Regular and stochastic dynamics of particles in the field of a wave packet

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In our analysis of particle dynamics we distinguish two limiting cases corresponding to motions with velocities much smaller and much larger than the group velocity of a wave packet. We obtain the conditions for the stochastization of the motion in these cases and show that in the second case the stochastic mechanism for increase in the particle energy has an upper bound. This restriction is universal. We construct and consider standard mappings generated by the motion in the field of a wave packet and establish the conditions for the transition to a quasilinear equation. We study the role played by dissipation. We obtain the mapping for both limiting cases, taking into account dissipation and we find the conditions for the occurence of stochastic attractors. We evaluate the limiting energies for particle acceleration.

1. INTRODUCTION

The study of the motion of particles in the field of wave packets is a typical problem for many problems of the dynamics of a continuous medium and, in particular, of a plasma. Since a perturbation, represented in the form of a wave packet, of the equilibrium state of the medium contains a large number of hamonics, the particle motion becomes extraordinarily complex. Under well defined conditions one can describe it using a kinetic equation which is known for a plasma as the quasilinear equation.[1] One can find various aspects of its applications in review articles.[2–4]

The main condition which allows us to change from the regular dynamics of particles to their kinetic description is that no particles be captured in some way from the waves of the packet.[1] In well-defined very simple situations this condition is connected with the condition of overlap of resonances. In this way it has been possible to find a quantitative criterion for the possibility of using the quasilinear equation[5,6] without assuming the existence of random phases for the waves constituting the wave packet.

The real situation is much more complicated, and this determines the large number of attempts to improve or reconsider in some way or other the quasilinear plasma theory. The difficulty in defining exact criteria for the applicability of the kinetic description lies in the fact that the waves in the packet have different phase velocities and dispersion effects may suppress the evolution of the stochastic particle dynamics. Those cases which might be subject to an exact analysis are therefore of particular interest.

We consider in the present paper two such cases: the case of a time-like packet and the case of a space-like packet; these are two different limiting situations in the structure of a wave packet. The stochastic dynamics evolves principally differently in these cases. This enables us to reveal the general physical picture of the occurrence of a stochastic description of particle dynamics using the quasilinear equation and to evaluate the time for the decoupling of the corresponding phase correlations.

The method used in this paper reduces the problem of particle dynamics to an exactly analyzable mapping. Moreover, this method also enables us to consider particle dynamics when there are friction forces present and to establish the conditions for the appearance of a stochastic attractor in that case.

2. THE TWO TYPES OF WAVE PACKETS AND PARTICLE DYNAMICS FOR A TIME-LIKE PACKET

We consider the one-dimensional motion of a particle determined by the equation

$$\ddot{x} = \frac{e}{m}\sum_n E_n \cos(k_n x - \omega_n t). \tag{2.1}$$

We make the following simplifing assumptions about the structure of the wave packet:

$$k_n = k_0 + n\Delta k, \quad \omega_n = \omega_0 + n\Delta\omega, \quad E_n \equiv E_0, \tag{2.2}$$

where n is an integer and where the summation in (2.1) goes from $-N$ to $+N$. Equations (2.2) mean that the dispersion effects are small and that the spectral characteristics of the packet are fairly uniform and symmetric. We can then write (2.1) in the form

$$\ddot{x} = \frac{eE_0}{m}\cos(k_0 x - \omega_0 t)\sum_{n=-N}^{N}\cos[n(\Delta k x - \Delta\omega t)]. \tag{2.3}$$

We introduce the parameter

$$\eta = \frac{x}{t}\frac{\Delta k}{\Delta\omega} \sim \frac{v}{v_g},$$

where $v = \dot{x}$ and v_g is the group velocity of the packet. We write also

$$k_0 x - \omega_0 t = \psi(x,t), \qquad \Delta k x - \Delta\omega t = \xi.$$

When $\eta \ll 1$ we can approximately put $\xi \approx -\Delta\omega t$ and in that case the packet will be called a t-packet. When $\eta \gg 1$ the opposite case occurs: $\xi \approx \Delta k x$ and we have an x-packet.

We consider first of all the first case ($\eta \ll 1$) and we shall assume that N in Eq. (2.3) is so large that we can put[1)] $N \to \infty$ in (2.3). Then, (2.3) takes the form

$$\ddot{x} = \frac{\Omega_0^2}{k_0}\cos\psi(x,t)\sum_{n=-\infty}^{\infty}\cos n\Delta\omega t$$

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$$=\frac{\Omega_0^2}{k_0}T\cos\psi(x,t)\sum_{n=-\infty}^{\infty}\delta(t-nT), \qquad (2.4)$$

where

$$T=2\pi/\Delta\omega,\quad \Omega_0^2=eE_0k_0/m, \qquad (2.5)$$

T is the period of the δ-pulse sequence and the distance $\Delta\omega$ between the modes of the packet can be estimated as follows:

$$\Delta\omega\sim[d\omega(k)/dk]\Delta k. \qquad (2.6)$$

We can write Eq. (2.4) in the form of a mapping $\hat{T}$ if we integrate it over a vanishingly small region around the point $t_n=nT$ and connect the quantities at times t_n and t_{n+1}. This gives

$$\hat{T}:\begin{cases}\bar{v}=v+(K/k_0T)\cos\psi\\ \bar{\psi}=\psi+\omega(\bar{v})T\end{cases}, \qquad (2.7)$$

where

$$\begin{gathered}K=\Omega_0^2T^2=eE_0k_0T^2/m,\quad v=\dot{x},\quad \omega(v)=k_0v-\omega_0,\\ v=v(t=nT-0),\quad \bar{v}=v(t=(n+1)T-0),\\ \psi=\psi[x(t=nT-0),\quad t=nT-0],\\ \bar{\psi}=\psi[x(t=(n+1)T-0),\quad t=(n+1)T-0].\end{gathered} \qquad (2.8)$$

The notation (2.8) introduces the dimensionless parameter K and the canonically conjugated pair of variables (v,ψ) taken at two successive points directly preceding the action of a δ-pulse.

Mappings of the kind (2.7) have been rather well studied.[5,7] When $K\gtrsim1$ there occurs a local instability leading to stochastic particle dynamics. This manifests itself in the fact that the phase correlation ψ is exponentially decoupled:

$$R[\psi;t]=\frac{1}{2\pi}\int_0^{2\pi}d\psi(0)\cos\psi(t)\cos\psi(0)$$

$$\sim\cos[(k_0v-\omega_0)t]\exp(-t/\tau_c), \qquad (2.9)$$

where the time for the decoupling of the correlation when $K\gg1$ equals[5,8]

$$\tau_c\sim2T/\ln K. \qquad (2.10)$$

One gets the condition for the local instability from (2.7) using the relation

$$K=\max|d\bar{\psi}/d\psi-1|\geqslant1. \qquad (2.11)$$

Equations (2.9) and (2.10) determine the nature of the particle dynamics on a short time scale $\sim T$. For long time scales there occurs in the space of the velocities v a slow diffusion described by the formula

$$\frac{\partial F}{\partial t}=\frac{1}{2}\frac{\partial}{\partial v}D\frac{\partial F}{\partial v}, \qquad (2.12)$$

where $F=F(v,t)$ is a velocity distribution function and $D=e^2E_0^2T/2m^2$ is the diffusion coefficient. Since the diffusion coefficient for the mapping (2.7) is independent of v, it follows from (2.12) that

$$\langle v^2\rangle=(K^2/2k_0^2T^2)(t/T), \qquad (2.13)$$

i.e., the particle energy increases. It will become clear presently that this property holds true also in the more general case.

Let the wave amplitudes E_n, the frequencies ω_n and the distances between the modes $\Delta\omega_n$ in (2.2) not be constant, but depend slowly on n. It is then clear that under the condition $\eta\ll1$ at which we obtained Eq. (2.12) a generalized quasilinear equation such as (2.12) will also hold, but now with a diffusion coefficient[8]

$$D=\frac{\pi e^2}{2m^2}\sum_k|E_k|^2\Delta\left(\frac{1}{\tau_c};\omega_k-kv\right), \qquad (2.14)$$

where the "smeared" δ-function

$$\Delta\left(\frac{1}{\tau_c};\xi\right)=\frac{1}{\pi}\frac{1/\tau_c}{(1/\tau_c)^2+\xi^2} \qquad (2.15)$$

has been introduced and where the τ_c is defined by Eqs. (2.9) and (2.10). We must now assume in them that E_0 is some characteristic value of the E_k averaged over the wave packet. The same also is true regarding Eqs. (2.5) and (2.6).

The average particle energy $\langle\mathscr{E}\rangle$ is equal to

$$\langle\mathscr{E}\rangle=\frac{1}{2}m\langle v^2\rangle=\frac{1}{2}m\int dvv^2F(v,t).$$

From (2.12) and (2.14) follows one of the main physical consequences when the stochasticity condition (2.11) is satisfied[5]: $d\langle\mathscr{E}\rangle/dt>0$. This inequality is established directly and means an unbounded increase in particle energy ("stochastic heating"). If we take into account that the number N of the modes in the wave packet is finite, we see that the maximum particle energy is determined by the maximum phase velocity of the wave contained in the packet.

The unbounded increase (with the reservation just made) in particle energy is caused by the inequality $\eta\ll1$, i.e., $v\ll|v_g|=|d\omega_k/dk|$. The appearance of particles with high velocities leads to a violation of that inequality and it is thus necessary as a matter of principle to consider the other limiting case $\eta\gg1$, i.e., the case of an x-packet.

Apparently, just this fact leads to a number of difficulties encountered in various papers where attempts are made to justify the quasi-linear equation. We shall show presently how one can obtain an exact description for the case $\eta\gg1$.

3. PARTICLE DYNAMICS IN A SPACE-LIKE PACKET

When $\eta\gg1$ we can use the approximation $\xi\simeq\Delta kx$ and as $N\to\infty$ Eq. (2.3) becomes

$$\ddot{x}=\frac{\Omega_0^2}{k_0}\cos\psi(x,t)\sum_{n=-\infty}^{\infty}\cos n\Delta kx$$

$$=\frac{\Omega_0^2}{k_0}\cos\psi(x,t)L\sum_{n=-\infty}^{\infty}\delta(x-nL), \qquad (3.1)$$

where we have introduced the characteristic length of the system $L=2\pi/\Delta k$ which determines the distance Δk between the modes of the wave packet.

Using (3.1) we write down an equation for the energy $\mathscr{E}$:

$$\dot{\mathscr{E}}=\frac{\Omega_0^2}{k_0}L\cos\psi(x,t)\dot{x}\sum_{n=-\infty}^{\infty}\delta(x-nL). \tag{3.2}$$

We can proceed with Eq. (3.2) in the same way as we obtained the $\hat{T}$-mapping (2.7), i.e., integrate (3.2) in the vicinity of the nth and the $(n+1)$st δ-pulses and connect the values of the variables near those points. In that case, the nth δ-pulse of the force, for instance, acts at the time $t=t_n$ which is found from the equation

$$x_n=x(t_n)=nL;\quad n=0,1,\dots\ . \tag{3.3}$$

We shall call the corresponding mapping the $\hat{L}$-mapping.

In actual fact, getting the $\hat{L}$-mapping is rather a delicate procedure. It follows from Eq. (3.2) that the quantity $\mathscr{E}$ and, hence, also $\dot{x}$, is discontinuous (it has a jump) in the point x_n. The right-hand side of Eq. (3.2) contains therefore a product of two generalized functions, of the kind $\theta(x)\delta(x)$, with singularities which coincide. Generally speaking, such a product is not defined uniquely (for a study of this problem, see Ref. 9). In a real physical situation, however, there should not be any ambiguity, as the particle trajectories are unique. For large values of N one can reason as follows.

Let N be finite and large. There (3.2) contains instead of the δ-functions "peaks" of width $\Delta x\sim L/N=2\pi/N\Delta k$. On the other hand, the function $\psi(x,t)$ has a characteristic variation scale $\sim 1/k_0$. Let this variation be very slow, i.e., let the inequality $1/N\Delta k\ll 1/k_0$ be satisfied. We have then a symmetric case in which we may take

$$\int\varphi(x)\delta(x-a)\,dx=\frac{1}{2}[\varphi^+(a)+\varphi^-(a)],$$

where $\varphi^{\pm}(a)$ are, respectively, the left-hand and right-hand values of the function $\varphi(x)$ which has a discontinuity at the point $x=a$.

Using these relations and integrating (3.2) in the vicinity of the points $x_n=nL$ and $\bar{x}_n=x_{n+1}=(n+1)L$, we find the $\hat{L}$-mapping:

$$\hat{L}:\begin{cases}\bar{\mathscr{E}}=\mathscr{E}+m(\Omega_0^2/k_0)L(\operatorname{sign}\bar{v}+\operatorname{sign}v)\cos\psi\\ \bar{\psi}=\psi+k_0L-\omega_0T(\mathscr{E})\end{cases}, \tag{3.4}$$

where $T(\mathscr{E})$ is the interval between successive times determined by Eq. (3.3); $(\mathscr{E},\psi)$ are a pair of canonically conjugate variables (the $\hat{L}$-mapping conserves measure) and one must complete (3.4) with an equation defining $\bar{v}$:

$$\bar{v}=v+\frac{1}{2}\frac{\Omega_0^2}{k_0}L\cos[\psi(x,t)]\left(\frac{1}{|\bar{v}|}+\frac{1}{|v|}\right). \tag{3.5}$$

We obtain Eq. (3.5) by integrating (3.1) in the vicinity of a point t_n and in it we have for the sake of simplicity omitted everywhere the index n. The values of $(\mathscr{E},v,\psi)$ and $(\mathscr{E},\bar{v},\bar{\psi})$ are determined in accord with (2.8).

We discuss some details of the $\hat{L}$-mapping. If v and $\bar{v}$ have the same sign, then

$$\hat{L}_s:\begin{cases}\bar{\mathscr{E}}=\mathscr{E}+mL(\Omega_0^2/k_0)\cos\psi\\ \bar{\psi}=\psi+k_0L-\omega_0T(\bar{\mathscr{E}})\end{cases}. \tag{3.6}$$

If v and $\bar{v}$ have opposite signs there occurs a simple rotation mapping:

$$\hat{L}_a:\begin{cases}\bar{\mathscr{E}}=\mathscr{E}\\ \bar{\psi}=\psi+k_0L-\omega_0T\end{cases}, \tag{3.7}$$
$$T=T(\mathscr{E})=T(\mathscr{E}_0)=\text{const.}$$

One easily establishes the function $T(\mathscr{E})$. It follows from (3.1) that outside the points x_n we have $v=$ const and, hence, the particle moves with a constant speed. Let L be the distance between two successive points x_n and x_{n+1}. It is traversed in a time

$$T=L/|v|=L/(2\mathscr{E}/m)^{1/2}. \tag{3.8}$$

We have further assumed, as is usually done, that the perturbation in (3.1) is rather small. This means, in particular, that the change in velocity under the action of the δ-pulse is also small, i.e.,

$$\varepsilon\equiv\Omega_0^2L/k_0v^2=eE_0L/2\mathscr{E}\ll 1. \tag{3.9}$$

We can therefore write Eq. (3.5) up to terms of order ε^2 in the form

$$\bar{v}=v+(\Omega_0^2L/k_0|v|)\cos\psi, \tag{3.10}$$

which is formally the same as the first line of Eq. (2.7). Although these formulae for the change in the velocity are outwardly similar, there is a principal difference between them: in (2.7) the mapping time $T=$ const whereas in (3.10) the mapping time $T=T(\mathscr{E})=L/|v|$ depends on the particle velocity (or energy).

In the same approximation (3.9) we may assume sign $\bar{v}=$ sign v, as is clear from (3.10), so that the problem of the particle dynamics at $\eta\gg 1$ reduces thus to the $\hat{L}_s$-mapping (3.6) where we must use Eq. (3.8) for $T(\mathscr{E})$:

$$\hat{L}_s:\begin{cases}\bar{\mathscr{E}}=\mathscr{E}+mL(\Omega_0^2/k_0)\cos\psi\\ \bar{\psi}=\psi+k_0L-\omega_0L/(2\bar{\mathscr{E}}/m)^{1/2}.\end{cases} \tag{3.11}$$

The remaining way of studying (3.11) is analogous to the analysis of the $\hat{T}$-mapping. Following (2.11) we consider the parameter

$$K=\max\left|\frac{d\bar{\psi}}{d\psi}-1\right|=\frac{\omega_0}{k_0}\frac{L^2\Omega_0^2}{|v|^3}=\frac{e}{m}E_0\frac{\omega_0L^2}{|v|^3}. \tag{3.12}$$

The condition for the occurrence of stochasticity $K\gtrsim 1$ leads to the inequality

$$|v|^3\leqslant u_0L^2\Omega_0^2,\quad u_0\equiv\omega_0/k_0, \tag{3.13}$$

whose physical character is completely different from the same condition for the $\hat{T}$-mapping. Indeed, the stochasticity of the phase ψ on the particle trajectory also leads as before, to stochastic heating and to a growth in the average value of $|v|$. However, this quantity cannot exceed a certain critical value $v_{\max}$ which is equal to

$$v_{max}=u_0^{1/3}(L\Omega_0)^{2/3}=(eE_0L^2\omega_0/m)^{1/3}. \tag{3.14}$$

When $v>v_{\max}$ the stochasticity (at least strong stochasticity) disappears and there is no further acceleration (at least over not too long time intervals). Of course, all these considerations are valid up to the small terms which we have dropped. A similar situation was met with in the Fermi acceleration model in Ref. 10.

The general picture of the particle dynamics now looks as follows:

1) If $v > v_g = |d\omega_k/dk|$ the motion occurs in the wave x-packet, and if condition (3.13) holds the particle gains energy on the average until its velocity reaches either the maximum phase velocity max (ω_k/k) for a wave in the packet, or $v_{\max}$ from Eq. (3.14). In the latter case the particle cannot reach resonance with waves having a phase velocity larger than $v_{\max}$.

2) If $v < v_{\max}$ the particle dynamics initially corresponds to the motion in a wave x-packet. If condition (2.11) holds

$$eE_0k_0T^2/m \sim eE_0k_0/m(\Delta\omega)^2 \gg 1, \tag{3.15}$$

the motion of the particle is stochastic and its energy grows. When it reaches a velocity $v \sim v_g$ there is a change in the nature of the dynamics. Combining conditions (3.15) and (3.13) we get

$$\frac{\omega_0}{k_0}\left(\frac{\Delta\omega}{\Delta k}\right)^2 \frac{1}{|v|^3} \gg 1 . \tag{3.16}$$

If (3.16) is not satisfied when $v \sim v_g$, i.e., $u_0 = \omega_0/k_0 < v_g$, stochastic heating is on the whole discontinued. If (3.16) is satisfied, i.e., $u_0 > v_g$ the further picture of the development corresponds to the first case.

3) The range of velocities v close to v_g is a special one. It corresponds to a special kind of motion which may be called group resonance. Indeed, from the definition of ξ it follows that the condition $\xi = 0$ is equivalent to the condition $v \approx v_g$. Hence and from Eq. (2.3) it is clear that a particle with $v \approx v_g$ is at resonance with the whole wave packet. It is therefore just the unification of the regions $\eta < 1$ and $\eta > 1$ (i.e., $v < v_g$ and $v > v_g$) which causes certain difficulties. We shall consider the problem of the group resonance separately.

We now turn to the kinetic description of the $\hat{L}_s$-mapping (3.11). The time for the decoupling of the phase τ_c is, if we take (3.8) and (3.10) into account, equal to[5]

$$\tau_c = \frac{2T(\mathscr{E})}{\ln K} = L\left[|v|\ln\left(\frac{e}{m}E_0\frac{\omega_0L^2}{|v|^3}\right)\right]^{-1} . \tag{3.17}$$

The kinetic equation has the following structure:

$$\frac{\partial F(\mathscr{E},t)}{\partial t} = \frac{1}{2}\frac{\partial}{\partial\mathscr{E}}D\frac{\partial F(\mathscr{E},t)}{\partial\mathscr{E}}, \tag{3.18}$$

where

$$D = \left\langle\!\!\left\langle \frac{(\Delta\mathscr{E})^2}{T(\mathscr{E})} \right\rangle\!\!\right\rangle, \quad \Delta\mathscr{E} = \bar{\mathscr{E}} - \mathscr{E} \tag{3.19}$$

and the averaging $\langle\langle \ldots \rangle\rangle$ is over the random phases ψ. If we neglect the finite correlation time we have from (3.19) and (3.6) simply $\langle\langle \cos^2\psi \rangle\rangle = \frac{1}{2}$ and

$$D = m^2L^2\Omega_0^4/2k_0^2T(\mathscr{E}) = Le^2E_0^2(\mathscr{E}/2m)^{1/2}. \tag{3.20}$$

From (3.20) we get, in particular, the law of stochastic heating in the velocity range $v \ll v_{\max}$:

$$\langle\mathscr{E}^{3/2}\rangle \sim Ct, \quad C = 3\cdot 2^{3/2}Le^2E_0^2/m^{1/2}. \tag{3.21}$$

We draw attention to the fact that the kinetic equation has a Fokker-Planck equation type structure for the distribution function F of different variables depending on whether the wave packet is a t-packet or an x-packet.

4. STANDARD MAPPINGS

The possibility to change in a number of physical problems from differential equations describing the dynamics of the system to a discrete mapping enables us to simplify the investigation. In the case of a system consisting of one degree of freedom and excited by an external force (3/2 degrees of freedom) one can indicate some typical ("standard") mappings which depend little on the concrete properties of the system. In this way problems about the dynamics, the stability, and the stochasticity in various physical objects obtain some universal structure and classification. In the present section we give simple physical considerations which distinguish standard mappings and connect their classification with the cases considered above.

We shall describe a mapping for a canonically conjugate pair of variables (p,x), where p is a generalized momentum (for instance, an action) and x a generalized coordinate (for instance, an angle).

Let, to start with, the time interval T which the mapping spans be constant and be independent of the number of the step. The simplest (standard) mapping then has the form[5,7]

$$\hat{T}: \bar{p} = p + \varepsilon\sin x, \quad \bar{x} = x + \alpha\bar{p}. \tag{4.1}$$

We briefly elucidate its meaning. The first equation describes the change in, for instance, the action under the effect of a single δ-pulse of the force. The change in the action $\Delta p = \bar{p} - p$ has the simplest form in its dependence on the phase x. The change in the phase $\Delta x = \bar{x} - x$ can be written in the form $\omega(\bar{p})T$ and, assuming for $\omega(p)$ the simplest form $\omega(p) = \text{const} + \omega_0 p$ we are led to (4.1) where $\omega_0 T = \alpha$ and the phase shifted by an unimportant constant.

The $\hat{T}$-type mapping (4.1) is the standard one. One can find a review of its properties in Ref. 7. It is the same as (2.7) (apart from the unimportant constant phase shift) and $K = \varepsilon\alpha$.

The first equation in (4.1) is rather universal. The change in the phase $\Delta x = \omega T$ in the second equation can have a different form. An alternative to (4.1) is $\omega = \omega_0 = \text{const}$ and $T = T(p)$. If p is the ordinary momentum, we have $T = \text{const}/p$ and the $\hat{L}_1$-mapping then occurs:

$$\hat{L}_1: \bar{p} = p + \varepsilon\sin x, \quad \bar{x} = x + \alpha/\bar{p}. \tag{4.2}$$

It appears in the problem of the Fermi acceleration[10] and when a particle moves in a billiards of "track" type.[5] From (4.2) we get the stochasticity condition in the form

$$K = \max\left|\frac{d\bar{x}}{dx} - 1\right| = \frac{\varepsilon\alpha}{p^2} \gg 1. \tag{4.3}$$

If p is a quantity proportional to the energy of the particle, $\Delta x = \text{const}/p^{1/2}$ and the $\hat{L}_2$-mapping then occurs:

$$\hat{L}_2: \bar{p}=p+\varepsilon \sin x, \quad \bar{x}=x+\alpha/p^{1/2}. \tag{4.4}$$

These are just the kinds of mapping (3.11) for a particle moving in an x-packet. It follows from (4.4) that stochasticity occurs when

$$K=\varepsilon\alpha/2p^{3/2}\gtrsim 1. \tag{4.5}$$

The $\hat{L}_1$-, $\hat{L}_2$-mappings arise when the spatial step of the mapping is fixed.

5. EFFECT OF DISSIPATION ON THE PARTICLE DYNAMICS

So far we have studied the particle dynamics when there are no dissipative factors whatever. In a real situation the role of such factors may be played by particle collisions or by dissipation caused by collective processes.

We shall assume that the magnitude of the dissipation is small. Taking it into account in its simplest form then reduces to changing Eq. (2.1) to the following one:

$$\ddot{x}+\gamma\dot{x}=\frac{e}{m}\sum_n E_n \cos(k_n x-\omega_n t), \tag{5.1}$$

where γ is the effective friction coefficient. Correspondingly in a wave t-packet Eq. (2.4) is replaced by

$$\ddot{x}+\gamma\dot{x}=\frac{\Omega_0^2}{k_0}T\cos\psi\sum_{n=-\infty}^{\infty}\delta(t-nT), \tag{5.2}$$

where we have used the same notation as in (2.5) and (2.6).

Equation (5.2) generates the following $(\hat{T}-\gamma)$-mapping:

$$(\hat{T}-\gamma):\begin{cases}\bar{v}=e^{-\Gamma}v+\dfrac{K(\Gamma)}{k_0 T}\cos\psi\\ \bar{\psi}=\psi-\omega_0 T+k_0 T\bar{v}\end{cases}, \tag{5.3}$$

where

$$\Gamma=\gamma T,\ K(\Gamma)=K\mu(\Gamma),\ \mu(\Gamma)=(1-e^{-\Gamma})/\Gamma,\ K=\Omega_0^2 T^2. \tag{5.4}$$

Equations (5.3) are analogous to Eqs. (2.7). When $\Gamma=0$ the $(\hat{T}-\gamma)$-mapping changes to the $\hat{T}$-mapping (2.7).

The properties of Eqs. (5.3) have been well studied.[11,12] When $K(\Gamma)\gtrsim 1$ there appears a stochastic attractor. The sequence of bifurcations of the transition from regular dynamics to the stochastic one was considered in Ref. 13.

The simplest properties of the stochastic dynamics when $K(\Gamma)\ll 1$ can be understood starting from Kolmogorov's kinetic equation:

$$\frac{\partial F}{\partial t}=-\frac{\partial}{\partial v}\left(\frac{\langle\langle\Delta v\rangle\rangle}{T}F\right)+\frac{1}{2}\frac{\partial^2}{\partial v^2}\left(\frac{\langle\langle(\Delta v)^2\rangle\rangle}{T}F\right),$$
$$F=F(v,t),\quad \langle\langle\ldots\rangle\rangle=\frac{1}{2\pi}\int_0^{2\pi}d\psi\ldots\ . \tag{5.5}$$

Since $\langle\langle\cos\psi\rangle\rangle=0$ and $\langle\langle\cos^2\psi\rangle\rangle=\frac{1}{2}$, it follows from (5.2) that

$$\langle\langle\Delta v\rangle\rangle=-(1-e^{-\Gamma})v,$$
$$\langle\langle(\Delta v)^2\rangle\rangle=(1-e^{-\Gamma})^2v^2+\frac{1}{2}\frac{K^2(\Gamma)}{k_0^2T^2}e^{-2\Gamma}. \tag{5.6}$$

Substitution of Eq. (5.6) into (5.5) gives

$$T\frac{\partial F}{\partial t}=(1-e^{-\Gamma})\frac{\partial}{\partial v}(vF)+\frac{1}{2}(1-e^{-\Gamma})^2\frac{\partial^2}{\partial v^2}(v^2F)+\frac{1}{4}\frac{K^2(\Gamma)}{k_0^2T^2}e^{-2\Gamma}\frac{\partial^2 F}{\partial v^2}. \tag{5.7}$$

Hence

$$T\frac{d}{dt}\langle v^2\rangle=-(1-e^{-2\Gamma})\langle v^2\rangle+\frac{1}{2}\frac{K^2(\Gamma)}{k_0^2T^2}e^{-2\Gamma}, \tag{5.8}$$

where the moments $\langle v^n\rangle$ are evaluated using a distribution function F which satisfies the kinetic Eq. (5.7). The solution of Eq. (5.8) has the form

$$\langle v^2\rangle=\frac{K^2(\Gamma)}{2k_0^2T^2\Gamma\mu(\Gamma)}\left[1-\exp\left(-\frac{\Gamma\mu(\Gamma)t}{T}\right)\right]+v_0^2\exp\left(-\frac{\Gamma\mu(\Gamma)t}{T}\right), \tag{5.9}$$

where v_0^2 is the value of $\langle v^2\rangle$ at $t=0$.

For not too long times ($\Gamma t/T\ll 1$) we get from (5.9) the usual result of stochastic increase of the particle energy:

$$\langle v^2\rangle\approx K^2(\Gamma)t/2k_0^2T^3+v_0^2=K^2\mu^2(\Gamma)t/2k_0^2T^3+v_0^2, \tag{5.10}$$

where K and $\mu(\Gamma)$ are given in Eq. (5.4). Formula (5.10) contains the same linear law for energy growth with time as (2.13). However, later on the increase in the average energy $\langle\mathscr{E}\rangle=m\langle v^2\rangle/2$ ceases and it reached saturation as $t\to\infty$ which according to Eq. (5.9) equals

$$\mathscr{E}_\infty=\frac{mK^2(\Gamma)}{4k_0^2T^2\Gamma\mu(\Gamma)}=\frac{mK^2}{4k_0^2T^2}\frac{\mu(\Gamma)}{\Gamma}=\frac{mK^2}{4k_0^2T^2\Gamma^2}(1-e^{-\Gamma}). \tag{5.11}$$

In particular, for small values $\Gamma\ll 1$ we get

$$\mathscr{E}_\infty=mK^2/4k_0^2T^2\Gamma. \tag{5.12}$$

For large values $\Gamma\gg 1$ we get

$$\mathscr{E}_\infty=mK^2/4k_0^2T^2\Gamma^2, \tag{5.13}$$

i.e., the power of the Γ-dependence of the limiting "heating energy" of the particles changes.

An effective method for evaluating correlators for $\hat{T}$- and $(\hat{T}-\gamma)$-type mappings was developed in Refs. 14 and 15. One can get Eqs. (5.12) and (5.13) from the results of Ref. 15 for $K\gg 1$. The main physical content of Eqs. (5.11) to (5.13) lies in the fact that they determine not only the possibility for stochastic heating of particles by the field of a wave packet when dissipation is present, but also the limiting magnitude of the heating as function of the parameters (amplitude, distances between the modes of the waves, dispersion) of the packet. To see this explicitly we rewrite, for instance, Eq. (5.12), using the notation:

$$\mathscr{E}_\infty=e^2E_0^2(4m\gamma|d\omega/dk|\Delta k)^{-1}. \tag{5.14}$$

Equation (5.14) can also be used for the general form of a wave packet in which the quantities E_0 and $d\omega/dk$ characterize, respectively, some values of E_k and $d\omega_k/dk$ averaged over the packet.

We now turn to a consideration of particle dynamics in an x-packet, taking dissipation into account. Instead of Eq. (5.2) we have

$$\ddot{x}+\gamma\dot{x}=\frac{\Omega_0^2}{k_0}L\cos\psi\sum_{n=-\infty}^{\infty}\delta(x-nL), \qquad (5.15)$$

which changes to (3.1) when $\gamma = 0$. Introducing $\mathscr{E} = mv^2/2$ as before, we have from (5.15)

$$\dot{\mathscr{E}}+2\gamma\mathscr{E}=2mL\frac{\Omega_0^2}{k_0}\dot{x}\cos\psi\sum_{n=-\infty}^{\infty}\delta(x-nL). \qquad (5.16)$$

To simplify further calculations we restrict ourselves to the case when inequality (3.9) holds, which presupposes the perturbation to be small. In that case sign $\dot{x}$ does not change and integrating (5.16) over a time interval T containing only a single δ-pulse gives

$$\bar{\mathscr{E}}=e^{-2\gamma T}[\mathscr{E}+2mL(\Omega_0^2/k_0)\cos\psi]. \qquad (5.17)$$

We choose the time interval $T = T(\mathscr{E})$ such that it equals the space between two successive δ-pulses of the force, i.e.,

$$x(t_n+T-0)-x(t_n-0)=L,$$
$$x(t_n)=nL, \quad x(t_{n+1})=x(t_n+T)=(n+1)L. \qquad (5.18)$$

Equation (5.18) determines $T = T(\mathscr{E})$. As the equation of motion has in the interval $(t_n + 0, t_n + T - 0)$ the simple form $\dot{\mathscr{E}} + 2\gamma\mathscr{E} = 0$, this gives

$$T=T(\bar{\mathscr{E}}_0)=-\frac{1}{\gamma}\ln\left[1-\frac{\gamma L}{(2\bar{\mathscr{E}}_0/m)^{1/2}}\right],$$
$$\bar{\mathscr{E}}_0=\mathscr{E}+2mL(\Omega_0^2/k_0)\cos\psi. \qquad (5.19)$$

Equation (5.19) presupposes the existence of the inequality

$$v=(2\mathscr{E}/m)^{1/2}>\gamma L, \qquad (5.20)$$

which means that particles with velocities $v\leqslant\gamma L$ cannot traverse a length of path L during a finite time because of the deceleration due to friction force.

It is now easy to use Eqs. (5.17) and (5.19) to write down the final equations of the $(\hat{L} - \gamma)$-mapping:

$$(\hat{L}-\gamma):\begin{cases}\bar{\mathscr{E}}=\exp[-2\gamma T(\bar{\mathscr{E}}_0)]\bar{\mathscr{E}}_0,\\ \bar{\mathscr{E}}_0=\mathscr{E}+2mL(\Omega_0^2/k_0)\cos\psi\\ \bar{\psi}=\psi+k_0L-\dfrac{\omega_0}{\gamma}\ln\left[1-\dfrac{\gamma L}{(2\bar{\mathscr{E}}_0/m)^{1/2}}\right].\end{cases} \qquad (5.21)$$

The main difference between it and the $(\hat{T} - \gamma)$-mapping is in that the time interval of the mapping depends on the particle energy and, as we shall see below, this can lead to interesting physical consequences.

We evaluate the parameter K characterizing the degree of local instability of the system:

$$K=\max\left|\frac{d\bar{\psi}}{d\psi}-1\right|=\frac{2\omega_0L^2\Omega_0^2}{k_0v^3(1-\gamma L/v)}. \qquad (5.22)$$

When $K\gtrsim 1$ there occurs an instability and the particle dynamics becomes stochastic. In that case the $(\hat{L} - \gamma)$-mapping generates a stochastic attractor. The detailed description of the corresponding dynamics will be given separately. Here we note merely that as γ increases the parameter K increases. The cause of this is the following. Decreasing γ leads to an increase in the time it takes to traverse a path of length L. The difference between the initial and final phases therefore increases. This, in turn, diminishes the degree of their correlation. The effect described here qualitatively operates in the velocity range bounded by inequality (5.20).

6. CONCLUSION

Our study of particle dynamics in the field of a wave packet has enabled us to distinguish between two typical limiting cases which differ in whether the wave packet is a t-packet or an x-packet. The packet properties themselves are in a certain sense arbitrary and depend not only on the spectral structure of the packet but also on the particle velocity.

The main feature of the limiting cases considered here lies in that the analysis of the instability of the trajectories and the condition for the occurrence of stochastic dynamics can be carried out exactly. Thus we establish a number of exact conditions under which the kinetic description of the particle dynamics which in plasma theory is well known as the quasilinear theory, is valid.

One of the important physical consequences is that a bound is imposed on the increase in the average particle energy by stochastization of their motion. It is also interesting to note that the problem considered generates some mappings which may be called standard ones by virtue of their typical nature. We shall consider separately the more detailed formal properties of those mappings.

[1)]A detailed discussion of all these conditions was given in Ref. 6 (see also Ref. 5).

[1]A. A. Vedenov, E. P. Velikhov, and R. Z. Sagdeev, Nuclear Fusion **1**, 82 (1961).
[2]A. A. Galeev and R. Z. Sagdeev, Vopr. Teor. Plazmy **7**, 3 (1973) [Rev. Plasma Phys. **7**, 1 (1979)].
[3]B. B. Kadomtsev, Vopr. Teor. Plazmy **4**, 188 (1964) [English translation published as a book, Plasma Turbulence, by Academic Press].
[4]A. A. Vedenov, Vopr. Teor. Plazmy **3**, 203 (1963) [Rev. Plasma Phys. **3**, 229 (1967)].
[5]G. M. Zaslavskiĭ, Stokhastichnost' dinamicheskikh sistem (Stochasticity of dynamical systems) Nauka, Moscow, 1984.
[6]G. M. Zaslavskiĭ, and R. Z. Sagdeev, Zh. Eksp. Teor. Fiz. **52**, 1081 (1967) [Sov. Phys. JETP **25**, 718 (1967)].
[7]B. V. Chirikov, Phys. Repts. **52**, 263 (1979).
[8]G. M. Zaslavskiĭ, S. S. Moiseev, R. Z. Sagdeev, A. V. Tur, G. A. Khomenko, and V. V. Yanovskiĭ Fiz. Plazmy **9**, 62 (1983) [Sov. J. Plasma Phys. **9**, 37 (1983)].
[9]N. N. Bogolyubov, A. A. Logunov, and I. T. Todorov, Osnovy aksiomaticheskogo podkhoda v kvantovoĭ teorii polya (Elements of the axiomatic approach in quantum field theory) Nauka, Moscow, 1969.
[10]G. M. Zaslavskiĭ and B. V. Chirikov, Dokl. Akad. Nauk SSSR **159**, 306 (1964) [Sov. Phys. Dokl. **9**, 989 (1965)].
[11]G. M. Zaslavsky, Phys. Lett. **69A**, 145 (1978).
[12]G. M. Zaslavskiĭ and Kh. -R. Ya. Rachko, Zh. Eksp. Teor. Fiz. **76**, 2052 (1979) [Sov. Phys. JETP **49**, 1039 (1979)].
[13]O. F. Vlasova and G. M. Zaslavsky, Phys. Lett. **105A**, 1 (1984).
[14]R. V. Jensen and C. R. Oberman, Phys. Rev. Lett. **46**, 1547 (1981).
[15]R. V. Jensen and C. R. Oberman, Physica **4D**, 183 (1982).

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