf{nE}_1], \quad \mathbf{H}_2=-[\mathbf{nE}_2],$$

we can rewrite (6) in the equivalent form

$$\mathbf{j}=\frac{16}{45}\frac{\alpha^2}{m^4}\{\omega_1[4(\mathbf{E}_1\mathbf{E}_2)\mathbf{E}_2+7(\mathbf{E}_1\mathbf{H}_2)\mathbf{H}_2]\sin^2 k_2x\times\cos k_1x+\omega_2[4(\mathbf{E}_1\mathbf{E}_2)\mathbf{E}_1+7(\mathbf{E}_2\mathbf{H}_1)\mathbf{H}_1]\sin^2 k_1x\cos k_2x\}. \qquad (6')$$

The correction to the electric field strength $\mathbf{E}^{(1)}$ necessitated by the wave interaction is calculated from the equations

$$\mathbf{E}^{(1)}=-\partial\mathbf{A}^{(1)}/\partial t, \quad \Box\mathbf{A}^{(1)}=-\mathbf{j}. \qquad (7)$$

The nonlinearity of the current $\mathbf{j}$ in the field strength produces in the correction $\mathbf{E}^{(1)}$ not only terms with the initial frequencies ω_1 and ω_2, but also terms with the combination frequencies $2\omega_1+\omega_2$, $\omega_1+2\omega_2$, etc. The most important, at least from the quantitative standpoint, are effects connected with a coherent wave interaction in which the wave frequency remains unchanged. To take these effects into account, it suffices to make in (6) the substitution

$$\sin^2 k_nx\rightarrow\overline{\sin^2 k_nx}=1/2 \quad (n=1, 2). \qquad (8)$$

It is easy to verify from (6) and (7) that in this case the equations for waves 1 and 2 separate, and each wave can be regarded independently as moving in the mean field produced by the other wave. The influence of this field reduces to the onset of a refractive index that depends on the wave-polarization direction. Denoting the refractive index, in the case of waves with mutually parallel (perpendicular) polarizations, by $n_\parallel$ ($n_\perp$), we get from (6) and (7)

$$n_\parallel=1+\frac{16}{45}\frac{\alpha^2E^2}{m^4}, \quad n_\perp=1+\frac{28}{45}\frac{\alpha^2E^2}{m^4}, \qquad (9)$$

where E is the electric field of the head-on wave.

The difference between the refractive indices $n_\parallel$ and $n_\perp$ causes, in particular, a linearly polarized wave to become elliptically polarized after negotiating a certain distance l. The principal axis of the polarization ellipse remains directed along the initial wave polarization, and the degree of ellipticity (the ratio of the minor and major axes) is given by

$$\varepsilon=\frac{2}{15}\sin 2\theta\frac{\alpha^2E^2}{m^4}\omega l, \qquad (10)$$

where θ is the angle between the polarization planes of the head-on waves. Obviously, the maximum wave-interaction effect is reached at $\theta=45°$.

We shall not discuss now the quantitative aspect of the effects connected with such a macroscopic approach to wave interaction. Greater interest, from the experimental viewpoint, attaches to the interaction of narrow optical laser beams. Effects connected with the transverse dimensions of the beam become significant in this case, and equations such as (10) yield only the order of magnitude of the effect.

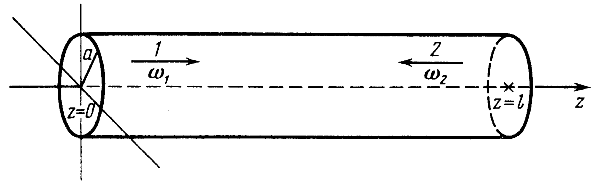

FIG. 1. Region of laser-beam interaction.

2. INTERACTION OF TWO LIGHT BEAMS

It can be seen from the preceding analysis that the effects of an interaction of electromagnetic waves depend strongly on the field strength in the wave. One of the methods of increasing this strength is to use narrow focused laser beams. Next, to enhance the effect, it is necessary to make the interaction length of the light beams as large as possible. We consider therefore cases with two head-on plane-polarized laser beams having frequencies ω_1 and ω_2 and whose interaction region is a cylinder of radius a and length $l\gg a$ (Fig. 1). Of course, we assume here that $\omega a\gg l$ and use the plane-wave approximation in the interaction region. The effective current density $\mathbf{j}$ is then given by Eq. (6) inside the cylinder, and $\mathbf{j}=0$ outside.

The correction to the vector potential $\mathbf{A}^{(1)}$, necessitated by the wave interaction, is calculated from the known formula for retarded potentials

$$\mathbf{A}^{(1)}(\mathbf{R},t)=\frac{1}{4\pi}\int_V d^3\mathbf{r}\frac{\mathbf{j}(\mathbf{r},t-|\mathbf{R}-\mathbf{r}|)}{|\mathbf{R}-\mathbf{r}|}, \qquad (11)$$

where the integration is over the volume of the cylinder.

We consider first interaction without change of frequency, when we can use the substitution (8) in (6). We assume also, for the sake of argument, that the wave 1 is more intense ($E_1>E_2$) and consider its influence on the polarization of wave 2. The correction to the vector potential of wave (2) is given, according to (6), (8), and (11), by

$$\mathbf{A}_2^{(1)}(\mathbf{R},t)=\frac{8}{45}\frac{\alpha^2E_1^2}{m^4}\omega_2E_2\{7\mathbf{e}_2-3\mathbf{e}_1(\mathbf{e}_1\mathbf{e}_2)\}I,$$
$$I=\frac{1}{4\pi}\int_V d^3\mathbf{r}|\mathbf{R}-\mathbf{r}|^{-1}\cos\omega_2(t+z-|\mathbf{R}-\mathbf{r}|), \qquad (12)$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are the polarization vectors of waves 1 and 2. We are primarily interested in the value of the field of wave 2 at the point $R=0$, i.e., after the wave has passed through the entire interaction region. At this point, the integral I can be easily calculated (see the Appendix). Putting $\omega_2a\gg1$ and $\omega_2a^2/l\sim1$, we get[3)]

$$I=\frac{l}{2\omega_2}\{\sin\omega_2tF_1(\omega_2)+\cos\omega_2tF_2(\omega_2)\},$$
$$F_1(\omega)\approx1-\cos f-f\,\mathrm{si}\,f, \quad F_2(\omega)\approx\sin f-f\,\mathrm{ci}\,f, \qquad (13)$$
$$f=a^2\omega/2l,$$

where si f and ci f are the integral sine and cosine. Using now Eqs. (7) and (12), we obtain the resultant intensity of the field of wave 2 at the origin

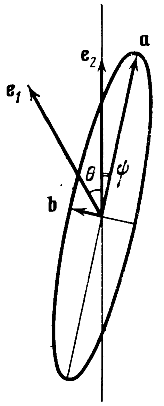

FIG. 2. Polarization of wave 2 after interaction.

$$\mathbf{E}=E_2\Big\{\mathbf{e}_2\sin\omega_2 t-\frac{4}{45}\frac{\alpha^2E_1^2}{m^4}l\omega_2[7\mathbf{e}_2-3\mathbf{e}_1(\mathbf{e}_1\mathbf{e}_2)]$$
$$\times[F_1(\omega_2)\cos\omega_2 t-F_2(\omega_2)\sin\omega_2 t]\Big\}. \qquad (14)$$

To within linear corrections, this expression can be rewritten in the form

$$\mathbf{E}=E_2\{\mathbf{a}\sin(\omega_2 t+\delta)+\mathbf{b}\cos(\omega_2 t+\delta)\},$$
$$\mathbf{a}=\mathbf{e}_2+A[7\mathbf{e}_2-3(\mathbf{e}_1\mathbf{e}_2)\mathbf{e}_1]F_2, \qquad (15)$$
$$\mathbf{b}=3A(\mathbf{e}_1\mathbf{e}_2)[\mathbf{e}_1-(\mathbf{e}_1\mathbf{e}_2)\mathbf{e}_2]F_1,$$
$$\delta=-A[7-3(\mathbf{e}_1\mathbf{e}_2)^2]F_1,\quad \mathbf{ab}=0,\quad A=\frac{4}{45}\frac{\alpha^2E_1^2}{m^4}\omega_2 l.$$

This form of the field corresponds to an elliptically polarized wave with the principal axis of the ellipse making an angle ψ with the initial polarization $\mathbf{e}_2$ (Fig. 2)

$$\psi=\frac{2}{15}\sin 2\theta\frac{\alpha^2E_1^2}{m^4}\omega_2 lF_2(\omega_2), \qquad (16)$$

and with an ellipticity given by

$$\varepsilon\equiv\frac{|\mathbf{b}|}{|\mathbf{a}|}=\frac{2}{15}\sin 2\theta\frac{\alpha^2E_1^2}{m^4}\omega_2 lF_1(\omega_2), \qquad (17)$$

where θ is the angle between the polarization planes of the head-on beams, and $\cos\theta=\mathbf{e}_1\cdot\mathbf{e}_2$. Thus, after acting with wave 1, the linearly polarized wave 2 becomes elliptically polarized, and the ellipse axis is rotated relative to the initial polarization plane. As a result, the angle between the polarizations of waves 1 and 2 increases somewhat (see Fig. 2). The effect is maximal if the angle between the polarization planes of the opposing beams is $\theta=45°$. We note also that the effect depends quadratically on the field intensity of wave 1 and is independent on its frequency.

The polarization-ellipse rotation is connected with the finite transverse dimension a of the beam. The analysis in Sec. 1 is suitable only for beams of unlimited width ($a\gg l$) and corresponds to the case $a\to\infty$ and $f\to\infty$, when $F_1=1$ and $F_2=0$.[4)] Under real experimental conditions, as will be shown below R_1 and F_2 differ noticeably from these limiting values.

We note also that the foregoing effects do not appear if wave 1 is circularly polarized. Indeed, such a wave can be resolved into two components that are polarized in mutually perpendicular planes. It can be verified that each of these components acts on wave 2 independently. The effects considered contain a factor sin 2 that reverses sign when the substitution $\theta\to\theta+\pi/2$ is made. Therefore the effects from the two mutually perpendicular components cancel each other and wave 2 remains linearly polarized in the initial direction after the interaction.

We consider now briefly the question of effects connected with the onset of combination frequencies such as $2\omega_1+\omega_2$ or $\omega_1+2\omega_2$ when waves 1 and 2 interact. These effects are due to those current terms in (6) which contain the factor $\cos 2k_1x\cos k_2x$ or $\cos k_1x\cos 2k_2x$. The difference between the directions of vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ causes these current components to be waves whose frequencies do not equal the length of the wave vector. The coherence of the radiation is thus upset in different segments of the interaction region. A radiation field of combined frequency turns out therefore to be weaker (by an approximate factor ωl) than the radiation at the fundamental frequencies.

We shall use the equations to explain how this takes place. If we are interested as before in the field at the point $\mathbf{R}=0$, calculation of the retarded potentials (11) leads to integrals of the form

$$\int_0^l dz\int_0^{a^2} d\rho^2(z^2+\rho^2)^{-1/2}\cos(\omega(z^2+\rho^2)^{1/2}-kz), \qquad (18)$$

in which $\omega a, ka\gg 1, l\gg a$. If $\omega\neq k$, the integrand in (18) oscillates rapidly with change of z, and it is this which leads to the indicated smallness. If, however, $\omega=k$ exactly, the oscillations drop out of the region $a\ll z\lesssim l$ and coherent addition of the radiation takes place.

The additional smallness of the radiation at combination frequencies notwithstanding, these field components may be of interest if sufficiently effective methods are devised for their detection. Without going into the details of the calculations, we present therefore some results.

At the point $\mathbf{R}=0$, the most substantial is the field of frequency $2\omega_2+\omega_1$, whose intensity is of the form

$$\mathbf{E}'=\frac{1}{90}\frac{\alpha^2E_1E_2^2}{m^4}(2\pi u_0)^{1/2}\Big(1+\frac{\omega_1}{2\omega_2}\Big)[7\mathbf{e}_1$$
$$-3(\mathbf{e}_1\mathbf{e}_2)\mathbf{e}_2]\cos\Big[(2\omega_2+\omega_1)t-u_0+\frac{3\pi}{4}\Big], \qquad (19)$$
$$u_0=2a(2\omega_1\omega_2)^{1/2}.$$

We have neglected in (19) the terms ~ 1 compared with the terms $\sim(\omega a)^{1/2}$. It can be seen that (19) corresponds to a field with a linear polarization whose direction differs substantially from the polarization directions $\mathbf{e}_1$ and $\mathbf{e}_2$ of the initial fields. In contrast to (14), the field (19) depends linearly on E_1 and quadratically on E_2, and it is important for its observation that wave 2 be the more intense.

3. NUMERICAL ESTIMATES AND DISCUSSION OF RESULTS

We discuss now the quantitative aspects of the effects connected with the wave interaction. It is convenient for this purpose to use a somewhat different notation for Eqs. (16) and (17).

We replace the field strength E_1 by the laser radiation power J averaged over the period. In the units employed, the relation between the two is

$$J = {}^1/_2\pi a^2 E_1^2. \tag{20}$$

The expressions for the ellipticity and for the rotation angle can then be written in the form

$$\begin{aligned} \varepsilon &= \frac{2\alpha^2}{15\pi}\sin 2\theta \frac{J}{J_0}\left(\frac{\omega_2}{m}\right)^2 \frac{1}{f} F_1(\omega_2), \\ \psi &= \frac{2\alpha^2}{15\pi}\sin 2\theta \frac{J}{J_0}\left(\frac{\omega_2}{m}\right)^2 \frac{1}{f} F_2(\omega_2), \\ f &\approx \frac{a^2\omega_2}{2l}, \quad J_0 = m^2 = 0{,}64\cdot 10^8 \text{ W}, \end{aligned} \tag{21}$$

the functions F_1 and F_2 are given by the equations in (13). We assume hereafter that the angle between the polarizations of the head-on beams is $\theta = 45°$, so that $\sin 2\theta = 1$. It can be seen that, to achieve the maximum effect, J and ω_2 must be as large as possible, and furthermore f should not be very large, since the functions F_1/f and F_2/f decrease with increasing f.

The power limit of modern pulsed lasers is $J \sim 10^{13}$ W (at a wavelength $\lambda_1 \sim 10^{-4}$ cm),[7] and the length of the interaction region is restricted to $l \sim 30$ cm by the pulse duration. Beam 2, whose polarization is analyzed, can be the fourth harmonic of the same laser ($\omega_2 = 4\omega_1$), corresponding to $\omega_2/m \sim 10^{-5}$. The value of f is bounded from below by the Fresnel conditions that stem from the uncertainty relation $\Delta k_x \Delta x \gtrsim 1$. Assuming $\Delta x \sim 2a$ and $\Delta k_x \sim \omega\vartheta \sim \omega a/l$, we obtain the condition $a^2\omega/2l \gtrsim 1/4$. Since this condition must be met not only for beam 2 but also for beam 1 with frequency $\omega_1 = \omega_2/4$, we get $f = a^2\omega_2/2l \gtrsim 1$. For the estimates we shall assume $f = 1$, which corresponds (at $l = 30$ cm) to a beam radius $a \sim 10^{-2}$ cm, in which case

$$F_1(\omega_2) \approx 1{,}08, \quad F_2(\omega_2) \approx 0{,}50.$$

It can be seen that $F_1 \sim F_2$, i.e., the ellipticity ε and the rotation angle ψ are of the same order, in contrast to the case considered in Sec. 1.

Using the values chosen above for the quantities in Eq. (21), we obtain for the expected polarization effects the estimates

$$\varepsilon \sim 4\cdot 10^{-11}, \quad \psi \sim 2\cdot 10^{-11}. \tag{22}$$

We examine now how feasible is the observation of such small effects.

Measurement of the change of a polarization state reduces, as we know, to measurement of the increment of the light intensity. The feasibility of this procedure is limited by purely statistical fluctuations of the photon flux. A sensitivity very close to the limit was realized in cw measurements with laser polarimeters.[8,9] In pulsed measurements, the major role is played by the relative fluctuation of the total number N of the photons that take part in the measurement. The maximum sensitivity attained in measurement of the rms polarization-plane rotation is, in terms of radiation, equal to $N^{-1/2}$ (under optimum recording conditions[10]). To achieve a sensitivity 10^{-11} we need therefore 10^{22} photons, which corresponds at 0.25 μm wavelength to a total control-radiation energy 10^4 J. If, for example, this power is 1% of the power of the ω_1 polarizing radiation, a minimum of 100 pulses is needed to observe the effect.

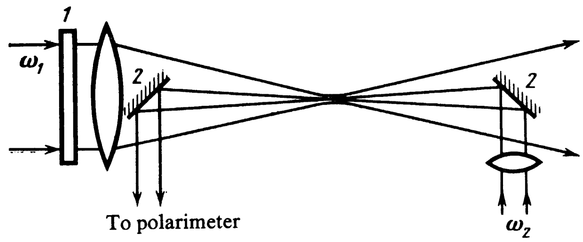

FIG. 3. Possible experimental setup.

Figure 3 shows by way of example one of the possible experimental setups. Powerful linearly polarized radiation passes through element 1 that controls the position of the polarization plane (this can be, e.g., a glass plate uniaxially strained to transform it into a $\lambda/2$ phase plate). A lens focuses the radiation into a narrow caustic. Mirrors 2 are mounted along the beam axis to pass through the caustic the focused control radiation ω_2. The latter is analyzed at the exit with a polarimeter. The polarization states of the beam ω_2 are compared as the polarization axis of beam ω_1 is successively deflected by 90° and back with the aid of element 1.

Without discussing the polarimeter variants, we note only that neither the high power of the control radiation nor the fluctuation of amplitude of the pulses are unsurmountable obstacles to attainment of maximum sensitivity.[11]

We conclude by discussing briefly the possibility of eliminating some false effects. In particular, the presence of residual matter in the laser-beam interaction region can lead to electro-optic and magneto-optic effects (such as the Kerr effect, the Faraday effect, and others) which mask effects connected with scattering of light by light. This calls for thorough evacuation of the interaction region. If we assume, by way of estimate, a residual-gas pressure $\sim 10^{-11}$ Torr, corresponding to a density $\sim 3\cdot 10^5$ atoms/cm^3, the difference between the refractive index and unity for wave 2 is of the order of $n - 1 \sim 10^{-17}$. Even if the difference between the refractive indices of waves with different polarizations were of the same order, $\Delta n \sim 10^{-17}$, the magnitude of the false effects on the residual matter would be comparable with the measured effects. In fact, all the electro-optic and magneto-optic phenomena yield $\Delta n \ll n - 1$, so that the false effects are negligibly small at the indicated pressure.

Thus, the highest level of contemporary technology of generating ultrapowerful light pulses, when combined with the maximum capabilities of optical measurements, makes feasible, for the first time ever, observation of optical polarization of vacuum. With the existing apparatus for laser thermonuclear fusion, the cost of the experiment in question is determined for the most part by the operating cost and amortization of equipment capable of withstanding many hundreds of pulses.

The authors thank M. I. D'yakonov for helpful discussion of the questions touched upon in this paper.

APPENDIX

To find the correction (11) to the field of wave 2, we must calculate the integral [see Eq. (12)]

$$I(\mathbf{R},t)=\frac{1}{4\pi}\int_V d^3\mathbf{r}\,|\mathbf{R}-\mathbf{r}|^{-1}\cos\omega(t+z-|\mathbf{R}-\mathbf{r}|),\quad (A.1)$$

where the integration is over the volume of a cylinder (Fig. 1). An analytic calculation of this integral for arbitrary $\mathbf{R}$ is a rather complicated task. If, however, we are interested only in the field along the cylinder axis (the z axis), the integral (A.1) can be expressed in terms of known functions. Denoting the coordinate of the observation point by Z, we obtain

$$I(Z)=\sin\omega(t+Z)\,[S(f_0)-S(f_l)]+\cos\omega(t+Z)\,[C(f_0)-C(f_l)]+I_1(Z).\quad (A.2)$$

Here

$$f_0(Z)=\omega\{(Z^2+a^2)^{1/2}+Z\},\qquad f_l(Z)=\omega\{((Z-l)^2+a^2)^{1/2}+Z-l\},\quad (A.3)$$

and the functions $S(x)$ and $C(x)$ are given by the formulas

$$S(x)=\frac{a^2}{4}\left\{\mathrm{si}\,x+\frac{\cos x}{x}-\frac{\sin x}{a^2\omega^2}\right\},\qquad C(x)=\frac{a^2}{4}\left\{\mathrm{ci}\,x-\frac{\sin x}{x}-\frac{\cos x}{a^2\omega^2}\right\},\quad (A.4)$$

where si x and ci x are the integral sine and cosine. The increment $I_1(Z)$ takes different forms inside the outside the cylinder:

$$I_1(Z)=\begin{cases}\dfrac{l}{2\omega}\sin\omega(t+Z) & (Z\leqslant 0),\\[2mm] \dfrac{l-Z}{2\omega}\sin\omega(t+Z)+\dfrac{1}{2\omega^2}\sin\omega t\sin\omega Z & (0\leqslant z\leqslant l),\\[2mm] \dfrac{1}{2\omega^2}\sin\omega(t+l-Z)\sin\omega l & (l\leqslant Z).\end{cases}\quad (A.5)$$

As $Z\to\pm\infty$ the integral $I(Z)\to 0$, as it should. If we are interested in the field at the point $Z=0$ and recognize that $l\gg a$ and $a\gg 1$, the foregoing expressions can be greatly simplified. Actually, in this case we have

$$f_0(0)=\omega a\gg 1,\quad S(f_0)\approx C(f_0)\approx 0,\qquad f_l(0)\equiv f=\frac{\omega a^2}{(l^2+a^2)^{1/2}+l}\approx\frac{\omega a^2}{2l}\sim 1.\quad (A.6)$$

In addition, at $fl\sim 1$ we can neglect in the functions $S(fl)$ and $C(fl)$ the last terms that contain $(a\omega)^{-2}$. As a result we obtain for $I(Z=0)$ the expression (13) above.

[1)]It must be noted, however, that a phenomenon of similar nature, Delbrück scattering[4)] of γ quanta by the Coulomb field of a nucleus, has already been observed and investigated in experiment (see, e.g., Ref. 5).

[2)]For waves traveling in the same direction $j_k=0$, i.e., such waves do not interact.

[3)]At very high frequencies it is necessary to use the more accurate value $f=a^2\omega/(l+(l^2+a^2)^{1/2})$ (see the Appendix).

[4)]If the more accurate value $f=a^2\omega/(l+(l^2+a^2)^{1/2})$ is used, the functions F_1-1 and F_2 contain as $a\to\infty$ also insignificant small terms of order $1/\omega l$, which strictly speaking do not decrease with increase of a.

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Translated by J. G. Adashko