Quantization of the Hall conductivity of a two-dimensional electron gas in a strong magnetic field

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The Hall conductivity of a two-dimensional system of noninteracting electrons in a strong magnetic field is considered. A general relation between the conductivity σ_{xy} and the properties of the current-carrying edge states is established. A new proof, based on the indicated relation, of the quantization of the Hall conductivity is presented. The important limiting case of a random potential $V(\mathbf{r})$ varying slowly in space is analyzed in detail. Formulas expressing the $\sigma_{ij}(\omega)$ in terms of the characteristics of the classical drift trajectories, which coincide with the equipotential lines $v(\mathbf{r}) = \text{const}$, are obtained in this limit. The effect of the frequency corrections on the Hall-conductivity quantization is also investigated.

1. INTRODUCTION

At low temperatures the magnetic-field dependence of the Hall resistance of a two-dimensional electron system has a stepped character in the region of strong magnetic fields B, the Hall conductivity in the plateau regions being, to a high degree of accuracy, quantized:

$$\sigma_{xy}=-ne^2/h, \qquad (1)$$

where n is a whole number[1] (we shall not discuss here fractional quantization[2]). Although great progress has been made in the construction of the theory of this phenomenon (see, for example, Refs. 3–17), there are still a number of problems here that have not been definitively solved. In particular, there is no reliable theoretical estimate for the corrections to the values of the conductivity (1). Also necessary is a detailed investigation of the structure of the delocalized states, the response at a finite frequency, the role of the inelastic processes. Finally, there is still a need for the clearest possible explanation of the quantization (1).

In the present paper we consider the Hall conductivity in a system of noninteracting two-dimensional electrons moving in an external potential $V(\mathbf{r})$. Although we shall be interested in the value of σ_{xy} in the thermodynamic limit, it is convenient to begin the computation of the Hall conductivity with an analysis of a system of finite dimensions. In a magnetic field the presence of edges leads to the appearance of current-carrying edge states corresponding to electrons "hopping" along the reflecting edge. The importance of the role of these states for the theory of the quantized Hall effect was pointed out by Halperin.[7] Further investigations[12,13,15,18–21] showed that the bulk Hall conductivity can be directly related to the characteristics of the edge states.

We present here a simple derivation of the general relation between σ_{xy} and the current carried by the edge states, this relation being valid in the case when there are no current-carrying states at the Fermi level inside the system. The derivation presented allows us to better understand the physical causes of the relation between the bulk value (of the conductivity) and the edge current. It turns out that this relation is simply a consequence of the equality to zero of the total transport current in a system with edges in thermodynamic equilibrium in an external electric field. The total equilibrium current in the situation in question vanishes as a result of the compensation of the volume Hall current by the currents flowing along the edges. This is precisely the reason why we can represent σ_{xy} in the form of a sum over only the edge states.

The fact that the Hall conductivity can be expressd in terms of the current in the edge states explain at the qualitative level the insensitivity of σ_{xy} to local changes in the potential $V(\mathbf{r})$ inside the system, since such changes have no effect on the properties of the states localized near the edges. In the thermodynamic limit σ_{xy} does not also depend on the properties of the edge, and assumes only the quantized values: integral multiples of e^2/h.

The above-noted relationships manifest themselves especially clearly in the case of a very strong magnetic field B, when the external potential $V(\mathbf{r})$ can be assumed to vary slowly over scales of the order of $l = (c\hbar/eB)^{1/2}$ (the Larmor radius for the lowest Landau level). A detailed analysis of the Hall conductivity in this limit will be carried out in Sec. 2. In the limit of strong fields we can neglect the electron transitions between the Landau levels, and the motion of the electron occupying some Landau level reduces to drift motion along an equipotential line $V(\mathbf{r}) = \text{const.}$[1)] The wave function of an electron with energy ε (measured from the center of the broadened Landau level) is then localized near the line $V(\mathbf{r}) = \varepsilon$, and decreases exponentially over a distance l from this line. The properties of the equipotential lines $V(\mathbf{r}) = \text{const}$ for a two-dimensional random potential are fairly well known in connection with the classical percolation problem.[24] If the mean value of the potential is equal to zero, then the equipotentials with $\varepsilon \neq 0$ are closed curves of finite length, which corresponds to localized states. The radius of localization increases without restriction as $\varepsilon \to 0$, and therefore the existence of a drift trajectory of infinite length is possible at $\varepsilon = 0$. According to this picture, all the states in a two-dimensional disordered system located in a strong magnetic field are localized, with the exception of the

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state at the center of the Landau level (other approaches led to a similar result[15,16,25]). The drift approximation was first applied to the analysis of the quantized Hall effect by Iordansky,[9] who obtained the formula (1) from an analysis of the delocalized drift trajectories in an external electric field. A similar picture is used to explain the quantization of σ_{xy} in Refs. 10 and 26 (see also Refs. 14 and 27).

A distinctive feature of the present investigation is the unification of the simple physical picture provided by the drift approximation with linear response theory, which allows us to obtain a new expression for the conductivity. Although the actual contribution to the Hall current in an external electric field is made only by the delocalized states with $\varepsilon = 0$, the conductivity σ_{xy} can be expressed solely in terms of the properties of the states at the Fermi level,[2)] i.e., in the present case, in terms of contour integrals along the drift trajectories $V(\mathbf{r}) = \varepsilon_F$. It then turns out that, if $\varepsilon_F > 0$ (i.e., if the Landau level is more than half-filled), then only the edge trajectories make a nonzero contribution to the expression for σ_{xy}. The existence of a plateau in the electron-concentration dependence of the Hall conductivity and the nondependence of σ_{xy} on the specific form of the random potential $V(\mathbf{r})$ turn out in this case to be simple consequences of the general topological properties of the edge trajectories.

The formulas obtained for the conductivity in the drift approximation allow us to also investigate the frequency dependence of $\sigma_{ij}(\omega)$. The frequency corrections lead to the destruction of the plateaus in the magnetic-field dependence of σ_{xy}. In the region of frequencies much higher than the characteristic frequencies of the drift motion, $\sigma_{ij}(\omega)$ coincides with the conductivity tensor of an ideal impurity-free system. For the estimation of the conductivity in the region of low and intermediate frequencies we put forward a simple model within the framework of which the response at a finite frequency can be expressed in terms of the geometric characteristics of the equipotential lines $V(\mathbf{r}) = \text{const}$. Let us note that the measurement of the conductivity at low frequencies (especially in the region $\varepsilon_F \approx 0$) could, in principle, provide important information about the properties of the random potential $V(\mathbf{r})$.

In the final section the results obtained for the statistical Hall conductivity within the framework of the drift approximation are generalized to the case of an arbitrary scattering potential. If there are no current-carrying states with energy ε_F inside the system, then σ_{xy} can be expressed in terms of the current $i(\varepsilon_F)$ carried by the edge states lying at the Fermi level. It will also be shown that the edge current $i(\varepsilon_F)$ does not depend on the specific properties of the edge, and assumes only universal quantized values. This explains the Hall-conductivity quantization, which is exact in an infinite system at zero temperature.

2. QUANTIZATION OF THE HALL CONDUCTIVITY IN A STRONG MAGNETIC FIELD

To compute the Hall conductivity, let us consider a system of large but finite dimensions $L_x \times L_y$, with periodic boundary conditions along the X axis. We shall assume that the electron motion along the Y axis is bounded by a confining potential that increases without restriction as the boundaries are approached, and is equal to zero inside the system. In fact we are considering a band of finite width rolled into a cylinder. If the electric field is directed along the Y axis, then the potential energy of an electron is equal to

$$U(x, y) = V(x, y) + eEy,$$

where $V(\mathbf{r})$ contains both a confining potential in the vicinity of the boundaries and a smoothly varying random potential with zero average value.

In a strong magnetic field, the transitions between the Landau levels can be neglected, and the motion of an electron in the lowest Landau level[3)] reduces simply to the drift of the guiding center of the cyclotron orbit. The coordinates (X, Y) of the center of the orbit are noncommuting operators (see, for example, Ref. 29), but the commutator $[X, Y] = il^2$ tends to zero in the limit as $B \to \infty$. Therefore, in strong fields we can use the classical equations of drifting motion[4)]

$$\dot{X} = \frac{l^2}{\hbar}\frac{\partial U}{\partial Y}, \quad \dot{Y} = -\frac{l^2}{\hbar}\frac{\partial U}{\partial X}. \tag{2}$$

These equations have the form of canonical equations of motion for a system with Hamiltonian $U(X, Y)$, with Y playing the role of momentum conjugate to the variable X. For the drift approximaation to be valid, it is necessary that the potential $V(\mathbf{r})$ vary slowly over scales of the order of l.

In a system of noninteracting electrons the equation for the distribution function $f(X, Y)$ at the lowest Landau level is a simple corollary of the drift equations (2), and has the form

$$\frac{\partial f}{\partial t} - \frac{l^2}{\hbar}\left[\frac{\partial U}{\partial \mathbf{R}} \times \frac{\partial f}{\partial \mathbf{R}}\right]_z = 0. \tag{3}$$

In the state of thermodynamic equilibrium $f = f_0(U(\mathbf{R}))$, where $f_0(\varepsilon)$ is the Fermi distribution function.

Let us now assume that the electron gas was initially in equilibrium in the absence of the electric field, and then the field was adiabatically switched on. The time-dependent solution to Eq. (3) with the initial condition $f = f_0(V(\mathbf{R}))$ at $t = -\infty$ can be written in the following form:

$$f(\mathbf{R}, t) = f_0\left(V(\mathbf{R}) + eE(t)Y - e\int_{-\infty}^{t} d\tau\, Y(\tau)\dot{E}(\tau)\right), \tag{4}$$

where $\mathbf{R}(\tau) = (X(\tau), Y(\tau))$ is that solution to the dirft equations (2) which is such that the trajectory corresponding to it passes through the point R at the moment of time t, i.e., $\mathbf{R}(t) = \mathbf{R}$. That (4) is a solution can be verified through direct substitution into (3), it being only necessary to take into account the fact that $Y(\tau)$ is an implicit function of t and $\mathbf{R}$. Notice that (4) differs from the equilibrium distribution function $f_0(U(\mathbf{R}))$ in an electric field. Substituting (4) into the expression for the total current:

$$\mathbf{I} = -e\int \frac{d\mathbf{R}}{2\pi l^2}\dot{\mathbf{R}} f(\mathbf{R}, t)$$

and taking account of Eq. (2), as well as the fact that, because of the presence of the confining potential, $f_0(V) = 0$ at the boundaries $y = \pm L_y/2$, we finally obtain in the approximation linear in E the expression

$$\mathbf{I}(t)=(e^2/h)\int_{-\infty}^{t} d\tau\dot{E}(\tau)\int d\mathbf{R}Y(\tau;t,\mathbf{R})\,\mathrm{rot}[\mathbf{e}_z f_0(V(\mathbf{R}))], \quad (5)$$

where $Y(\tau)$ is now determined by the equations (2) with $U = V(\mathbf{R})$ and $\mathbf{e}_z$ is the unit vector along the Z axis. At zero temperature the distribution function has the form

$$f_0(V)=\theta(\varepsilon_F-V(\mathbf{R})), \quad (6)$$

where ε_F is the Fermi energy measured from $\hbar\omega_c/2$, and therefore

$$\mathrm{rot}[\mathbf{e}_z f_0(V)]=[\mathbf{e}_z\times\nabla V]\delta(\varepsilon_F-V(\mathbf{R}))=\mathbf{t}(\mathbf{R})\delta_F(\mathbf{R}). \quad (7)$$

Here the δ function $\delta_F(\mathbf{R})$ is centered on the equipotential $V(\mathbf{R}) = \varepsilon_F$, and

$$\mathbf{t}(\mathbf{R})=[\mathbf{e}_z\times\nabla V]/|\nabla V|$$

is the unit tangent vector to this curve at the point $\mathbf{R}$. Now the current (5) can be expressed solely in terms of the properties of the drift trajectoreis with energy ε_F, i.e., the trajectories lying at the Fermi level.

Let us, to begin with, consider the response to a constant field. Notice that the expression (5), which can be derived directly from the the Kubo formula in the limit $B\to\infty$ (see Appendix I), does not vanish even when $\dot{E}(\tau)\to 0$, which indicates the absence of ergodicity.[21,31] The reason for this is that, as will be seen below, because of the presence of the confining potential, two distinct (edge) states occur at the Fermi level. The contribution of these states to the various averages does not satisfy the condition for the weakening of the correlations, and leads to a nonzero universal expression for the Hall current.

If in the distant past the electric field $E(\tau)$ was adiabatically switched on from zero to some value E, then the current (5) does not depend on the time t. This can most easily be seen if we choose the dependence $E(\tau)$ in the form of a linearly increasing function in the time interval $[-T,0]$, and set $E(\tau) = 0$ for $\tau < -T$ and $E(\tau) = E$ for $\tau > 0$. Then for $T\to\infty$ we have

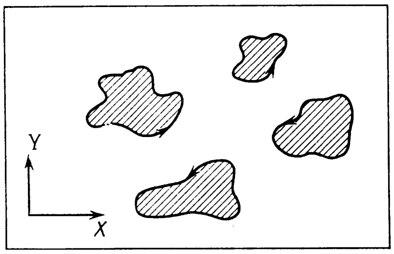
$$\int_{-\infty}^{t} d\tau\dot{E}(\tau)Y(\tau;\mathbf{R})=\lim_{T\to\infty}\frac{E}{T}\int_{-T}^{0} Y(\tau;\mathbf{R})\equiv E\overline{Y}(\mathbf{R}).$$

For periodic trajectories, which are the ones that will be considered below, $\overline{Y}(\mathbf{R})$ is simply the average over the period of the motion. The same result is obtained in the case when the field is switched on in the usual way, i.e., for $E(\tau) = E\exp\delta\tau$, $\delta\to 0$. Thus, the final expression for the current at zero temperature assumes the form

$$\mathbf{I}=\frac{e^2}{h}E\oint_{\Gamma} ds\mathbf{t}(s)\overline{Y}(s), \quad (8)$$

where Γ is the boundary of the region occupied by the electrons, i.e., the equipotential line on which $V(\mathbf{r}) = \varepsilon_F$.

Let us now consider how the current behaves as the Landau level is systematically filled. Let $\varepsilon_F < 0$ initially.



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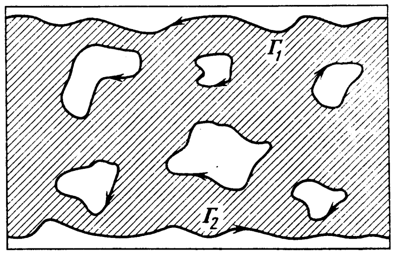

b

FIG. 1. The hatched regions are the regions for which $V(\mathbf{r}) < \varepsilon_F$ in the cases when a) $\varepsilon_F < 0$ and b) $\varepsilon_F > 0$. The continuous curves depict the equipotentials $V(\mathbf{r}) = \varepsilon_F$.

Then the regions with $V<\varepsilon_F$ form isolated "lakes" (the hatched regions in Fig. 1a), which, from the standpoint of percolation theory, are finite clusters. The curve Γ in this case consists of a set of closed contours Γ_i to which correspond periodic drift trajectories. Since for a periodic trajectory the mean quantity $\overline{Y}(\mathbf{R})$ does not depend on the location of the point $\mathbf{R}$ on the contour,

$$\mathbf{I}=\frac{e^2}{h}E\sum_i \overline{Y}_i\oint_{\Gamma_i} ds\mathbf{t}(s)=0,$$

as it should be, since, for $\varepsilon_F < 0$, all the occupied states are localized.

As the electron concentration is increased further, we reach the percolation threshold $\varepsilon_F = 0$, and the picture qualitatively changes when $\varepsilon_F > 0$. There are now in the interior only isolated "islands," for which $V > \varepsilon_F$ (see Fig. 1b), since simultaneous flow through the dark and white regions is impossible in two dimensions. But in addition to that there are still electron-free regions in the immediate neighborhood of the edges $y = \pm\frac{1}{2}L_y$, where $V > \varepsilon_F$ because of the presence of the confining potential. There are now two types of trajectories at the Fermi level. First, there are localized states inside the system, which do not make a contribution to the total current, and, secondly, there are extended edge states Γ_1 and Γ_2. Because of the periodic boundary conditions along the X axis, the motion along the trajectories Γ_1 and Γ_2 is periodic (these trajectories are closed on the cylinder). Therefore, the total current is equal to

$$I_x=\frac{e^2}{h}E\sum_{\alpha=1,2}\overline{Y}_\alpha\oint_{\Gamma_\alpha} dst_x(s). \quad (9)$$

The expression (9) does not imply that the Hall current under the conditions in question actually flows along the edges.

It simply indicates that I_x can be expressed in terms of only the characteristics of the edge states (see below, as well as Refs. 12, 13, 18–20).

The contour integrals in (9) in fact do not depend on the specific form of the contours Γ_α. This is due to the fact that

$$q=\frac{1}{L_x}\oint_\gamma ds\,t_x(s)=\frac{1}{L_x}\oint_\gamma dx$$

is a topological invariant distinguishing between two classes of homotopically nonequivalent contours existing in the geometry in question. These classes correspond to localized and extended states.[10] For the contours that can be contracted to a point through continuous deformation (to them correspond the localized states), we have $q=0$, whereas $q=\pm 1$ for the contours that traverse the entire system from $x=0$ to $x=L_x$. Since the edge states are delocalized states, we obtain from (9) the expression

$$I_x=-(e^2/h)EL_x(\overline{Y}_1-\overline{Y}_2).$$

The trajectories Γ_1 and Γ_2 are located near the edges of the system; therefore, for $L_y\to\infty$, the difference between the mean values of the Y coordinate is simply equal to the dimension of the system: $\overline{Y}_1-\overline{Y}_2\to L_y$. Therefore, in an infinite system the Hall conductivity is equal to

$$\sigma_{xy}=-e^2/h, \qquad (10)$$

where $\varepsilon_F>0$. Similar analyses can be performed for the higher Landau levels. As a result, we arrive at the ideal quantization (1): it turned out that the localization of the volume states lying at the Fermi level is sufficient for this quantization, the existence of a gap in the spectrum of the states being unnecessary.

But the main result of the analysis performed is not so much the formula (10) itself, which has been derived before by other methods in the drift approximation,[9,10,27] as the new explanation of the quantization on the basis of the representation of the Hall current in the form of (9) and the general topological properties of the edge trajectories. As we shall see below, the proof given admits of a generalization to the case of an arbitrary potential $V(\mathbf{r})$. Let us also note that, since the Hall conductivity was expressed only in terms of the properties of the states at the Fermi level, we were able to avoid the consideration of the delocalized state with $\varepsilon=0$ (which actually carries the current), for the description of which the drift approximation is, apparently, not directly applicable. The point is that, in the immediate neighborhood of the percolation threshold $\varepsilon=0$ (in the absence of an electric field) tunneling between neighboring equipotentials having the same energy is an important process (see Appendix II).

3. RESPONSE AT A FINITE FREQUENCY

The general expression (5) for the current allows us to also compute the conductivity at finite frequencies $\omega\ll\omega_c=eB/mc$. If we take account of the fact that $Y(\tau;t,\mathbf{R})$ actually depends on the difference $\tau-t$ (on account of the homogeneity of the equations (2) with respect to the time), then after simple calculations we obtain from (5) the expressions

$$\sigma_{xy}(\omega)=-i\omega\frac{e^2}{hS}\int d\mathbf{R}\tilde{Y}_\omega(\mathbf{R})\frac{\partial f_0(V)}{\partial Y},$$
$$\sigma_{yy}(\omega)=i\omega\frac{e^2}{hS}\int d\mathbf{R}\tilde{Y}_\omega(\mathbf{R})\frac{\partial f_0(V)}{\partial X}; \qquad (11)$$

here $S=L_xL_y$ is the area of the system and

$$\tilde{Y}_\omega(\mathbf{R})=\lim_{\delta\to+0}\int_0^\infty d\tau\tilde{Y}(\tau;\mathbf{R})\exp(i\omega\tau-\delta\tau), \qquad (12)$$

where $\tilde{Y}(\tau)$ denotes the time-reversed solution of the drift equations (2) that satisfies the initial condition $\tilde{\mathbf{R}}(0)=(\mathbf{R})$. At zero temperature we can again use the equality (7), and express the conductivity in terms of only the drift trajectories at the Fermi level:

$$\sigma_{xy}(\omega)=-i\omega\frac{e^2}{hS}\oint_\Gamma dX\tilde{Y}_\omega(\mathbf{R}) \qquad (13)$$

($\sigma_{yy}(\omega)$ is given be a similar formula with $dX\to-dY$). Here, as before, Γ is the equipotential curve $V(\mathbf{r})=\varepsilon_F$.

In the region of high frequencies, the computation of the conductivity can be carried through. Indeed, at the limit[5] $\omega\to\infty$ the dominant contribution to the integral (12) is made by the region of small τ, where $\tilde{Y}(\tau)\approx\tilde{Y}(0)+\tau\dot{\tilde{Y}}(0)$. Taking account of the explicit form of the equations of motion (2), we obtain the following expansion in powers of the parameter $1/\omega$:

$$\tilde{Y}_\omega(\mathbf{R})=\frac{i}{\omega}Y-\frac{1}{\omega^2}\frac{l^2}{\hbar}\frac{\partial V(\mathbf{R})}{\partial X}+\dots. \qquad (14)$$

Substituting (14) into the formula (11), we obtain

$$\sigma_{xy}(\omega)\approx-\frac{e^2}{h}\nu-\frac{i}{\omega}\frac{e^2}{hS}\int d\mathbf{R}\frac{\partial f_0}{\partial\varepsilon_F}\frac{l^2}{\hbar}\frac{\partial V}{\partial X}\frac{\partial V}{\partial Y},$$
$$\sigma_{xx}(\omega)\approx\frac{i}{\omega}\frac{e^2}{hS}\int d\mathbf{R}\frac{\partial f_0}{\partial\varepsilon_F}\frac{l^2}{\hbar}\left(\frac{\partial V}{\partial X}\right)^2, \qquad (15)$$

where ν is the degree of occupancy of the Landau level (in the $B\to\infty$ limit ν is equal to the ratio of the area occupied by the electrons to S). At zero temperature the derivative of the distribution function can be represented in the form

$$\frac{\partial f_0(V)}{\partial\varepsilon_F}=\delta(\varepsilon_F-V(\mathbf{R}))=\int\frac{dt}{2\pi}\exp[it(\varepsilon_F-V(\mathbf{R}))],$$

after which the averaging over the random potential in (15) can be carried out without difficulty. For potentials $V(\mathbf{R})$ distributed according to the Gaussian distribution with

$$\langle V(\mathbf{R})\rangle=0,\ \langle V(\mathbf{R})V(0)\rangle=V_0^2\exp(-R^2/a^2),$$

the averages in (15) can be computed with the aid of the equalities

$$\left\langle\frac{\partial V}{\partial X}\frac{\partial V}{\partial Y}\exp(-iVt)\right\rangle=0,\quad\left\langle\left(\frac{\partial V}{\partial X}\right)^2\exp(-iVt)\right\rangle=\frac{2V_0^2}{a^2}\exp(-t^2V_0^2/2).$$

If in the expression for $\sigma_{xy}(\omega)$ we also take account of the next term of the $1/\omega$ expansion, then after a series of computations we finally obtain

$$\sigma_{xy}(\omega)\approx-\frac{e^2}{h}\nu-\frac{4e^2}{h}\left(\frac{\Omega_0}{\omega}\right)^2 V_0\rho'(\varepsilon_F),$$
$$\sigma_{xx}(\omega)\approx\frac{2ie^2}{h}\left(\frac{\Omega_0}{\omega}\right)\rho(\varepsilon_F), \qquad (16)$$

where $\Omega_0=l^2V_0/\hbar a^2$ and the function

$$\rho(\varepsilon_F)=(2\pi)^{-1/2}\exp(-\varepsilon_F^2/2V_0^2)$$

is proportional to the density of states at the lowest Landau level in the limit $B\to\infty$. It can be seen from the formulas (16) that, at high frequencies, the electron practically does not feel the potential $V(\mathbf{r})$, and $\sigma_{ij}(\omega)$ tends to the conductivity tensor for the pure system.

It is not possible to compute the conductivity exactly at lower frequencies, and we limit ourselves to a qualitative analysis. We shall assume that the trajectories at the Fermi level are circles with radii of the order of the localization length $L(\varepsilon_F)$, and that the motion along them is harmonic, with frequency ω_j. In the case when $\varepsilon_F<0$ we find for the motion along the j-th circle that

$$\tilde{Y}(\tau;R)=Y\cos\omega_j\tau+X\sin\omega_j\tau,$$

after which a calculation with the aid of the formulas (12) and (13) yields the contribution of this circle to the conductivity:

$$\sigma_{xy}(\omega)=\frac{e^2}{h}\frac{\omega^2}{\omega_j^2-(\omega+i0)^2}\frac{S_j}{S},$$
$$\sigma_{xx}(\omega)=\frac{e^2}{h}\frac{-i\omega\omega_j}{\omega_j^2-(\omega+i0)^2}\frac{S_j}{S}, \qquad (17)$$

where S_j is the area enclosed by the circle.

In order to take account of the irregularity of the potential, let us further assume that the frequencies ω_j are randomly distributed around some characteristic frequency $\Omega(\varepsilon_F)$. A simple estimation yields $\Omega\sim v/t(\varepsilon_F)$, where $v\sim l^2V_0/\hbar a$ is the mean drift velocity and $t(\varepsilon_F)$ is a typical length of the drift trajectory lying at the Fermi level. For ε_F not too close to zero, $t(\varepsilon_F)\sim a$, and we have from (16) that $\Omega(\varepsilon_F)\sim\Omega_0$. In real systems it is unlikely that this frequency will be much lower than the width of the Landau level, i.e., we can expect that $\Omega_0\sim10^{10}$–10^{11} sec^{-1}. Let us, for simplicity, assume that the frequencies ω_j occur with probability

$$P(\omega_j)=2\Omega^2\omega_j(\omega_j^2+\Omega^2)^{-2}. \qquad (18)$$

Other distributions lead to qualitatively close results. The summation over all the states can now be carried out in two stages. First we sum (17) over all the circles with fixed frequency ω_j. The result will be proportional to the total area enclosed by these circles, which is equal to $\nu SP(\omega_j)$. After this, the summation over all the frequencies reduces to the averaging of (17) with weight $P(\omega_j)$. For the continuous distribution (18) we obtain as a result the expressions

$$\sigma_{xx}=-iz\left(\frac{1+iz}{1+z^2}\right)^2\frac{e^2\nu}{4\hbar},\qquad z=\frac{\omega}{\Omega(\varepsilon_F)}>0,$$
$$\sigma_{xy}=-\frac{e^2}{h}\frac{\nu z^2}{1+z^2}\left(1+\frac{2\ln z-i\pi}{1+z^2}\right), \qquad (19)$$

where $\varepsilon_F<0$, i.e., with $0<\nu<0.5$.

Similar calculations can be performed for the $\varepsilon_F>0$ case as well. The only difference (besides the existence of edge trajectories) consists in the fact that the regions bounded by the trajectories lying at the Fermi level are now free of electrons, and the direction of motion along these trajectories is reversed. If we take into account the fact that the tangent vector to the trajectory also changes sign (according to (7) its direction correlates with the direction of ∇V), then it can be shown that, for $\varepsilon_F>0$, the contribution of the individual circular trajectories to σ_{xx} coincide with (17), while the contribution to σ_{xy} differs from (17) only in sign. Taking these remarks into account, we find that, in the region $0.5<\nu<1$,

$$\sigma_{xy}=-\frac{e^2}{h}+\frac{e^2(1-\nu)}{h}\frac{z^2}{1+z^2}\left(1+\frac{2\ln z-i\pi}{1+z^2}\right), \qquad (20)$$

and σ_{xx} is given by the formula (19) with ν replaced by $1-\nu$. This replacement, as well as the fact that the frequency-dependent corrections in (20) and (19) have opposite signs, indicates that the frequency response in the region $\varepsilon_F>0$ is determined by the contribution of the positively charged holes in the population of the Landau level.

The dependence of $\sigma_{xy}(\omega)$ on the degree ν of occupancy in the model under consideration is schematically depicted in Fig. 2. It can be seen that the plateaus in σ_{xy} are destroyed by the frequency-dependent corrections $\delta\sigma_{xy}(\omega)$, whose sign depends on both ν and the relation between ω and $\Omega(\varepsilon_F)$.[6] For $\omega\gg\Omega$ the Hall conductivity given by the formulas (19) and (20) tends to the conductivity $\sigma_{xy}^0=-e^2\nu/h$ in the pure system, which agrees with the exact result (16). The dependences obtained qualitatively agree with the results obtained in Ref. 22, in which δ-function impurities are considered.

Interesting phenomena should occur near the percolation threshold $\varepsilon_F\approx0$. In this region a typical equipotential curve has a complicated fractal structure, and bears absolutely no resemblance to a circle. Therefore, for $\varepsilon_F\to0$, the above-developed harmonic approximation cannot be used to analyze the conductivity in the entire frequency range. In the region of small ε_F values a trajectory lying at the Fermi level

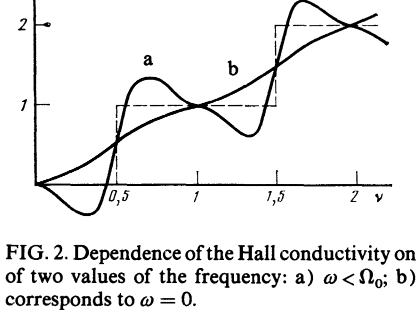

FIG. 2. Dependence of the Hall conductivity on the degree ν of occupancy of two values of the frequency: a) $\omega<\Omega_0$; b) $\omega>\Omega_0$. The dashed line corresponds to $\omega=0$.

is characterized by an entire set different scales, starting from the smallest, which is of the order of the correlation length a of the potential, and ending with the localization length $L(\varepsilon_F)$, which tends to infinity as $\varepsilon_F \to 0$. In this connection we should distinguish several frequency intervals.

In the region of high frequencies $\omega \gg \Omega_0$, the result (16) is valid. The high-frequency conductivity is determined by the smallest spatial scale a, and therefore the divergence of the localization length at $\varepsilon_F \to 0$ is not manifested in (16) in any way. For sufficiently small ε_F there exists a region of intermediate frequencies

$$\Omega(\varepsilon_F) < \omega < \Omega_0 = l^2 V_0/\hbar a^2, \quad (21)$$

in which the fractal structure of the trajectories should be exhibited. From the standpoint of percolation theory, the region occupied by the electrons in the case when $\varepsilon_F < 0$ is a set of finite clusters, and since a typical cluster has a complicated structure in the vicinity of the percolation threshold, its contribution to the frequency-dependent conductivity will differ from (17). In the frequency region (21) we can expect for the response a universal frequency dependence, determined, for example, by the fractal dimensions of the cluster or its boundary.

At low frequencies $\omega \ll \Omega(\varepsilon_F)$ the response should be determined by the entire period of the motion. Therefore, we have at the $\omega \to 0$ limit the estimate

$$\sigma_{xx}(\omega) \sim -i\omega/\Omega(\varepsilon_F), \ \mathrm{Re}\,(\sigma_{xy}+ne^2/h) \sim (\omega/\Omega(\varepsilon_F))^2. \quad (22)$$

The equipotential curve having a length equal to $t(\varepsilon_F)$ is the outer boundary of the cluster, and for large clusters $t_j \sim S_j^{0.9}$, where S_j is the area of the cluster.[34] According to the scaling theory of percolation,[35] the area of a typical cluster behaves in the region of small ε_F values like $\varepsilon_F^{-1/\sigma}$, where in the two-dimensional case $\sigma \approx 0.39$. Then for the characteristic frequency $\Omega(\varepsilon_F)$ we obtain in the $\varepsilon_F \to 0$ limit the estimate

$$\Omega(\varepsilon_F) \sim t^{-1} \sim \varepsilon_F^{\gamma}, \quad (23)$$

where $\gamma \approx 2.3$. Since at low frequencies $\sigma_{xx} \sim -i\omega L^2(\varepsilon_F)$, where $L(\varepsilon_F)$ is the localization length, we find from (23) that $L(\varepsilon_F) \sim \varepsilon_F^{-\nu}$, and that $\nu \approx 1.15$. This result is quite close to the value, $\nu = 1.35$, obtained when we take as $L(\varepsilon_F)$ the correlation length for two-dimensional percolation theory.[27]

These estimates cannot, apparently, be extended too close to the point $\varepsilon_F \approx 0$, where the quantum effects are important (see Appendix II). Therefore, we can expect the occurrence in the energy region $\varepsilon_F \sim (l/a)^2 V_0$ a transition from the value of the critical exponent ν given by percolation theory to another ν value characteristic of the quantum localization regime. It should also be emphasized that, at very low frequencies $\omega < V_0 \exp(-a^2/2l^2)$, the tunneling effects are the controlling effects even at $\varepsilon_F \sim V_0$. In this frequency region, which is very narrow when $a \gg l$, the use of the well-known Mott arguments leads to the result[36] $\mathrm{Re}\,\sigma_{xx} \sim \rho^2(\varepsilon_F)\omega^2 \ln \omega^2$.

4. GENERAL RELATION BETWEEN THE HALL CONDUCTIVITY AND THE EDGE STATES. QUANTIZATION OF THE EDGE CURRENT

The proof, given for the case of a smooth potential in Sec. 2, of the quantization of the Hall conductivity is based on the relation between σ_{xy} and the characteristics of the edge trajectories (the formula (9)). Let us now show that this proof can be generalized to the case of an arbitrary random potential $V(\mathbf{r})$.

Let us consider the previous system of finite dimensions, located in an external electric field E, which we shall now describe with the aid of the Hamiltonian

$$H = \frac{\hbar^2}{2m}\left[-\frac{\partial^2}{\partial y^2} + \left(-i\frac{\partial}{\partial x} + \frac{eB}{c\hbar}(y+\xi)\right)^2\right] + eEy + V(x,y), \quad (24)$$

where $V(\mathbf{r})$ is the sum of a random and a confining potential and ξ is an arbitrary parameter. In the case of periodic boundary conditions, $\psi(0,y) = \psi(L_x, y)$, for the motion along the X axis we can assume that the electron motion occurs on the surface of a cylinder. In this configuration the introduction of the parameter ξ corresponds to the addition to the system of a solenoid placed on the axis of the cylinder, and containing a magnetic flux proportional to ξ (Refs. 4, 13, and 37). The parameter ξ can be eliminated from the Hamiltonian by means of the gauge transformation

$$\psi(x, y) \to \psi(x, y) \exp(-i\xi x/l^2). \quad (25)$$

In that case, however, the new wave function will satisfy modified boundary conditions: $\psi(0,y) = \psi(L_x, y) \times \exp(i\xi L_x/l^2)$. Only when $\xi = 2\pi l^2 k/L_x$ (k is a whole number) will the boundary conditions remain unchanged and (25) be an admissible gauge transformation.

Let us further note that, in the limit $L_x \to \infty$, the observable quantities computed for a fixed temperature and a fixed chemical potential μ should not depend on the gauge variable[7)] ξ (cf. Ref. 17). Indeed, any physical quantity assumes the same value at the points $\xi = \xi_k = 2\pi l^2 k/L_x$, since in this case the parameter ξ can be eliminated by means of the gauge transformation (25). In the limit $L_x \to \infty$, when the system under consideration turns into a strip of infinite length, the points ξ_k become infinitely closely bunched up, and any dependence on ξ disappears. From this it follows, in particular, that in the state of thermodynamic equilibrium the total current flowing along the X axis should be equal to zero (even in the presence of an external electric field). Indeed, the equilibrium current can be represented in the form

$$I_x = -e\sum_{\alpha'} \langle \alpha' | v_x | \alpha' \rangle f_0(\varepsilon_{\alpha}') = -\frac{c}{B}\sum_{\alpha'} \frac{\partial \varepsilon_{\alpha}'}{\partial \xi} f_0(\varepsilon_{\alpha}') = -\frac{c}{B}\left(\frac{\partial \Omega}{\partial \xi}\right)_{T,\mu}, \quad (26)$$

where v_x is the velocity operator, the $|\alpha'\rangle$ are the exact eigenstates of the Hamiltonian (24) with the eigenvalues ε_α', $f_0(\varepsilon)$ is the Fermi distribution function, and $\Omega(\mu, T)$ is the thermodynamic potential of the system of noninteracting electrons. Since in the $L_x \to \infty$ limit Ω does not depend on ξ, the equilibrium current (26) vanishes.

Let us now expand the current (26) in a series in powers of the electric field (in the case of finite L_y), and let us limit ourselves to the terms linear in E. Then from the fact that the equilibrium current is equal to zero we obtain

$$-e^2\sum_{\alpha\neq\beta}\frac{f_0(\varepsilon_\alpha)}{\varepsilon_\alpha-\varepsilon_\beta}[\langle\alpha|v_x|\beta\rangle\langle\beta|y|\alpha\rangle+\text{c.c.}]$$

$$-e\sum_\alpha\langle\alpha|v_x|\alpha\rangle\frac{\partial f_0}{\partial\varepsilon_\alpha}\left(\frac{\partial\varepsilon_\alpha'}{\partial E}\right)_{E=0}=0, \qquad (27)$$

where $|\alpha\rangle$ and $|\beta\rangle$ are the $E=0$ eigenstates with energies ε_α and ε_β respectively. The first term in (27) is simply the Kubo formula for the static Hall conductivity. Since the electric field enters into the electron Hamiltonian only in the form $\delta H = eEy$, we have in the $E\to 0$ limit

$$\partial\varepsilon_\alpha'/\partial E=\langle\alpha'|\partial H/\partial E|\alpha'\rangle\approx e\langle\alpha|y|\alpha\rangle.$$

Taking these observations into account, we find from (27) that at zero temperature

$$\sigma_{xy}=-\frac{e^2}{S}\sum_\alpha\langle\alpha|v_x|\alpha\rangle\langle\alpha|y|\alpha\rangle\delta(\varepsilon_\alpha-\varepsilon_F). \qquad (28)$$

Only the current-carrying states (for which $\langle\alpha|v_x|\alpha\rangle\neq 0$) lying at the Fermi level make a contribution to the right-hand side of the formula (28). If all the $\varepsilon_\alpha=\varepsilon_F$ states inside the system are localized, or if ε_F falls within the gap in the spectrum of the volume states, then such current-carrying states can exist only at the edges of the system.[8)] The physical meaning of Eq. (27) then become clear: at equilibrium the bulk current $j_x=\sigma_{xy}E$ carried by the delocalized states with $\varepsilon_\alpha<\varepsilon_F$ is fully compensated by the total edge current that arises because of the difference in the electron concentrations at the opposite edges. It is precisely because of this that we are able to represent the bulk conductivity σ_{xy} in the form of a sum over only the edge states. The direct analog of the expression (28) in the strong-magnetic-field limit is the formula (9) obtained earlier.

Let us now take into account the fact that the edge states are localized near the edges of the system, and therefore $y|\alpha\rangle\approx\pm(L_y/2)|\alpha\rangle$ and the edge currents flowing along opposite edges in the case $E=0$ are equal in absolute value, but have different directions. Then we obtain from (28) in the limit $L_y\to\infty$ the expression

$$\sigma_{xy}=-\frac{e^2}{l}{\sum_\alpha}'\langle\alpha|v_x|\alpha\rangle\delta(\varepsilon_\alpha-\varepsilon_F)\equiv-ei(\varepsilon_F). \qquad (29)$$

The prime on the summation sign indicates that the summation in (29) is over only the edge states near one of the boundaries ($y=-L_y/2$), i.e., virtually over the delocalized states in a semifinite system. We denote by $i(\varepsilon)$ the contribution of the states with energy ε to the total current (per unit length) flowing near the edge.

Since the thermodynamic limit σ_{xy} does not depend on the specific properties of the edge, the current $i(\varepsilon_F)$ should also possess this property. This is most easily proved in the case when there is a gap in the spectrum of the volume states, and the Fermi level lies in the gap. Then the current $i(\varepsilon_F)$ turns out to be connected with the magnetic moment M of the system (see Appendix III), and we obtain from (29) the following well-known formula for the Hall conductivity[8,19,38,39]:

$$\sigma_{xy}=-ec\partial M/\partial\varepsilon_F=-ec(\partial n/\partial B)\varepsilon_F,$$

where n is the mean electron density at a fixed Fermi energy. In the more general case, when there is no gap in the spectrum, this formula is no longer applicable, but the current $i(\varepsilon_F)$ carried by the states at the Fermi level in a semifinite system all the same assumes only universal quantized values.

To prove this assertion, let us introduce some new concepts. Let us consider the Schrödinger equation with the Hamiltonian (24) with $E=0$. Let us eliminate the parameter ξ from the Hamiltonian with the aid of the transformation (25). Then, as noted above, the dependence on ξ will show up in the boundary conditions

$$\psi(0,y)=\psi(L_x,y)\exp(i\xi L_x/l^2).$$

Since when ξ is changed by an amount that is an integral multiple of $2\pi l^2/L_x$, the boundary conditions remain unchanged, any solution of the Schrödinger equation will, when subjected to the transformation

$$T:\xi\to\xi+2\pi l^2/L_x, \qquad (30)$$

which consists in the adiabatic variation of the parameter ξ, again go over into a solution of the same equation with the same boundary conditions. For localized states the explicit form of the boundary conditions is unimportant, and they do not depend on ξ at all in the limit $L_x\to\infty$, but this is not the case for extended states.[13,37,40] Therefore, the group $G=\{T^k; k=0, \pm 1, \pm 2,...\}$ (the transformation that is the inverse of (30), and that is defined as the decrease of ξ by $2\pi l^2/L_x$) has a nontrivial effect on the set of all delocalized states.

If now we choose some delocalized state $|\alpha_0\rangle$, then we obtain through the successive application of the operation T a set of delocalized states $|\alpha_0(k)\rangle=T^k|\alpha_0\rangle$, called the orbit of the state $|\alpha_0\rangle$ with respect to the group G. The entire set of extended states then splits up into nonintersecting equivalence classes (orbits) under the action of the group G (two states $|\alpha\rangle$ and $|\beta\rangle$ are equivalent if $|\alpha\rangle=g|\beta\rangle$ for some $g\in G$). For example, in the $V(\mathbf{r})=0$ case all the states of a given Landau belong to one orbit, since the transformation T in this case simply changes the y coordinate of the center of the cyclotron circle. In the general case several orbits can correspond to each Landau level.

As a result we obtain a convenient classification of the extended states, in which each such state is completely specified by giving the orbit to which this state belongs and the integer k specifying the position of the state in question on this orbit.

Let us now return to the expression (29) of the current, which contains a sum over all the extended states in a semifinite system (since it is assumed that there are no extended states at the Fermi level in the interior, only the edge states actually make a contribution to the sum). We can, in accordance with the above-proposed classification of such states, rewrite the sum in (29) in the form

$$i(\varepsilon_F)=\frac{e}{L_x}\sum_N\sum_k\langle\alpha_N(k)|v_x|\alpha_N(k)\rangle\delta(\varepsilon_F-\varepsilon_N(k)), \qquad (31)$$

where we perform first the summation over the states belonging to the N-th orbit, and then the summation over all the orbits. In the limit $L_x \to \infty$ the transformation (30) corresponds to an infinitesimal change in the parameter ξ, which leaves the expression under the summation sign almost unchanged. Therefore, the summation over k can be replaced by integration. If next we use the fact that $dk=(L_x/2\pi l^2)d\xi$, and $\langle\alpha|v_x|\alpha\rangle=(l^2/\hbar)\partial\varepsilon_\alpha/\partial\xi$, then from (31) we obtain

$$i(\varepsilon_F)=\frac{e}{h}\sum_N\int d\xi\frac{\partial\varepsilon_N(\xi)}{\partial\xi}\delta(\varepsilon_F-\varepsilon_N(\xi))=\frac{e}{h}(N_+-N_-), \tag{32}$$

where N_+ is the number of those intersections of the Fermi level by the orbits of the group G for which $\partial\varepsilon_N/\partial\xi>0$ when $\varepsilon_N(\xi)=\varepsilon_F$ and N_- is the number of intersections for which this derivative is negative. If there is no scattering potential, then $N_-=0$, and N_+ coincides with the number of filled Landau levels. Although this is not so in the general case (see Refs. 11, 12, and 20, in which the quantization of σ_{xy} in a periodic potential is considered), the current $i(\varepsilon_F)$ always assumes only quantized values.

Since the bulk Hall conductivity is equal to $-ei(\varepsilon_F)$, we have thus shown that, in the case when there are no current-carrying states at the Fermi level inside the system, σ_{xy} in an infinite system and at zero temperature can assume only values that are integral mutiples of e^2/h. It should be noted that an expression containing an integral similar to (32) is obtained in Ref. 13 for the Hall current in a disordered system. The distinctive feature of our approach consists in the fact that the quantization of σ_{xy} is explained as being the result of the quantization of the current carried by the edge states, in accordance with the clear picture obtained within the framework of the drift approximation.

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APPENDIX I.

From the Kubo formula we can, following Ref. 41, obtain for the Hall conductivity in the limit $B\to\infty$ the expression

$$\sigma_{xy}=-ecn/B+\delta\sigma_{xy}, \tag{33}$$

$$\delta\sigma_{xy}=-\frac{e^2}{S}\int_0^\infty d\tau e^{-\delta\tau}\int\frac{dX\,dY}{2\pi l^2}\beta\frac{l^2}{\hbar}\frac{\partial V}{\partial Y}f_0(V)[1-f_0(V)]\dot{\tilde{Y}}(\tau),$$

where $\tilde{Y}(\tau)$ is the solution to the drift equations (2) with reversed time and β is the reciprocal temperature (the notation here is some-what different from the one used in Ref. 41). Using the identity

$$\beta(\partial V/\partial Y)f_0(V)[1-f_0(V)]=-\partial f_0(V)/\partial Y,$$

and integrating by parts in (33), we finally obtain

$$\sigma_{xy}=-\frac{e^2}{hS}\int[Y_2f_0(Y_2,X)-Y_1f_0(Y_1,X)]dX$$
$$+\delta\frac{e^2}{hS}\int_0^\infty d\tau e^{-\delta\tau}\int d\mathbf{R}\frac{\partial f_0(V)}{\partial Y}\tilde{Y}(\tau), \tag{34}$$

where Y_2 and Y_1 are the coordinates of the edges of the system. The first term in (34) is obtained in Ref. 41, but in our case it is equal to zero, since $f_0(V)=0$ at the edges. The second term is not obtained in Ref. 41, since ergodicity is assumed there, and δ is taken to be equal to zero from the very beginning. But in the presence of a confining potential, because of the contribution of the edge states, the integrand in (34) does not decrease at $\tau\to\infty$, if we set $\delta=0$. Therefore, the last term in (34) possesses a finite, nonzero limit at $\delta\to 0$. It is easy to verify that (34) leads to the expression (5) for the current.

APPENDIX II.

Let us estimate the energy region around the percolation threshold $\varepsilon=0$ where the processes of tunneling between neighboring equipotentials having the same energy are important. Let us assume that two such equipotentials come closest to each other in the vicinity of the saddle point lying on the trajectory with $\varepsilon=0$. If we assume that the separatrices of the saddle coincide with the coordinate axes, then in the neighborhood of the saddle point we have

$$V(X,Y)\approx cXY,\quad c\sim\nabla^2V,$$

and the drift trajectories have the form of hyperbolas. The shortest distance between two trajectories with energy ε in this case is $d=(8\varepsilon/c)^{1/2}$. Tunneling is important when $d\lesssim l$. But in the region of energies

$$|\varepsilon|\gg\Delta\varepsilon\sim l^2\langle|\nabla^2V|\rangle,$$

when $\langle d\rangle\gg l$, the quantum effects can be neglected. Of course in the case of a purely periodic potential the tunneling process is important at any energy because of the resonance effects, which, however, disappear in the presence of even a slight disorder. Let us note that, in a strong magnetic field, the energy region $\Delta\varepsilon$ is much smaller than the total Landau-level width, which is of the order of $\langle|V|\rangle$. The transition from the classical picture of the localization to the purely quantum picture can be expected to occur in the region $|\varepsilon|<\Delta\varepsilon$.

APPENDIX III

The quantity $i(\varepsilon)$ is equal to the contribution of the edge states with energy ε to the diamagnetic current, $I_D=cM$ (M is the magnetization), per unit length flowing in the vicinity of one of the edges. The fact that the diamagnetic current flows alongside the edges is an obvious consequence of the equality $j=c\,\mathrm{curl}\,M$, since the current density is nonzero only at the edges, where M decreases to zero. It should be emphasized that, unlike the total transport current, the current flowing near one edge includes a contribution from the localized volume states; therefore, in the general case

$$I_D=\int_{-\infty}^{\varepsilon_F}d\varepsilon[i(\varepsilon)+i_0(\varepsilon)],$$

where $i_0(\varepsilon)$ is the contribution of the volume states. Only in the case when the Fermi level lies in the gap in the spectrum of the volume states do we have $i_0(\varepsilon_F)=0$ and

$$i(\varepsilon_F)=\partial I_D/\partial\varepsilon_F=c\partial M/\partial\varepsilon_F,$$

from which we immediately obtain the formula for σ_{xy} given in the text. The fact that the Hall conductivity in the situation under consideration can be expressed in terms of the magnetic moment has been pointed out by Widom[38] and Streda and Smrcka.[39]

[1)]This drift approximation describes the opposite limiting case with respect to the δ-function impurity models.[3,22] Since a real random potential contains both short-wavelength fluctuations with $a<l$ (a is the correlation length of the potential) and fluctuations with $a\gg l$ (Ref. 23), both limiting cases are equally worthy to be studied.

[2)]This assertion is valid in the general case as well.[28]

[3)]The entire subsequent analysis can be generalized without difficulty to the case of an arbitrary Landau level.

[4)]The $B\to\infty$ limit is the classical limit in a system of interacting electrons as well. It turns out that we can also obtain general expressions for the thermodynamic quantities in the form of power series in $1/B$, the coefficients of which can be expressed in terms of the total potential energy and its derivatives averaged over the classical Boltzmann distribution.[30]

[5)]This limit indicates that ω is much higher than the characteristic frequencies of the drift motion, but, of course, $\omega\ll\omega_c$.

[6)]Pepper and Wakabayashi[32] have experimentally observed the destruction of the plateuas in the very low frequency region (1–10 kHz). The dependencies obtained by them cannot be explained by the present model, and are apparently of different origin (see, in this connection, Ref. 33).

[7)]Let us emphasize that we are speaking of equilibrium values computed at fixed ξ. The energy an the other physical quantities vary in the course of the adiabatic variation of ξ.

[8)]Let us recall that, in a magnetic field, there always occur near the edges delocalized states (corresponding to "hopping" electrons) that survive in the presence of disorder.[7,15]

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