Resonant excitation of quasistatic multipole oscillations of a plasma ellipsoid

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A general theory is derived for the natural and forced quasistatic oscillations of a homogeneous and isotropic plasma with the shape of a triaxial ellipsoid. Closed analytic representations are derived for the frequencies, radiative damping constants, and multipole moments, including the magnetic and toroidal moments of the polarization currents which result from the nonspherical shape of the plasma, for dipole, quadrupole, and octupole oscillations. The force exerted on the plasma in an incident plane wave is derived. In the case of a double resonance, this force has a lateral component comparable to the longitudinal component.

1. INTRODUCTION

Resonant effects play a special role in the interaction of a high-frequency electromagnetic field with a small volume of plasma, and they become progressively more intense as the dimensions of the plasma become smaller in comparison with the wavelength of the external field. An analysis of these effects requires that we first study the natural oscillations of the plasma, i.e., find the frequencies, damping rates, and spatial structures of these oscillations. So far, this problem has been solved rigorously only for the simplest model, that of a uniform plasma sphere.[1] In the case of a small plasma in which we are interested here, the approximate quasistatic theory works quite well. That theory was used in Ref. 2 to derive dispersion relations for the real natural frequencies of a plasma spheroid.

The symmetry (the geometric degeneracy), of the model of a spheroid or, especially, a sphere however, may conceal certain features which are characteristic of plasmas of arbitrary shape. A flexible geometric model for such plasmas is a triaxial ellipsoid with arbitrary ratios of axes. Our purpose in this paper is to derive a theory of the resonant effects in a homogeneous and isotropic plasma ellipsoid.

We use the following general method for analyzing quasistatic oscillations of plasma formations of arbitrary shape. We denote by $\mathbf{P}(\mathbf{r}, t)$ the polarization distribution in the plasma, and we denote by $\mathbf{v}(\mathbf{r}, t)$ the velocity field of the plasma electrons. We assume that the plasma is uniform, isotropic, and cold (there is no spatial dispersion). We then obviously have $\dot{\mathbf{P}} = ne\mathbf{v}$, where n is the density, and e is the charge of an electron. The polarization $\mathbf{P}$ creates an electric field $\mathbf{E}_P$, which in the quasistatic theory is simply the Coulomb field of the exchange polarization charges, $\rho = \operatorname{div}\mathbf{P}$, and the surface charges, $\sigma = P_n$ where $\mathbf{n}$ is the outward normal to the surface of the plasma.

If, in addition to the field produced by the plasma itself, $\mathbf{E}_P$, there is a field produced by external sources, $\mathbf{E}_{\text{ext}}(\mathbf{r}, t)$, the equation of motion of an electron can be written

$$\ddot{\mathbf{P}}+\nu_{eff}\dot{\mathbf{P}}=\frac{\omega_0^2}{4\pi}(\mathbf{E}_P+\mathbf{E}_{ext}), \tag{1.1}$$

where $\omega_0 = (4\pi ne^2/m)^{1/2}$ is the plasma frequncy, and ν_{eff} the effective collision rate. In particular with $\nu_{\text{eff}} = 0$ and $\mathbf{E}_{\text{ext}} = 0$, Eq. (1.1) becomes the quasistatic equation

$$(\omega_0^2/4\pi)\mathbf{E}_P=-\bar{\omega}^2\mathbf{P},$$

which determines both the real frequencies $\bar{\omega}$ and the distributions $\mathbf{P}(\mathbf{r})$ of the natural oscillations. For these oscillations, the vectors $\mathbf{v}$ and $\mathbf{E}_P$ are $\pi/2$ out of phase, and the dynamics of the quasistatic natural oscillations involve pumping the energy of the electric field, $U_E = U_E^0 \cos^2(\bar{\omega}t + \psi)$, into the kinetic energy of the particles, $K(t) = K^0\sin^2(\bar{\omega}t + \psi)$. Here

$$U_E^0=U_i+U_e=\frac{1}{8\pi}\int|\mathbf{E}_P|^2dV_i+\frac{1}{8\pi}\int|\mathbf{E}_P|^2dV_e,$$

$$K^0=\frac{nm}{2}\int|\mathbf{v}|^2dV_i=\frac{1}{8\pi}\frac{\omega_0^2}{\bar{\omega}^2}\int|\mathbf{E}_P|^2dV_i=2\pi\frac{\bar{\omega}^2}{\omega_0^2}\int|\mathbf{P}|^2dV_i$$

are the peak values of these energies. Obviously, a necessary condition for the occurrence of a natural oscillation is $K^0 = U_E^0$, from which we immediately find $\bar{\omega}\leqslant\omega_0$. The equality here holds only in the case of completely localized oscillations, in which the field vanishes in the volume V_e outside the plasma. In general, introducing the dimensionless natural frequency $\Omega = \bar{\omega}/\omega_0$, we have

$$K^0=2\pi\Omega^2\int|\mathbf{P}|^2dV,\quad U_i=\Omega^2K^0,\quad U_e=(1-\Omega^2)K^0, \tag{1.2}$$

where K^0 is the total (peak kinetic) energy of the natural oscillation.

Since the quasistatic fields of the natural oscillations are electrostatic, $\mathbf{P} = -\nabla\Pi$, $\mathbf{E}_P = -\nabla\Phi$, and since we have

$$\Phi=\int\frac{\rho dV}{R}+\oint_i\frac{\sigma dS}{R}=\int\nabla^2\Pi\frac{dV_i}{R}-\oint\frac{\partial\Pi}{\partial n}\frac{dS}{R}$$

in free space, by introducing the operator

$$\mathscr{L}\{f\}=\frac{1}{4\pi}\left(\oint\frac{\partial f}{\partial n}\frac{dS}{R}-\int\nabla^2 f\frac{dV_i}{R}\right) \tag{1.3}$$

we can transform from the vector equation $\mathbf{E}_P = -4\pi\Omega^2\mathbf{P}$ to the standard scalar equation of the eigenvalue problem:

$$\mathscr{L}\{\Pi\}=\Omega^2\Pi. \tag{1.4}$$

For unlocalized oscillations, with $\Omega < 1$, the dielectric constant of the plasma, $\varepsilon = 1 - \Omega^{-2}$, is nonzero, so that the fields are harmonic: $\nabla^2\Pi = 0$. The operator $\mathscr{L}$ thus contains only a surface integral.

If problem (1.4) has been solved, we need only find the damping constants to obtain a complete description of the

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natural oscillations. The collisional loss is evidently $\gamma_{CT} = 1/2\ \nu_{eff}$. In the case of a collisionless plasma, on the other hand, we would also have to take into account the small radiative loss, which gives rise to a radiative damping rate γ_{rad}. This rate can be found by, for example, the method of Ref. 3, which is based on simple energy considerations. Corresponding to the oscillatory polarization $\mathbf{P}\exp(-i\omega t)$ is a polarization current $\mathbf{j} = -i\omega\mathbf{P}$, which creates in the wave zone in free space (this method also works in regions with boundaries) a radiation field

$$\mathbf{H}=k^2\frac{e^{ikr}}{r}[\mathbf{vG}],\quad \mathbf{E}=[\mathbf{Hv}],$$

where $k = \omega/c$, $\mathbf{v}$ is a unit vector along the direction to the observation point, with $\mathbf{G}$ is the interference vector

$$\mathbf{G}(\mathbf{v})=\int\mathbf{P}(\mathbf{r})e^{-ik\mathbf{vr}}dV$$

$$=\int\mathbf{P}dV-ik\int(\mathbf{vr})\mathbf{P}dV-\frac{k^2}{2}\int(\mathbf{vr})^2\mathbf{P}dV+\ldots. \qquad (1.5)$$

Since the dimensions of the plasma are small, we need to consider only the first nonvanishing term of expansion (1.5). The radiated power is

$$J=\frac{c}{8\pi}\oint|\mathbf{H}|^2dS_r=\frac{ck^4}{2}\overline{|[\mathbf{vG}]|^2}, \qquad (1.6)$$

where the superior bar means an average over the solid angle. Calculating J, we immediately also find the radiative damping rate $\gamma_{rad} = J/(2K^0)$, where K^0 is given by (1.2).

2. NATURAL OSCILLATIONS OF A PLASMA ELLIPSOID

For an object of arbitrary shape, numerical methods would have to be used to solve the functional equation (1.4). The situation simplifies dramatically, however, in the case of an ellipsoid, where—as follows from results derived in the 19th century by Ferrers[4]—a polynomial distribution of the function $f(x,y,z)$ corresponds to an operator $\mathscr{L}\{f\}$ which has the form of a polynomial of the same degree,[5] with coefficients which contain internal potential factors

$$M_{lmn}=(2l-1)!!(2m-1)!!(2n-1)!!\frac{abc}{2}$$

$$\times\int_0^\infty(a^2+\lambda)^{-l}(b^2+\lambda)^{-m}(c^2+\lambda)^{-n}\frac{d\lambda}{D(\lambda)},$$

$$D(\lambda)=[(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)]^{1/2}. \qquad (2.1)$$

Here a, b, c are the semiaxes of the ellipsoid, whose center coincides with origin of a Cartesian coordinate system x, y, z, whose axes are oriented along those of the ellipsoid. In particular, $M_{100} = M_a$, $M_{010} = M_b$, $M_{001} = M_c$ are the ordinary depolarization factors ($M_a + M_b + M_c = 1$), while all the other M_{lmn} can be expressed in terms of M_a, M_b, and M_c by means of the recurrence relations given in Ref. 5.

In Ref. 5, using the results derived by Ferrers, we solved the problem of a dielectric ellipsoid in a nonuniform static field, and we found certain eigenfrequencies of a plasma ellipsoid. In Ref. 5 we took a "head-on" approach to these problems: We studied linear algebraic equations for the coefficients of the polynomial distributions $\Pi(x, y, z)$ which arise when these distributions are substituted into equations like (1.4). In this head-on approach, the calculations are extremely complicated, and the final results are excessively complicated in places. Fortunately, the mathematical arsenal of the 19th century furnishes essentially a ready-made solution of this problem. Specifically, it follows from the Liouville equations (see Ref. 6, for example) for Lamé ellipsoidal harmonic functions that the internal Lamé harmonics are eigenfunctions of Eq. (1.4). Although both the Liouville equations and the Lamé harmonics themselves are conventionally written in ellipsoidal coordinates, they can be rewritten in Cartesian coordinates, where (as Niven has shown[7]) the Lamé functions have a polynomial structure and are described by

$$\left\{\begin{matrix} & x & xy & \\ 1 & y & yz & xyz \\ & z & zx & \end{matrix}\right\}\cdot1\cdot\Theta_1\cdot\Theta_2\ldots\Theta_k, \qquad (2.2)$$

where each Θ_i is given by

$$\Theta_i=1-\alpha_i x^2-\beta_i y^2-\gamma_i z^2,$$

$$\alpha_i=\frac{1}{a^2-\theta_i},\quad \beta_i=\frac{1}{b^2-\theta_i},\quad \gamma_i=\frac{1}{c^2-\theta_i}, \qquad (2.3)$$

and the values of the quantities θ_i are found from the harmonic condition. Expression (2.3) differs from that in Ref. 7 in that the signs of the θ_i have been changed (so that the latter become positive). From the Liouville equations we also find the following general expression for the eigenvalues of Eq. (1.4) corresponding to the eigenfunctions in (2.2):

$$\Omega^2=\frac{abc}{2}\left[\frac{dR(\lambda)}{d\lambda}\right]_{\lambda=0}\int_0^\infty\frac{d\lambda}{R(\lambda)D(\lambda)}. \qquad (2.4)$$

Here $R(\lambda)$ is the product of the factors $(a^2+\lambda)$, $(b^2+\lambda)$ and $(c^2+\lambda)$, each of which corresponds to x, y, and z braces in (2.2), and factors $(\theta_i+\lambda)^2$, which correspond to quadratic functions of Θ_i in (2.2).

Let us examine in more detail the dipole, quadrupole, and octupole oscillations, designating them by the following, sequence of numbers (in addition to suitable indices): 1–3 for dipole modes, 4–8 for quadrupole modes, and 9–15 for octupole modes. Furthermore, we supplement the dimensionless frequency $\Omega = \omega/\omega_0$ by the dimensionless damping rate $\Gamma = \gamma/\omega_0$, and we write all the polarization potentials of the natural modes in dimensionless form. All the factors

$$L=\int|\mathbf{P}|^2dV=\int(\nabla\Pi)^2dV$$

will then have the dimensions of length.

Dipole oscillations (which were studied back in Ref. 3) correspond to a uniform polarization. There are three such modes, and they are described by the polarization potentials

$$\Pi_1=\Pi_a=-x/a,\ \Pi_2=\Pi_b=-y/b,\ \Pi_3=\Pi_c=-z/c.$$

For the potential Π_a in expansion (1.5), the very first term, $G_x = V/a$, is nonzero; this term is equal to the total dipole moment of the plasma. In this case we have $L_a = V/a^2$, $R(\lambda) = a^2 + \lambda$, so that for a mode with potential Π_1 we have

$$\frac{J_1}{\omega}=\frac{1}{3}\frac{k^3V^2}{a^2},\quad \Omega_1{}^2=\Omega_a{}^2=M_a,\quad \Gamma_a=\frac{\Omega_a{}^2}{12\pi}k_0{}^3V. \qquad (2.5)$$

Cyclic substitution leads to expressions for the two other dipole modes.

Of the five quadrupole modes, three are described by the potentials

$$\Pi_4=\Pi_{ab}=-\frac{xy}{ab},\quad \Pi_5=\Pi_{bc}=-\frac{yz}{bc},\quad \Pi_6=\Pi_{ca}=-\frac{zx}{ca}. \tag{2.6}$$

For these potentials, the first term in expansion (1.5) is zero, while the second term, like the other integrals of the powers of the coordinates over the volume of the ellipsoid, is evaluated with the help of the Lagrange formula:

$$\int\left(\frac{x}{a}\right)^{2l}\left(\frac{y}{b}\right)^{2m}\left(\frac{z}{c}\right)^{2n}dV =3V\frac{(2l-1)!!(2m-1)!!(2n-1)!!}{(2l+2m+2n+3)!!}. \tag{2.7}$$

Substituting the calculated results into (1.6), and taking an average over the solid angle, we find the following results for the potential $\Pi_4=\Pi_{ab}$:

$$\frac{J_4}{\omega}=\frac{k^5V^2}{500}\frac{(a^2+b^2)^2}{a^2b^2}+\frac{k^5V^2}{300}\frac{(a^2-b^2)^2}{a^2b^2}, \tag{2.8}$$

where the first term corresponds to the radiation of an electric quadrupole whose tensor components are given by the general formula

$$D_{ij}=\int \mathbf{P}\nabla(3x_ix_j-r^2\delta_{ij})dV.$$

In our case, the only nonvanishing components are

$$D_{xy}=D_{yx}=3V(a^2+b^2)/5ab.$$

The second term is the radiation of a magnetic dipole produced by the polarization currents

$$\mathbf{j}=-i\omega\mathbf{P},\quad \mathbf{m}=-\frac{ik}{2}\int[\mathbf{rP}]dV$$

for which the only nonvanishing component is

$$m_z=-\frac{ikV}{10}\frac{a^2-b^2}{ab}.$$

Furthermore, for $\Pi_4=\Pi_{ab}$, it is easy to show that

$$L_4=\frac{V}{5a^2b^2}(a^2+b^2),\quad R(\lambda)=(a^2+\lambda)(b^2+\lambda),$$

so that

$$\Omega_4^2=\Omega_{ab}^2=(a^2+b^2)M_{110},\quad \Gamma_{ab}=\frac{\Omega_{ab}^4}{300\pi}k_0{}^5V\frac{2a^4+2b^4-a^2b^2}{a^2+b^2}. \tag{2.9}$$

Expressions analogous to (2.8) and (2.9) for the modes Π_5 and Π_6 can be found through cyclic substitution.

Two other quadrupole modes are described by potentials of the form

$$\Pi=1-\alpha x^2-\beta y^2-\gamma z^2, \tag{2.10}$$

where α,β, and γ have the structure in (2.3) ($\theta_i=\theta$). The harmonic condition $\nabla^2\Pi=0$ gives us the relation $\langle\alpha\rangle=\alpha+\beta+\gamma=0$, from which we find a quadratic equation for θ:

$$\theta^2-{}^2/_3\langle a^2\rangle\theta+{}^1/_3\langle b^2c^2\rangle=0. \tag{2.11}$$

Here and below, the angle brackets mean the sum of the three terms found through cyclic substitution. The roots θ' and θ'' of this equation correspond to the two sets of numbers α',β',γ' and $\alpha'',\beta'',\gamma''$, which give us two independent solutions of problem (1.4): $\Pi'=\Pi_7$ and $\Pi''=\Pi_8$. On occasion below we will omit the primes, with the understanding that any expression without primes stands for two expressions, referring to the Π' and Π'' modes.

From (2.3) we find

$$a^2\alpha=1+\theta\alpha,\quad b^2\beta=1+\theta\beta,\quad c^2\gamma=1+\theta\gamma, \tag{2.12}$$

so that α, β, and γ satisfy the relations $\langle a^2\alpha\rangle=3$, $\langle a^2\alpha^2\rangle=\theta\tau$, where $\tau=\langle\alpha^2\rangle$; these relations will simplify the calculations below.

For oscillations with potential Π_7 and Π_8, expansion (1.5) also begins with the second term, which corresponds to the nonvanishing components of the quadrupole electric tensor:

$$D_{xx}=\frac{12}{5}V\theta\alpha,\quad D_{yy}=\frac{12}{5}V\theta\beta,\quad D_{zz}=\frac{12}{5}V\theta\gamma.$$

The magnetic moment of the polarization currents is zero. Furthermore, in this case we have $L=(4/5)V\theta\tau$ and $R=(\theta+\lambda)^2$, so we can finally write

$$\frac{J}{\omega}=\frac{2}{125}k^5V^2\theta^2\tau,\quad \Omega^2=\left\langle\frac{a^2M_a}{a^2-\theta}\right\rangle,\quad \Gamma=\frac{\Omega^4}{200\pi}k_0{}^5V\theta. \tag{2.13}$$

Finally, of the seven octupole oscillations, six are described by three pairs of potentials: $\Pi_9=\Pi'_{aaa}$, $\Pi_{10}=\Pi''_{aaa}$; $\Pi_{11}=\Pi'_{bbb}$, $\Pi_{12}=\Pi''_{bbb}$, and $\Pi_{13}=\Pi'_{ccc}$, $\Pi_{14}=\Pi''_{ccc}$, where, say

$$\Pi_{aaa}=\frac{x}{a}(1-\alpha_ax^2-\beta_ay^2-\gamma_az^2), \tag{2.14}$$

and α_a,β_a and γ_a are given by (2.3) with $\theta_i=\theta_a$. It follows from the harmonic condition $\nabla^2\Pi_{aaa}=0$ that the relation $3\alpha_a+\beta_a+\gamma_a=0$ holds; this relation gives us a quadratic equation for θ'_a and θ''_a:

$$\theta_a{}^2-{}^2/_5(a^2+2b^2+2c^2)\theta_a+{}^1/_5(a^2b^2+a^2c^2+3b^2c^2)=0. \tag{2.15}$$

From relations of the type in (2.12) we easily find the relations

$$3a^2\alpha_a+b^2\beta_a+c^2\gamma_a=5,\quad 3a^2\alpha_a{}^2+b^2\beta_a{}^2+c^2\gamma_a{}^2=\theta_a\tau_a,$$
$$3a^4\alpha_a{}^2+b^4\beta_a{}^2+c^4\gamma_a{}^2=5+\theta_a{}^2\tau_a,\quad \tau_a=3\alpha_a{}^2+\beta_a{}^2+\gamma_a{}^2,$$

which substantially simplify the calculations. For the octupole potentials, expansion (1.5) begins with the third term, and for potential (2.14) the radiative power is

$$\frac{J_{aaa}}{\omega}=\frac{12k^7V^2}{(7!!)^3}\frac{\theta_a{}^2}{a^2}\times\left\{\tau_a(2a^2+\theta_a)^2+\frac{7}{4}[2\tau_a(a^2-\theta_a)^2-15]+105\right\}. \tag{2.16}$$

The first term in braces here corresponds to the radiation of an electric octupole

$$O_{ijk}=\int\mathbf{P}\nabla[5x_ix_jx_k-r^2(x_i\delta_{jk}+x_j\delta_{ki}+x_k\delta_{ij})]dV,$$

for which the following components are nonvanishing in our case:

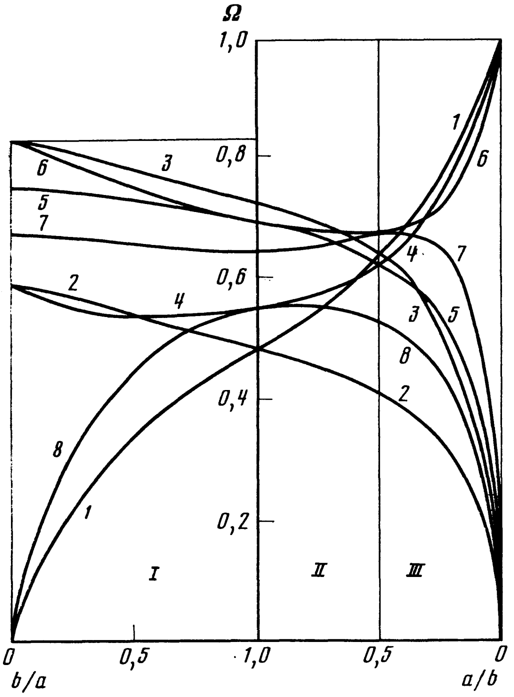

FIG. 1. Dipole and quadrupole frequencies of a plasma ellipsoid ($b = 2c$).

$$(O_{xxx}, O_{xyy}, O_{xzz}) = \frac{3V}{7a}\theta_a(2a^2+\theta_a)(3\alpha_a, \beta_a, \gamma_a).$$

The second term corresponds to the radiation of a magnetic quadrupole produced by the polarization currents $\mathbf{j} = -i\omega\mathbf{P}$. Its tensor

$$Q_{ij} = \frac{1}{c}\int\{[\mathbf{rj}]_i x_j + [\mathbf{rj}]_j x_i\}dV$$

has the following nonvanishing component in our case:

$$Q_{yz} = Q_{zy} = -\frac{2ikV}{35}\frac{\theta_a}{a}(\theta_a - a^2)(\beta_a - \gamma_a).$$

The third term in (2.16) results from the radiation of the so-called toroidal or anapole moment,[8,9]

$$\mathbf{T} = \frac{1}{10c}\int\{(\mathbf{rj})\mathbf{r} - 2r^2\mathbf{j}\}dV,$$

which is equivalent to the radiation of an electric dipole with a dipole moment $\mathbf{p}^{\mathrm{eq}} = -(1/c)\dot{\mathbf{T}} = ik\mathbf{T}$. For currents $\mathbf{j} = i\omega\nabla\Pi_{aaa}$ the only nonzero component is

$$p_x^{\mathrm{eq}} = (2k^2V/35)(\theta_a/a).$$

The potential Π_{aaa} corresponds to

$$L_{aaa} = \frac{2V\theta_a}{35a^2}(2a^2+\theta_a)\tau_a, \qquad R = (a^2+\lambda)(\theta_a+\lambda)^2,$$

so that we finally find

$$\Omega_{aaa}^2 = (2a^2+\theta_a)\left(\frac{a^2M_{200}}{a^2-\theta_a} + \frac{b^2M_{110}}{b^2-\theta_a} + \frac{c^2M_{101}}{c^2-\theta_a}\right), \tag{2.17}$$

$$\Gamma_{aaa} = \frac{3\Omega_{aaa}^6}{8\pi(7!!)^2}k_0^7V\theta_a\frac{2\tau_a(5a^4+3\theta_a^2-2a^2\theta_a)+105}{\tau_a(2a^2+\theta_a)}.$$

Cyclically permuting the indices, and restoring the primes, we find a complete description of the six octupole modes 9–14.

For the seventh octupole mode the polarization potential is $\Pi_{15} = \Pi_{abc} = -xyz/abc$, and we have

$$\frac{J_{abc}}{\omega} = \frac{6k^7V^2}{(7!!)^3a^2b^2c^2}\left\{\langle a^2b^2\rangle^2 + \frac{7}{4}\langle a^4(b^2-c^2)^2\rangle\right\}, \tag{2.18}$$

where the first term corresponds to the radiation of an electric octupole

$$O_{xyz} = (V/7abc)\langle a^2b^2\rangle,$$

and the second to that of a magnetic quadrupole

$$\{Q_{xx}, Q_{yy}, Q_{zz}\} = -\frac{2ikV}{abc}\{a^2(b^2-c^2), b^2(c^2-a^2), c^2(a^2-b^2)\}.$$

In this case the toroidal momentum is zero.[1)] For Π_{abc} we have

$$L_{abc} = (V/35a^2b^2c^2)\langle a^2b^2\rangle, \quad R(\lambda) = (a^2+\lambda)(b^2+\lambda)(c^2+\lambda),$$

so that

$$\Omega_{15}^2 = \Omega_{abc}^2 = \langle a^2b^2\rangle M_{111}, \tag{2.19}$$

$$\Gamma_{abc} = \frac{3\Omega_{abc}^6}{4\pi(7!!)^2}k_0^7V\frac{3\langle a^4b^4\rangle - a^2b^2c^2\langle a^2\rangle}{\langle a^2b^2\rangle}.$$

Figures 1 and 2 show curves of the resonant frequencies of a plasma ellipsoid with a semiaxis ratio $b/c = 2$. The dipole and quadrupole frequencies are shown in Fig. 1, and the quadrupole and octupole frequencies in Fig. 2. Plotted along the left half of the abscissa in each figure is the ratio b/a, while a/b is plotted along the right half. The range of the reduced resonant frequencies Ω is from 0 to 1 and is shown entirely in Fig. 1. Figure 2 shows only the most important

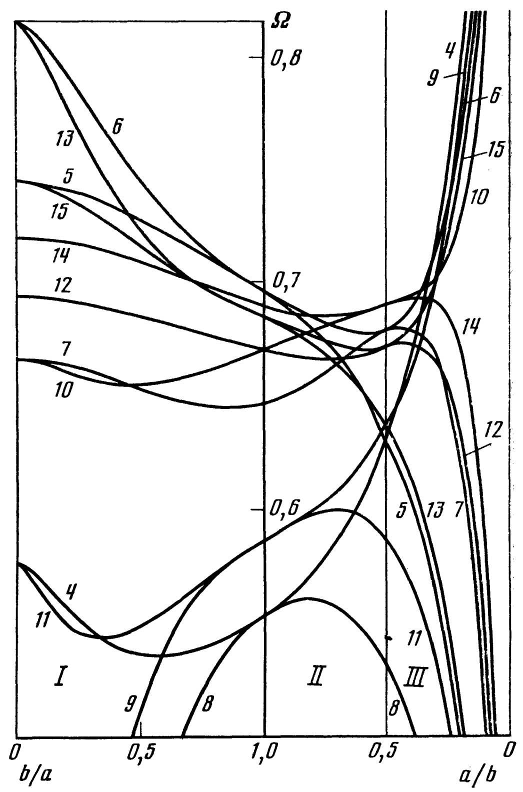

FIG. 2. Quadrupole and octupole frequencies of a plasma ellipsoid ($b = 2c$).

part of this range: that which contains all the curve intersections. The curves are designated in accordance with the notation used in the text proper. The vertical lines separating regions I, II, and III correspond to the degeneracy of the ellipsoid into a spheroid ($a = b$ or $a = c$).

3. RESONANCE EXCITATION OF A PLASMA ELLIPSOID

The Lamé-Niven ellipsoidal harmonics, the simplest of which were discussed in the preceding section, have the orthogonality property

$$\int \nabla\Pi_r \nabla\Pi_s \, dV = \int \mathbf{P}_r \mathbf{P}_s \, dV = 0 \quad \text{for} \quad r \neq s. \tag{3.1}$$

This property can be proved in general form by returning to the original Lamé notation for all the Π_s in ellipsoidal coordinates. For the 15 potentials which we have been discussing here, written in Niven's polynomial form, the validity of (3.1) can be established by a straightforward check. For most of the pairs $\mathbf{P}_r$, $\mathbf{P}_s$ the orthogonality can be seen immediately from the fact that the integrand is of odd parity with respect to one of the coordinates. Nontrivial cases are represented by pairs of the type $\Pi_a = \Pi_1$, $\Pi_{aaa} = \Pi_{9,10}$ (the orthogonality follows from the relation $3a^2\alpha_a + b^2\beta_a + c^2\gamma_a = 5$) and pairs of the type $\Pi' = \Pi_7$, $\Pi'' = \Pi_8$ and $\Pi'_{aaa} = \Pi_9$, $\Pi''_{aaa} = \Pi_{10}$, for which the orthogonality is a consequence of the relations

$$\langle\alpha'\alpha''\rangle = 0, \quad 3\alpha_a'\alpha_a'' + \beta_a'\beta_a'' + \gamma_a'\gamma_a'' = 0, \tag{3.2}$$

which follow from definitions (2.3) and Eqs. (2.11) and (2.15).

In the class of harmonic functions, the Lamé harmonics form a complete system, so that the polarization distribution inside the ellipsoid can be written in the form

$$\mathbf{P}(\mathbf{r}, t) = \sum_s f_s(t) \mathbf{P}_s(\mathbf{r}). \tag{3.3}$$

It then follows from $\mathbf{v} = \dot{\mathbf{P}}/ne$ and (3.1) that the kinetic energy of the plasma is equal to the sum of partial energies:

$$K(t) = \frac{2\pi}{\omega_0^2} \sum_s L_s \dot{f}_s^2, \qquad L_s = \int |\mathbf{P}_s|^2 \, dV.$$

The complete electric energy can also be written as a sum of partial energies:

$$U_E(t) = 2\pi \sum_s L_s \Omega_s^2 f_s^2.$$

Furthermore, a dissipative function describing the internal loss due to collisions will be additive, since the rate of this loss is $2\nu_{\text{eff}} K(t)$. If we ignore the loss due to radiation, we conclude that the natural oscillations of the ellipsoid have all the properties of the normal modes of analytic dynamics, and each can be analyzed indepenently. Since for the vector eigenfunctions $\mathbf{P}_s$ we can write an equation $\mathbf{E}_P\{\mathbf{P}_s\} = -4\pi\Omega_s^2 \mathbf{P}_s$, substitution of (3.3) into the general equation (1.1) gives us

$$\sum_s (\ddot{f}_s + \nu_{eff}\dot{f}_s + \omega_s^2 f_s)\mathbf{P}_s = \frac{\omega_0^2}{4\pi}\mathbf{E}_{ext},$$

from which we find the following system of independent equations, where we are making use of the orthogonality:

$$\ddot{f}_s + \nu_s \dot{f}_s + \omega_s^2 f_s = \omega_0^2 \mathscr{E}_s, \qquad \mathscr{E}_s = \frac{1}{4\pi L_s} \int \mathbf{E}_{ext} \mathbf{P}_s \, dV. \tag{3.4}$$

The formal substitution

$$\nu_{eff} \to \nu_s = \nu_{eff} + 2\gamma_{\text{rad}} = \nu_{eff} + 2\omega_0 \Gamma_s$$

makes it possible to also take into account the radiative loss in the case of a monochromatic external field ($\sim e^{-i\omega t}$) with a frequency close to the resonant frequency. This approach is legitimate, of course, only for an isolated resonance in a single mode.

If, for certain values of the parameters b/a and c/a, the frequencies of two modes agree, then a mixed term may appear in the expression for the total radiated power, and the radiative decay constants derived above would be replaced by some new ones, found by solving the problem of two oscillators with a weak dissipative coupling. We will not pursue that case here, since the radiative loss is additive in the new ponderomotive effective with which we will be concerned below. We simply note that in the class of dipole and quadrupole modes there is no mixed loss for any of the possible pairs oscillating at a common frequency, while for the octupole modes a mixed loss arises in the pairs (9, 10) (11, 12), and (13, 14), since in each of these pairs there are identical tensor components of the magnetic moment and of the vector toroidal moment. Furthermore, for the pairs (9, 1), (10, 1); (11, 2), (12, 1); and (13, 3), (14, 3) there will be a mixed loss due to products of an equivalent dipole (or toroidal) moment of an octupole mode and a dipole moment of a dipole mode.

If collisions are infrequent, and the condition $\nu_{\text{eff}} \ll \gamma_{\text{rad}}$ holds for all of the modes of interest (for the higher-order multipole modes, this inequality is violated sooner or later), at the exact resonance $\omega = \omega_s$ we have

$$f_s = i\frac{\omega_0^2}{\omega_s \nu_s}\mathscr{E}_s = \frac{i\mathscr{E}_s}{2\Omega_s \Gamma_s},$$

and the reradiated power is

$$|f_s|^2 J_s = \frac{\pi\omega_0}{\Gamma_s} L_s |\mathscr{E}_s|^2.$$

Let us assume that the external field is a linearly polarized plane wave of unit amplitude:

$$\mathbf{E}_{ext} = \mathbf{e} \exp(ik\varkappa\mathbf{r}) = \mathbf{e}\{1 + ik(\varkappa\mathbf{r}) - {}^1\!/_2 k^2 (\varkappa\mathbf{r})^2 + \ldots\},$$

where $\mathbf{e}$ and $\varkappa$ are mutually orthogonal unit vectors. The integral in $\mathscr{E}_s$ then takes the following forms for the dipole, quadrupole, and octupole modes, respectively:

$$(\mathbf{e}\mathbf{P}_s) V \quad (s=1-3), \qquad ik \int (\mathbf{e}\mathbf{P}_s)(\varkappa\mathbf{r}) dV \quad (s=4-8),$$
$$-\frac{k^2}{2}\int (\mathbf{e}\mathbf{P}_s)(\varkappa\mathbf{r})^2 dV \quad (s=9-15).$$

Since the reradiated power is, by virtue of the definition of the scattering cross S, equal to cwS (c is the velocity of light, and w is the average energy density of the incident wave, equal in our case to $w = 1/8\pi$), we can write the following expression for the case of an exact single resonance, $\omega = \omega_s$:

$$S_s = \frac{8\pi^2 k_0}{\Gamma_s} L_s |\mathscr{E}_s|^2 = \frac{\lambda_s^2}{2\pi}\sigma_s \quad . \tag{3.5}$$

Here $\lambda_s = 2\pi/k_s = 2\pi/k_0\Omega_s$ are the resonant wavelengths, and the functions σ_s depend only on the ratios of the axes of the ellipsoid and its orientation. The explicit expressions are

$$\sigma_1=3e_x^2,\quad \sigma_4=15\frac{(a^2\varkappa_x e_y+b^2\varkappa_y e_x)^2}{2a^4+2b^4-a^2b^2},\quad \sigma_{7,8}=10\frac{\langle \alpha\varkappa_x e_x\rangle^2}{\tau},$$

$$\sigma_{9,10}=105[e_x+e_x\theta_a(\alpha_a\varkappa_x^2+\beta_a\varkappa_y^2+\gamma_a\varkappa_z^2)+2a^2(\alpha_a\varkappa_x e_x+\beta_a\varkappa_y e_y+\gamma_a\varkappa_z e_z)]^2[2\tau_a(5a^4+3\theta_a^2-2a^2\theta_a)+105]^{-1},\quad (3.6)$$

$$\sigma_{15}=105\frac{\langle a^2b^2\varkappa_x\varkappa_y e_z\rangle^2}{3\langle a^4b^4\rangle-a^2b^2c^2\langle a^2\rangle},$$

while the other σ_s can be found from the expressions for σ_1, σ_4, and $\sigma_{9,10}$ by cyclic permutation.

In the case of a single resonance, a force $\mathbf{F}_s = wS_s\varkappa$ is exerted on a plasma in the field of an incident wave. If instead the resonant frequencies of two modes are the same, then we would have

$$\mathbf{F}=wS_{tot}\varkappa-\mathbf{\Pi}, \quad (3.7)$$

where $S_{\rm tot}$ is the total scattering cross section, and $\mathbf{\Pi}$ is the total momentum flux carried off by the scattered field. For several multipole radiators, this moment flux is generally nonzero. In the wave zone ($r\gg\lambda$) we have

$$d\mathbf{\Pi}=\frac{dS_r}{16\pi}(|\mathbf{E}|^2+|\mathbf{H}|^2)\mathbf{v},$$

and since $|\mathbf{E}| = |\mathbf{H}|$ we have

$$\mathbf{\Pi}=\oint \mathbf{v}|\mathbf{H}|^2\frac{dS_r}{8\pi}, \quad (3.8)$$

where the integral is over a sphere of radius r. For the wave zone, the multipole expansion of the magnetic vector is

$$\mathbf{H}=k^2\frac{e^{ikr}}{r}\left[\mathbf{v},\left(\mathbf{p}-\frac{ik}{6}\mathbf{D}-\frac{k^2}{30}\mathbf{O}+\left[\mathbf{m}-\frac{ik}{6}\mathbf{Q},\mathbf{v}\right]\right)\right]. \quad (3.9)$$

Here $\mathbf{p}$ is the total electric dipole moment (which includes $\mathbf{p}^{\rm eq}$, due to the toroidal moment), and the vectors $\mathbf{D}$, $\mathbf{O}$, and $\mathbf{Q}$ are related to the tensors of the electric quadrupole and octupole moments (D_{ij} and O_{ijk}) and of the magnetic quadrupole moment (Q_{ij}) by $D_i = \nu_j D_{ji}$, $O_i = \nu_j\nu_k O_{jki}$, $Q_i = \nu_j Q_{ji}$. Substituting (3.9) into (3.8), and taking an average over the solid angle, $do = dS_r/r^2$, we finally find

$$\Pi_l=\frac{k^4}{3}\mathrm{Re}[\mathbf{p}^*\mathbf{m}]_l+\frac{k^5}{30}\mathrm{Im}(p_i^*D_{il}+m_i^*Q_{il})$$
$$+\frac{k^6}{540}\mathrm{Re}(\varepsilon_{lij}D_{ik}^*Q_{jk})+\frac{2k^7}{4725}\mathrm{Im}(D_{ij}^*O_{ijl}), \quad (3.10)$$

where ε_{ijk} is the completely antisymmetric unit tensor.

We find immediately from (3.10) that $\mathbf{\Pi}_{\rm res}$ may arise when the resonant frequencies of only a pair of dipole and quadrupole modes or of a pair of quadrupole and octupole modes coincide. For such pairs there is no mixed loss in terms of radiation energy, so that all the results of §2 of this paper can be applied without change. The total number of possible versions is very large. For example, it follows from Fig. 1 that for $b = 2c$ there are nine intersections of dipole (1–3) and quadrupole (4–8) curves for which the vector

$$\Pi_l=\frac{k^4}{3}\mathrm{Re}[\mathbf{p}^*\mathbf{m}]_l+\frac{k^5}{30}\mathrm{Im}(p_i^*D_{il}) \quad (3.11)$$

is nonzero. The number of quadrupole-octupole intersections in Fig. 2 is far larger.

We will accordingly content ourselves with a single example: $\Omega_b = \Omega_{ab}$. For a dipole mode we then have $p_y = (V/b)f_2$, and for a quadrupole mode we have

$$m_z=-\frac{ikV}{10}\frac{a^2-b^2}{ab}f_4,\qquad D_{xy}=\frac{3V}{5ab}(a^2+b^2)f_4.$$

Substitution of these expressions into (3.11) leads to the following result at resonance:

$$\mathbf{\Pi}=w\frac{\lambda^2}{2\pi}\boldsymbol{\sigma}_{\rm int},\quad \boldsymbol{\sigma}_{\rm int}=3(4a^2-b^2)e_y\frac{a^2\varkappa_x e_y+b^2\varkappa_y e_x}{2a^4+2b^4-a^2b^2}\mathbf{x}_0. \quad (3.12)$$

We can thus write

$$\mathbf{F}=w\sigma\lambda^2/2\pi,\quad \boldsymbol{\sigma}=(\sigma_2+\sigma_4)\varkappa-\boldsymbol{\sigma}_{\rm int}. \quad (3.13)$$

With $b = 2c$, curves 2 and 4 in Fig. 1 intersect at $a\approx 2b$, so that we have

$$\sigma_4={}^1/_2(4\varkappa_x e_y+\varkappa_y e_x)^2,\quad \boldsymbol{\sigma}_{\rm int}={}^3/_2(4\varkappa_x e_y+\varkappa_y e_x)e_y\mathbf{x}_0.$$

If, for example, an ellipsoid is oriented with its axes along the "bisectors" of the octants formed by the vectors $\mathbf{E}$, $\mathbf{H}$, and $\varkappa$ of the incident wave, with $\varkappa_x = \varkappa_y = \varkappa_z = e_x = -e_y = e_z = 1/\sqrt{3}$ we would have $\sigma = 1/2$, $\sigma_{\rm int} = (\sqrt{3}/2)\mathbf{x}_0$. If we rotate the ellipsoid through an angle $\pi/2$ around the $\mathbf{x}_0$ axis, we have $\varkappa_x = -\varkappa_y = \varkappa_z = e_x = -e_y = -e_z = 1/\sqrt{3}$ and $\sigma_4 = 25/18$, $\sigma_{\rm int} = (5/2\sqrt{3})\mathbf{x}_0$. In these two cases we thus have $\sigma_\parallel = 1$, $\sigma_\perp = 1/\sqrt{2}$ and $\sigma_\parallel = 14/9$, $\sigma_\perp = 5\sqrt{2}/6$ for the accelerating force (parallel to $\varkappa$) and the lateral force, respectively.

4. CONCLUSION

In summary, some effects disappear when an ellipsoid degenerates into a sphere. In th case of a sphere, the polarization currents corresponding to the quadrupole and octupole modes no longer produce either magnetic or toroidal moments.[2)] Consequently, the total power radiated by several modes at a single frequency does not contain cross terms, so that all the modes are also independent of each other in terms of radiative loss. Since oscillations of the same multipolarity l have a single common resonant frequency, $\Omega_l = (2 + 1/l)^{-1/2}$, in the case of a sphere, a double resonance at modes of adjacent multipolarity (§3) is not possible, and the resonant momentum flux $\mathbf{\Pi}$ is zero.

We have yet another general comment. The problem of finding the resonant frequencies of quasistatic oscillations of a uniform plasma can be reduced to a macroscopic boundary-value problem for the Laplace equation in two regions:

$$\Delta\Phi^i=0,\qquad \Delta\Phi^e=0,$$

$$\Phi_\Gamma{}^i=\Phi_\Gamma{}^e,\qquad \varepsilon_i\left(\frac{\partial\Phi^i}{\partial n}\right)_\Gamma=\left(\frac{\partial\Phi^e}{\partial n}\right)_\Gamma. \quad (4.1)$$

This problem has nontrivial solutions for a discrete spectrum of negative values of the dielectric constant of the plasma: $\varepsilon_i = 1 - \Omega_i^{-2}$. At $\varepsilon_e = 1/\varepsilon_i$ these solutions are obviously also solutions of the problem which differs from (4.1) in the replacement of the second boundary condition by

$$(\partial\Phi^i/\partial n)_\Gamma=\varepsilon_e(\partial\Phi^e/\partial n)_\Gamma.$$

This other problem is the problem of the natural modes of a homogeneous plasma in which there is a vacuum cavity with the same shape as that of the plasma in the original problem.

Since $\varepsilon_e = 1 - \Omega_e^{-2}$, the condition $\varepsilon_i \varepsilon_e = 1$ leads immediately to the relation $\Omega_i^2 + \Omega_e^2 = 1$, which relates the resonant frequencies of the oscillations of identical structure for these two additional problems.

[1)] Unfortunately, the radiation by the magnetic and toroidal moments produced by the potential polarization currents was not considered in Ref. 3 in the calculation of the radiation constants of a spheroid. This expansion was carried out in a nonsystematic way in a methodological paper ("The multipole expansion revisited") by Van Bladel,[10] and several of the results of the paper are wrong.

[2)] When we take the limit of a sphere, the polarization potentials Π containing a factor Θ should be renormalized, since in the case $a = b = c$ we have $\theta = a$, and the coefficients α, β, and γ become infinite.

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