Fluctuating current-voltage characteristic and activation phase reversals of a high-Q Josephson junction

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The phase and voltage in a Josephson junction fluctuate in full analogy with the motion of a Brownian particle in a tilted periodic potential. It is shown that if the junction has high Q and the potential energy is large compared with the temperature the Fokker-Planck equation for these fluctuations can be written in the form of a system of integral equations that can be solved by the Wiener-Hopf method. An exact expression is obtained for the current-voltage characteristic (IVC) of the junction. It is shown that at the threshold current, which corresponds in the absence of fluctuations to the bifurcation point of the solution of the equation for the phase, the fluctuation IVC of the junction has a break at which the logarithmic derivative has a jump of order unity. The decay of the superconducting state of the junction is determined. The relative probabilities of the different phase flips and the probability of junction activation into the resistive state are calculated. Expressions are given for the activation time and for the lifetime of the resistive state, and their connection with the noise characteristics of the junction is discussed.

1. INTRODUCTION

The current-voltage characteristic (IVC) of a Josephson junction is determined in definite range of parameters by thermal fluctuations. Of greatest interest to us is the case of high-Q junctions under conditions when the temperature is low compared with the energy barriers between the potential-energy minima of the junction. The calculation of the static IVC of such a junction is mathematically equivalent to finding the average rate of untwisting of a low-friction physical pendulum by a small torque under conditions when the reversal of the pendulum rotation calls for overcoming a potential barrier greatly exceeding the temperature. The last process can be most lucidly presented as motion of a Brownian particle in a tilted periodic potential. The dependence of the average particle velocity on the inclination is connected, subject to simple substitutions, with the IVC of a Josephson junction.

Research into Brownian motion in the presence of potential barriers seems to have been initiated by Kramers,[1] who developed a theory of absolute rates of chemical reactions on the assumption that thermal dissociation of a molecule is similar to the escape of a Brownian particle from a deep potential well. Kramers's results pertain, first, to the case of strong friction, when only the particle motion near the top of the barrier is significant, and second, to the case of extremely weak friction, when the particle oscillates almost freely in the potential well and diffuses slowly in energy space. This limit is realized if the energy loss per oscillation is small compared with the temperature. A number of attempts were made to go outside the diffusion approximation and obtain results valid for arbitrary dissipation. It was necessary then to invoke unwarranted assumptions,[2] to use model-dependent equations,[3] or confine oneself to numerical results.[4]

We have shown in an earlier paper[5] that in an actual low-frequency region, where the energy loss per oscillation is small compared with the well depth but can be lower as well as higher than the temperature, the Fokker-Planck equation for the motion of a Brownian particle can be reduced to an integral equation in the energy variable, or else to a system of such equations, and the quantum transparency of the potential barrier can be taken into account in a natural manner. These equations are solved in the classical limit by a somewhat modified Wiener-Hopf method that yields also the solution for the quantum case.

The procedure proposed made it possible to find the complete solution of the Kramers problem[1] of the Brownian-particle metastable-state lifetime in a deep potential well (the strong-friction region was investigated in Refs. 6 and 7). In addition, in Ref. 5 we calculated the frequency of the transitions of a Brownian particle between the minima of a two-well potential. The results can have a bearing on the destruction of the superconductivity of a Josephson junction by thermal fluctuations,[8] and on activation transitions between neighboring states of a superconducting ring closed by such a junction.

We have reported briefly[10] an investigation of the motion of a Brownian particle in a slightly tilted periodic potential. As applied to a Josephson junction this means that the current through the junction is less than the threshold current[10] at which (neglecting the fluctuations) a stationary solution of the time-dependent equation for the order-parameter phase can be obtained in addition to the trivial static solution.

The present article is devoted to a calculation of the fluctuation IVC of a Josephson junction at arbitrary values of the current, as well as to an investigation of the probabilities of activated phase flip of the order parameter. The basic concepts and notation and an investigation of the fluctuation IVC junction in the exponential approximation are dealt with in the next two sections. In Sec. 4 we write down the

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basic equation and calculate the IVC below the threshold, where the exponential approximation is not valid. In Sec. 5 is described a method of continuing the IVC above the threshold values of the current, after which the relative value of the IVC jump at the threshold is obtained in Sec. 6. The section that follows solves the problem of activation relaxation of a spatially inhomogeneous distribution of Brownian particles in a tilted periodic potential. The solution is used to find the probabilities of the phase flips and the lifetime of the junctions's superconducting state. In Sec. 8 are given expressions for the activation time and the lifetime of the resistive state, and the connection between these quantities and the noise characteristic of the junction is indicated. The Conclusion contains a brief discussion of the results.

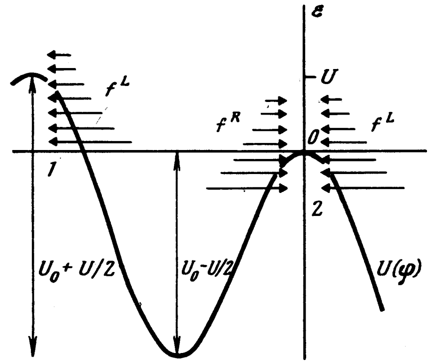

FIG. 1.

2. VOLTAGE ON JUNCTION IN THE ABSENCE OF FLUCTUATIONS

In the resistive model of a lumped Josephson junction, the current through the junction is the sum of the supercurrent $I_c \sin\varphi$ and of the normal current $V/R + CdV/dt$, where I_c is the critical current, φ is the order-parameter phase difference, V is the voltage on the junction, R is the junction resistance in the normal state, and C is its capacitance. The values of V and φ are connected by the Josephson relation, so that the system of equations for them

$$CdV/dt+V/R+I_c \sin\varphi=I, \quad d\varphi/dt=2eV, \tag{1}$$

is equivalent to the single equation

$$\frac{d^2\varphi}{dt^2}+\frac{1}{RC}\frac{d\varphi}{dt}+\frac{2e}{C}(I_c \sin\varphi-I)=0. \tag{2}$$

If the current I is assumed given, the problem of the junction's static IVC reduces to solving Eq. (2) for $\varphi(t)$ and averaging the Josephson relation

$$2eV(I)=\langle d\varphi/dt\rangle. \tag{3}$$

The parameters of Eq. (2) can be combined to form two quantities with dimension of frequency: $\omega = (2eI_c/C)^{1/2}$ and $\gamma = 1/RC$, where ω is the frequency of the small phase oscillations at $I=0$, and γ is the friction coefficient for these oscillations. We assume that the junction has high Q. Neglecting friction ($\gamma=0$), Eq. (2) describes the motion of a particle with coordinate φ in the potential

$$U(\varphi)=-\frac{I_c}{2e}\left(\cos\varphi+1+\frac{I\varphi}{I_c}\right), \tag{4}$$

shown in Fig. 1. At $\gamma=0$ the particle leaving the potential well is accelerated in the tilted potential. When friction is taken into account the particle either remains in one of the minima, or its motion becomes stationary.

When displaced over a period 2π of the potential, the particle acquires an energy $U=\pi I/e$. The total particle energy is of the form

$$\varepsilon=U_0\left[\frac{1}{4\omega^2}\left(\frac{d\varphi}{dt}\right)^2-\frac{1}{2}\left(\cos\varphi+1+\frac{I}{I_c}\varphi\right)\right], \tag{5}$$

where $U_0=I_c/e$ is the depth of the potential wells at $I=0$, and the energy ε is reckoned from the peaks of the potential at $I=0$. Friction causes ε to decrease in accordance with the relation

$$\frac{d\varepsilon}{dt}=-\frac{1}{4e^2R}\left(\frac{d\varphi}{dt}\right)^2.$$

At $\omega \gg \gamma$ the governing current is $I \sim \gamma I_c/\omega \ll I_c$, so that the potential tilt can be neglected in the energy-dissipation calculation. The energy lost by a particle of energy ε after one period of the potential is given by

$$\delta(\varepsilon)=\frac{1}{4e^2R}\int_0^{2\pi}\frac{d\varphi}{dt}d\varphi=\frac{4\gamma}{\omega}U_0\left(1+\frac{\varepsilon}{U_0}\right)^{1/2}E\left(\frac{U_0}{U_0+\varepsilon}\right), \tag{6}$$

where $E(x)$ is a complete elliptic integral. The energy in the stationary regime, at a given current I, is obtained from the balance condition

$$\delta(\varepsilon_0)=U\equiv\pi I/e. \tag{7}$$

Assuming ε_0 known, we get $d\varphi/dt$ from (5) and, taking (3) into account, we obtain the junction IVC in the form

$$I=I_0E(z^2)/z, \quad V=V_0\pi^2/4zK(z^2), \quad I_0=4\gamma I_c/\pi\omega, \quad V_0=RI_0, \tag{8}$$

where E and K are complete elliptic integrals. As $z\to1$ we obtain $I=I_0$ and $V=0$. This means that when the fluctuations are neglected Eq. (2) has at $I<I_0$ only a solution in the form $\varphi=$ const, while at $I>I_0$ a solution appears with $d\varphi/dt \neq 0$, for which the static IVC is given by (8).

At $I \gg I_0$ we get from (8)

$$V(I)\approx RI\left[1-\frac{\pi^4}{512}\left(\frac{I_0}{I}\right)^4\right]. \tag{9}$$

A plot of $V(I)$ is shown in Fig. 2. From this, as well as from the asymptotic form of (9), it follows that the IVC becomes linear quite rapidly. We shall distinguish hereafter between below-threshold $I<I_0$ and above-threshold $I>I_0$ values of the current.

3. FLUCTUATION CVC OF JUNCTION (EXPONENTIAL APPROXIMATION)

Thermal fluctuations affect the junction IVC in two ways. At $I<I_0$, when $V=0$ in the absence of fluctuations, the latter produce a finite voltage on the junction. The IVC for this region will be calculated in Sec. 4. At $I>I_0$ the fluctuations cause transitions between the junction states with

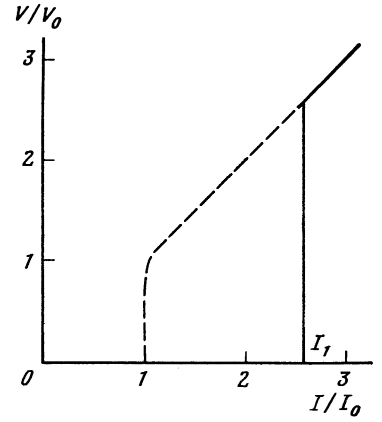

FIG. 2. Static IVC of junction. The dashed section can be observed at times on the order of the resistive-state lifetime.

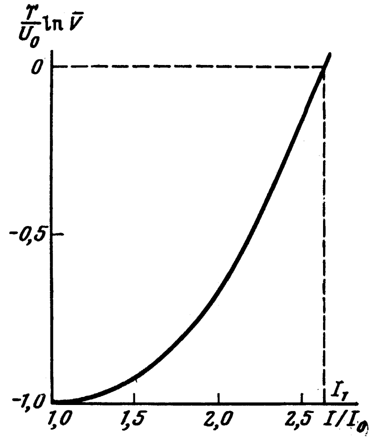

FIG. 3. Fluctuation IVC of a junction in the exponential approximation.

$V=0$ and $V=V(I)$ (see Fig. 2). The physical picture is in this case the following: Let initially $V=0$, which corresponds to finding the particle at the bottom of the potential well. After a time $\tau_a \sim \omega^{-1}\exp(U_0/T)$, the particle is expected to be ejected from this minimum and be set in motion at an average energy ε_0 determined by Eq. (7). We note that the activation time τ_a can be regarded as weakly dependent on the current I. On the contrary, the time τ_t of trapping a particle from an above-barrier state depends strongly on the energy ε_0, meaning also on the current I, so that $\tau_t \sim \omega^{-1}\exp[a(I/I_0)U_0/T]$, where $a(x)$ is a function defined below. Thus, the instrument that averages $V(t)$ over the time intervals $\tau_a, \tau_t \gg t \gg \omega^{-1}$, will read either $V=0$ or $V=V(I)$. The transition from $V=0$ to $V=V(I)$ is a Poisson random process with a characteristic time τ_a, while the reverse transition time is on the average τ_t. Averaging over a time $t \gg \tau_a$, τ_t we get

$$\bar{V}(I)=V(I)\tau_t/(\tau_a+\tau_t). \tag{10}$$

Clearly, at $\tau_a \ll \tau_t$ the fluctuations change the junction voltage little. On the contrary, at $\tau_a \gg \tau_t$ we get in the exponential approximation $\ln \bar{V}(I) \approx U_0[a(I/I_0)-1]/T$. It follows hence that when the fluctuations are taken into account a characteristic current appears, defined by the condition $a(I_1/I_0)=1$. At $I<I_1$ the $\bar{V}(I)$ dependence is exponential, and at $I>I_1$, when $t \gg a$, the fluctuations have little effect and $\bar{V}(I)=V(I)$. To find the function $a(x)$ and to determine the current I_1 we must solve the Boltzmann kinetic problem.

In the presence of fluctuations, the distribution $f(\varepsilon)$ of the particles in energy is concentrated in two regions--near the energy ε_0 and near the bottom of the well. We determine $f(\varepsilon)$ from the following considerations. At a particle displacement equal to the period of the potential, it loses an energy $\delta(\varepsilon)$ to friction and gains an energy U because of the tilt of the potential. In addition, the thermal fluctuations broaden the distribution function by an amount $(\delta(\varepsilon)T)^{1/2}$, so that the periodicity condition of the stationary function $f(\varepsilon)$ takes the form

$$(\delta(\varepsilon)-U)\frac{\partial f}{\partial \varepsilon}+T\delta(\varepsilon)\frac{\partial^2 f}{\partial \varepsilon^2}=0. \tag{11}$$

The variation scale of the function $\delta(\varepsilon)$ is $\varepsilon \sim U_0 \gg T$, it follows therefore from (11), with exponential accuracy,

$$f(\varepsilon)\approx C_1 \exp\left[-\int_0^{\varepsilon}\left(1-\frac{U}{\delta(\varepsilon)}\right)\frac{d\varepsilon}{T}\right].$$

The maximum of $f(\varepsilon)$ is reached at an energy ε_0 determined by the relation (7). We obtain the constant C_1 from the condition $f(0) \sim \exp(-U_0/T)$. In the upshot we obtain for the density of the particles of energy ε_0

$$\ln f(\varepsilon_0)\approx -\frac{U_0}{T}+\int_0^{\varepsilon_0}\left(\frac{U}{\delta(\varepsilon)}-1\right)\frac{d\varepsilon}{T}. \tag{12}$$

Taking (6), (7), and (12) into account we obtain for the junction IVC in the fluctuation region the parametric expression

$$\frac{T}{U_0}\ln \bar{V}(I)=-1+2\int_z^1\left[1-\frac{xE(z^2)}{zE(x^2)}\right]\frac{dx}{x^3}, \tag{13}$$

$$I=I_0E(z^2)/z. \tag{14}$$

The previously introduced function $a(I/I_0)$ is given by the second term of (13) if z is expressed in terms of I/I_0 and (14) is used.

The current I_1 and the voltage V_1 corresponding to the departure of the IVC from the fluctuation region are determined from relations (8) by substituting in them that value of z which causes the right-hand side of (13) to vanish:

$$I_1=2.63I_0=3.36\gamma I_c/\omega, \quad V_1=2.63V_0=1.68\omega/e.$$

It follows from these results that $V \ll V(I)$ so long as $I<I_1$. As applied to Fig. 1 this means that $V(K)$ is zero so long as $I<I_1$, and increases jumpwise to V_1 at this point, after which it follows the plot shown in Fig. 2. The dependence of $\ln V$ on I in the region $I_0<I<I_1$ is illustrated in Fig. 3.

4. FLUCTUATION IVC OF A JUNCTION BELOW THE THRESHOLD

In the preceding section the thermal fluctuations were taken into account on the basis of Eq. (11), in which the coefficient of the second term was chosen to satisfy the condition that the solution (11) have the Boltzmann form at $U=0$. A more rigorous approach calls for inclusion of the fluctuation current in Eq. (2), which is transformed thereby into the Langevin equation

$$\frac{d^2\varphi}{dt^2}+\frac{1}{RC}\frac{d\varphi}{dt}+\frac{2e}{C}(I_c\sin\varphi-I+I_f(t))=0. \quad (15)$$

We assume that the fluctuation current I_1 is Gaussian with a correlator

$$\langle I_f(t)I_f(t')\rangle=\frac{2T}{R}\delta(t-t').$$

Equation (15) is equivalent to the Fokker--Planck equation for the distribution function $f(\varphi,\dot\varphi)$ in φ and in $\dot\varphi\equiv d\varphi/dt$:

$$\dot\varphi\frac{\partial f}{\partial\varphi}-\frac{4e^2}{C}\frac{\partial U(\varphi)}{\partial\varphi}\frac{\partial f}{\partial\dot\varphi}=\frac{1}{RC}\frac{\partial}{\partial\dot\varphi}\left(\frac{4e^2}{C}T\frac{\partial f}{\partial\dot\varphi}+\dot\varphi f\right), \quad (16)$$

where $U(\varphi)$ is given by Eq. (4). The time derivative is omitted, for only the steady state is of interest. The function $f(\varphi,\dot\varphi)$ must be periodic in φ and normalized:

$$f(\varphi+2\pi,\dot\varphi)=f(\varphi,\dot\varphi), \quad \int_0^{2\pi}d\varphi\int_{-\infty}^{\infty}d\dot\varphi f(\varphi,\dot\varphi)=1. \quad (17)$$

The average junction voltage is then given by the integral

$$\bar V=\frac{1}{2e}\int_{-\infty}^{\infty}\dot\varphi\, d\dot\varphi\int_0^{2\pi}f(\varphi,\dot\varphi)\,d\varphi=\frac{\pi}{e}\int_{-\infty}^{\infty}\dot\varphi f(\varphi,\dot\varphi)\,d\dot\varphi,$$

where account is taken of the fact that under stationary conditions the flux is independent of the phase.

In a high-Q junction the energy is dissipated after a large number of oscillations, and the energy (5) can in first-order approximation be regarded as a conserved quantity. We propose the existence of two small parameters γ/ω and $T/U_0 \ll 1$. Their ratio is assumed arbitrary, so that the energy loss $\delta\sim\gamma U_0/\omega$ per oscillation is comparable with T and the solutions given below depend on the parameter δ/T.

We replace Eq. (16) by a simpler integral equation, using the following reasoning. If the dissipation and the potential tilt are neglected, i.e., at $\delta=U=0$, the motion of the representative point on the $(\varphi,\dot\varphi)$ plane is periodic: the particles with $\varepsilon<0$ move on closed trajectories and those with $\varepsilon>0$ on open ones. Since the tilt of the potential U and the energy dissipation δ are small compared with the energy U_0, the real motion of the particle per period will differ little from periodic. As follows from Fig. 1, a particle moving over the barrier close to its top either was reflected one period earlier (when its phase was approximately smaller or larger by 2π) from a neighboring barrier, or else passed over the latter. If the particle had an energy ε' at that instant, the particle distribution in energy is nearly Gaussian, in view of the Gaussian character of the random force, so long as the energy changes are relatively small. This means that the distribution at the top of the neighboring barrier is of the form

$$g(\varepsilon-\varepsilon')=[4\pi T\delta(\varepsilon)]^{-1/2}\exp[-(\varepsilon+\delta(\varepsilon)-\varepsilon')^2/4T\delta(\varepsilon)].$$

The function $\delta(\varepsilon)$ changes over energy intervals $\varepsilon\sim U_0 \gg T$, so that either ε or ε' can be its argument. We neglect for now this dependence, introducing the notation $\delta\equiv\delta(0)$. For a Josephson junction we have

$$\delta=\frac{1}{R}\left(\frac{2}{e}\right)^{3/2}\left(\frac{I_c}{C}\right)^{1/2}=\frac{\pi I_0}{e}=\frac{2\omega}{e^2R}=\frac{4\gamma I_c}{\omega e}.$$

We introduce the functions $f^R(\varepsilon)$ and $f^L(\varepsilon)$, which give the number of particles the move over the barrier with velocities directed to the right and to the left. In the stationary case these functions are identical for any barrier if the energy ε is reckoned from its top (see Fig. 1). The function $f^R(\varepsilon)$ at the barrier 2 is formed from particles that have passed over barrier 1 and of particles reflected from this same barrier. These particles are described by the functions $f^R(\varepsilon')\theta(\varepsilon')$ and $f^L(\varepsilon')\theta(-\varepsilon')$, and the distance between the points of reference of ε' and ε' is equal to U. The periodicity conditions for $f^R(\varepsilon)$ and $f^L(\varepsilon)$ take then the form of integral equations

$$f^R(\varepsilon)=\int_{-\infty}^{\infty}g(\varepsilon-\varepsilon'-U)[f^R(\varepsilon')\theta(\varepsilon')+f^L(\varepsilon')\theta(-\varepsilon')]d\varepsilon'$$
$$f^L(\varepsilon)=\int_{-\infty}^{\infty}g(\varepsilon-\varepsilon'+U)[f^R(\varepsilon')\theta(-\varepsilon')+f^L(\varepsilon')\theta(\varepsilon')]d\varepsilon', \quad (18)$$

where the shift of the argument of the function g by $\pm U$ takes into account the different points of energy reference of the different barriers. The normalization conditions (17) correspond to the presence of one particle at each potential minimum. At $-\varepsilon \gg T$, $f^R(\varepsilon)$ should be a Boltzmann function, so that we get the boundary condition

$$f^{R,L}(\varepsilon)\approx\frac{\omega}{2\pi T}\exp\left[-\frac{1}{T}\left(U_0+\varepsilon\mp\frac{U}{2}\right)\right], \quad -\varepsilon\gg T. \quad (19)$$

Solution of the system (18) with the boundary condition (19) allows us to express the junction IVC in the form

$$\bar V=\frac{\pi}{e}\int_0^{\infty}[f^R(\varepsilon)-f^L(\varepsilon)]d\varepsilon. \quad (20)$$

We solve the integral-equation system (18) by the Wiener-Hopf method. The unilateral Fourier transformation

$$\varphi_\pm^{R,L}(\lambda)=\int_{-\infty}^{\infty}f^{R,L}(\varepsilon)\theta(\pm\varepsilon)\exp(i\lambda\varepsilon/T)\,d\varepsilon \quad (21)$$

transforms the system (18) into

$$\varphi_+^R+\varphi_-^R=g_+(\varphi_+^R+\varphi_-^L), \quad \varphi_+^L+\varphi_-^L=g_-(\varphi_+^L+\varphi_-^R), \quad (22)$$

where the argument λ has been left out of all the functions, and

$$g_\pm(\lambda)\equiv\exp[-\delta\lambda^2/T-i\lambda(\delta\pm U)/T]. \quad (a)$$

The junction voltage is connected with $\varphi^{R,L}(\varepsilon)$ by the relation $\bar V=(\pi/e)\cdot[\varphi_+^R(0)-\varphi_+^L(0)]$, which is obtained from (20) when (21) is taken into account. For our purposes it suffices thus to find the difference $\varphi^R(\lambda)-\varphi^L(\lambda)\equiv\varphi(\lambda)$.

Solving Eqs. (22) for φ_+^R and φ_+^L and taking the difference $\varphi_+^R-\varphi_+^L$, we obtain an equation for $\varphi(\lambda)$:

$$\varphi_+(\lambda)=-G(\lambda)\varphi_-(\lambda), \quad (23)$$

where

$$G(\lambda)\equiv(1-g_+g_-)/(1-g_+)(1-g_-). \quad (24)$$

The condition means that $\varphi(\lambda)$ has a pole of the form

$$\varphi_-(\lambda) \approx -\frac{i\omega}{\pi}\frac{\mathrm{sh}(U/2T)\exp(-U_0/T)}{\lambda+i}, \quad |\lambda+i| \ll 1. \tag{25}$$

To solve (23) we express the kernel $G(\lambda)$ as

$$G(\lambda)=G_+(\lambda)G_-(\lambda), \tag{26}$$

where $G_+(\lambda)$ and $G_-(\lambda)$ are analytic in the upper and lower halves of the λ plane, respectively, and their analyticity regions overlap in a certain band. Using the Cauchy formula we obtain

$$G_\pm(\lambda)=\exp\left[\pm\int_{-\infty}^{\infty}\frac{d\lambda'}{2\pi i}\frac{\ln G(\lambda')}{\lambda'-\lambda\mp i0}\right]. \tag{27}$$

The singular points of $G(\lambda)$ that are closest to the real λ axis are located at $\lambda = 0$ and $\lambda = -i(1-U/\delta)$, and it is this which determines the common region of analyticity of $G_+(\lambda)$ and $G_-(\lambda)$.

The solution of (23) follows from the factorization condition, which is written simultaneously with the boundary condition (25):

$$\frac{\varphi_+(\lambda)}{G_+(\lambda)}=-G_-(\lambda)\varphi_-(\lambda) =\frac{i\omega}{\pi}\frac{\mathrm{sh}(U/2T)G_-(-i)\exp(-U_0/T)}{\lambda+i}. \tag{28}$$

The voltage $\bar V$ is equal to $\pi\varphi_+(0)/e$, so that

$$\bar V=\frac{\omega}{e}G_+(0)G_-(-i)\,\mathrm{sh}\left(\frac{U}{2T}\right)\exp\left(-\frac{U_0}{T}\right). \tag{29}$$

Using the multiplicative structure of $G(\lambda)$ [see (24)] we can rewrite the expressions for $G_\pm(\lambda)$ as follows:

$$G_+(\lambda)=\frac{\Phi(2\delta, 1-2i\lambda, 1)}{\Phi(\alpha-2i\lambda,\alpha)\,\Phi(\beta-2i\lambda,\beta)}, \quad \mathrm{Im}\,\lambda>-\frac{\beta}{2}, \tag{30}$$

$$G_-(\lambda)=\frac{\Phi(2\delta, 1-2i\lambda, 1)}{\Phi(\alpha-2i\lambda,\alpha)\,\Phi(\beta-2i\lambda,\beta)}, \quad \mathrm{Im}\,\lambda<-\frac{\alpha}{2}, \tag{31}$$

where $\alpha\equiv 1+U/\delta\equiv 1+I/I_0$, $\beta\equiv 1-U/\delta\equiv 1-I/I_0$, the function Φ is defined by the relation

$$\ln\Phi(\mu,\nu)=\int_0^{\pi/2}\frac{dx}{\pi}\ln\left\{1-\exp\left[-\frac{\delta}{4T}(\mu^2\,\mathrm{tg}^2 x+\nu^2)\right]\right\},$$

and $\Phi(2\delta,\mu,\nu)$ is given by the same expression, but with δ replaced by 2δ. Substitution of (30) and (31) in (29) yields

$$\bar V=\frac{\omega}{e}\frac{\mathrm{sh}(U/2T)A(2\delta)\exp(-U_0/T)}{\Phi(\alpha,\alpha)\,\Phi(\beta,\beta)\,\Phi(\beta,\alpha)\,\Phi(\alpha,\beta)}, \quad 1-\frac{U}{\delta}\gg\left(\frac{T}{U_0}\right)^{1/2}, \tag{32}$$

where A is the factor preceding the exponential in the problem of the decay of a metastable state of a Brownian particle in a single potential well[5]

$$\ln A(\delta)=\frac{2}{\pi}\int_0^{\pi/2}dx\ln\left[1-\exp\left(-\frac{\delta}{4T\cos^2 x}\right)\right]\equiv 2\ln\Phi(1,1).$$

The criterion indicated in (32) will be explained below. At $U\ll\delta$ ($I\ll I_0$), $\bar V(I)$ has ohmic conductance:

$$\bar V=\frac{\omega I}{e^2T}\frac{\exp(-I_c/eT)}{\sigma(\delta)}, \quad \sigma(\delta)=\frac{2A^2(\delta)}{\pi A(2\delta)}.$$

In the limiting cases we have the expansions

$$A(\delta)\approx\frac{\delta}{T}\left[1+\zeta\left(\frac{1}{2}\right)\left(\frac{\delta}{\pi T}\right)^{1/2}\right], \quad \sigma(\delta)\approx\frac{\delta}{\pi T}, \quad \delta\ll T,$$

where $\zeta(x)$ is the Riemann function, $\zeta(1/2) = -1.46$, and

$$A(\delta)\approx 1-\left(\frac{4T}{\pi\delta}\right)^{1/2}\exp\left(-\frac{\delta}{4T}\right), \quad \sigma(\delta)\approx\frac{2}{\pi}, \quad \delta\gg T.$$

At low dissipation we have

$$\Phi(\mu,\nu)\approx(\delta/4T)^{1/2}(\mu+\nu), \quad \delta\ll T,$$

so that we get from (32)

$$\bar V=\frac{\pi}{2}\frac{IR\exp(-I_c/eT)}{1-(I/I_0)^2}, \quad \delta\ll T,$$

where the dependence on the junction parameters is explicitly indicated. We note that in the low-dissipation model the ohmic conductance of the junction does not depend on its capacitance.

In the opposite limiting case we have

$$\bar V=\frac{\omega}{e}\frac{\mathrm{sh}(I/2eT)\exp(-I_c/eT)}{A^{1/2}[\delta(1-I/I_0)^2/T]}, \quad \delta\gg T,$$

from which it follows that the IVC depends on δ only near the threshold, when $I_0-I\sim I_0(T/\delta)^{1/2}$. As $I\to I_0$, the IVC of the junction has, under our assumption (that δ is independent of energy), a singularity of the form $(1-I/I_0)^{-1}$. Figure 4 shows plots of V against the reduced current I/I for various δ/T.

We consider now how the IVC can be extended to the threshold current. The appearance in the IVC of a singularity of the $(1-I/I_0)^{-1}$ type is due to the integration of the function $f^R(\varepsilon)$ which, if the dependence of δ on ε is neglected, is proportional to $\exp[-\varepsilon(1-U/\delta)/T]$, as follows from Sec. 3. In (32) this singularity follows from the asymptotic relation

$$\Phi(\beta,\beta)\approx\beta(\delta/T)^{1/2}=(1-I/I_0)(\delta/T)^{1/2}. \tag{33}$$

We introduce the function

$$V_{exp}(U) \equiv\int_0^{\infty}\frac{d\varepsilon}{T}\exp\left[-\frac{U_0}{T}-\int_0^{\varepsilon}\left(1-\frac{U}{\delta(\varepsilon')}\right)\frac{d\varepsilon'}{T}\right],$$

which determines the contribution of $f^R(\varepsilon)$ to the junction

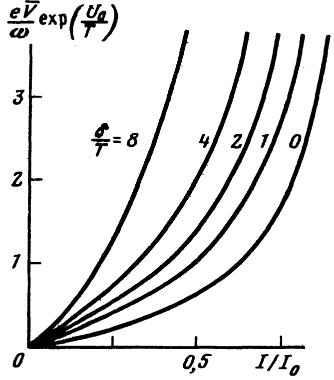

FIG. 4. Fluctuation IVC of junction below threshold.

voltage at values of U close to δ. Account is taken here also of the dependence of δ on ε, so that the expresion is valid up to the threshold $U=\delta$, at which

$$V_{exp}(\delta)\approx 4\pi^{1/2}\left[\frac{U_0}{T\ln(U_0/T)}\right]^{1/2}\exp\left(-\frac{U_0}{T}\right)$$

and does not depend on δ. At $1\gg(1-U/\delta)\gg(T/U_0)^{1/2}$ we have

$$V_{exp}(U)\approx(1-U/\delta)^{-1}\exp(-U_0/T).$$

Comparison with (32) and (33) shows that to continue (32) to the near-threshold region we must replace $\Phi^{-1}(\beta,\beta)\exp(-U_0/T)$ by $(T/\delta)^{1/2}V_{\exp}(U)$, and substitute $U=\delta$, $\alpha=2$, and $\beta=0$ in the remaining functions. The result is

$$\overline{V}=\frac{\omega}{e}\left(\frac{T}{\delta}\right)^{1/2}\frac{\mathrm{sh}(\delta/2T)A(2\delta)V_{exp}(U)}{\Phi(2,2)\Phi(2,0)\Phi(0,2)}. \tag{34}$$

This expression jointly with (32) determines the IVC of the junction in the region below threshold

5. FLUCTUATION IVC OF A JUNCTION ABOVE THRESHOLD

We examine now how the solution of (23) must be modified for the region $U>\delta$. Neglecting the dependence of δ on ε, the equation

$$f^R(\varepsilon)=\int_{-\infty}^{\infty}g(\varepsilon-\varepsilon'-U)f^R(\varepsilon')d\varepsilon'$$

has for sufficiently large $\varepsilon\gg U$, δ, and T the solutions

$$f^R\infty\mathrm{const},\quad f^R\infty\exp[-\varepsilon(1-U/\delta)/T],$$

which correspond to the zeros of $1-g_-(\lambda)$ at $\lambda=0$ and $\lambda=\lambda_0\equiv i(U/\delta-1)$. It was shown above that an exponential solution for $f^R(\varepsilon)$ with account taken of the energy dependence of $\delta(\varepsilon)$ introduces a factor $V_{\exp}(U)$ in the expression for the voltage. At the same time, the solution $f^R\propto$ const corresponds to a nonnormalizable distribution function and must be discarded. As applied to the function $\varphi(\lambda)$ this means that $\varphi_+(\lambda)$ should have a pole at the point λ_0 and be finite at $\lambda=0$. Factoring of (28) does not satisfy either condition.

The point is that the inequalities $\beta<0$ and $\alpha>2$ hold at $U>\delta$, so that expressions (30) and (31) are insufficient to determine $G_+(0)$ and $G_-(-i)$. To continue $G_\pm(\lambda)$ to the vital regions of λ it is necessary to return to the original notation (27), which results in

$$G_+(\lambda)=\frac{\Phi(2\delta,1-2i\lambda,1)\Phi(\beta-2i\lambda,\beta)}{\Phi(\alpha-2i\lambda,\alpha)[1-g_-(\lambda)]},$$

$$-\frac{\beta}{2}>\mathrm{Im}\,\lambda>-\frac{1}{2}, \tag{35}$$

$$G_-(\lambda)=\frac{\Phi(2\delta,1-2i\lambda,1)\Phi(\alpha-2i\lambda,\alpha)}{\Phi(\beta-2i\lambda,\beta)[1-g_+(\lambda)]},$$

$$-\frac{1}{2}>\mathrm{Im}\,\lambda>-\frac{\alpha}{2}. \tag{36}$$

Assuming that $\varphi_+(\lambda)\propto G_+(\lambda)/(\lambda+i)$, as follows from (26), the function $\varphi_+(\lambda)$ will be finite at $\lambda=\lambda_0$, meaning in the region (30). On the other hand in the vicinity of $\lambda=0$ the factor $G_+(\lambda)$ is given by (35) and has a pole singularity. It is therefore necessary to choose in place of (28) a different factorization of (23), such that $\varphi_+(\lambda)$ has a pole at $\lambda=\lambda_0$ and the pole of $G_+(\lambda)$ at $\lambda=0$ is eliminated. These requirements are satistied if we write, taking the boundary condition (25) into account,

$$\frac{\varphi_+(\lambda)}{G_+(\lambda)}=-\varphi_-(\lambda)G_-(\lambda)$$

$$=-\frac{iU}{\delta}\frac{\omega\,\mathrm{sh}(U/2T)\lambda G_-(-i)\exp(-U_0/T)}{\pi(\lambda+i)(\lambda-\lambda_0)}.$$

We have thus found the solution of (23) for the region $U>\delta$. It is clear from the foregoing that to calculate the voltage we must match this solution to the function $f^R(\varepsilon)$ at $\varepsilon\gg U$, δ, T. To this end it suffices to write

$$f^R(\varepsilon)=\frac{C_1}{T}\exp\left[-\frac{U_0}{T}+\int_0^{\varepsilon}\left(\frac{U}{\delta(\varepsilon')}-1\right)\frac{d\varepsilon'}{T}\right], \tag{37}$$

and to note that when the dependence of δ on ε is neglected we obtain hence a pole of the form

$$\varphi_+(\lambda)=iC_1e^{-U_0/T}/(\lambda-\lambda_0).$$

The coefficient C_1 is thus determined by the residue of $\varphi_+(\lambda)$ at the point λ_0, after which integration of (37) with respect to ε yields the junction voltage $\overline{V}=\pi C_1V_{\exp}(U)/e$. Since $\beta<0$, we have $\mathrm{Im}\,\lambda_0=-\beta>-\beta/2$, the residue must be determined by using Eq. (30) for $G_+(\lambda)$ and (36) for $G_-(\lambda)$. The result is

$$\overline{V}=(\omega/e)B(\delta,U)V_{exp}(U), \tag{38}$$

where

$$B(\delta,U)$$

$$=\frac{1}{2}\exp\left(\frac{U}{2T}\right)\frac{|\beta|\Phi(2\delta,1,1)\Phi(2\delta,2\alpha-3,1)\Phi(\beta,\alpha)}{\Phi(3\alpha-4,\alpha)\nu(\alpha,\beta)\Phi(\beta,\beta)}. \tag{38}$$

In the limiting cases we have near the threshold

$$B\approx\frac{1}{2},\quad\delta\ll T;\qquad B\approx\frac{|\beta|\exp(U/2T)}{2A^{1/2}(\delta\beta^2/T)},\quad\delta\gg T.$$

Expression (38) is matched to (34) at the point $U=\delta$. The relation (38) was obtained, with exponential accuracy, by Vollmer and Risken,[12] but their exponential is preceded by an incorrect factor.

The transition of the expression for the IVC from $\overline{V}(I)$ to $V(I)$ near the point $I=I_1$ can be easily tracked if account is taken, when the function $f(\varepsilon)$ is normalized, of the contribution made by the positive-energy particles.

6. BREAK IN THE IVC AT THE THRESHOLD VALUE OF THE CURRENT

The fact that the formulas for $\overline{V}$ are different at $U<\delta$ and $U>\delta$ suggests that the IVC of a Josephson junction has a break at $U=\delta$, i.e., at $I=I_0$. To find the value of this break, which is given by the jump of the logarithmic derivative, we note that near the threshold the principal part of V is given by an expression such as (38) also at $U < \delta$, the function $B(\delta, U)$ being determined as before by the residue of $\varphi_+(\lambda)$ at the point $\lambda = -i\beta$. Recognizing the $\beta > 0$ below the threshold, the residue must be determined by using expression (35) for $G_+(\lambda)$ and expression (31) for $G_-(\lambda)$. We then obtain IVC

$$B(\delta, U)=\frac{T}{U}\,\mathrm{sh}\,\frac{U}{2T}\,\frac{\Phi(2\delta,1,1)\,\Phi(2\delta,2\alpha-3,1)\,\Phi(\beta,\beta)}{\beta\Phi(\alpha,\beta)\,\Phi(\beta,\alpha)},$$

$$U<\delta. \tag{4}$$

Since $V_{\rm exp}(U)$ has no singularity at $U=\delta$, to find the derivative jump we must differentiate (39) and (40). It is important here that β reverses sign at $U=\delta$. The calculations show that only the function $\Phi(3\alpha-4,\alpha)$ contributes to the derivative jump, so that

$$\left(\frac{d\ln\bar V}{d\ln I}\right)_+-\left(\frac{d\ln\bar V}{d\ln I}\right)_-=-D(\delta),$$

where

$$D(\delta)=\left.\frac{\partial\ln\Phi(3\alpha-4,\alpha)}{\partial\alpha}\right|_{\alpha=2}=\frac{\delta}{T}\int_0^{\pi/2}\frac{dx}{\pi}\,\frac{3\,\mathrm{tg}^2x+1}{\exp(\delta/T\cos^2x)-1}.$$

In the limiting cases we have

$$D\approx1+\frac{3}{2}\zeta\left(\frac{1}{2}\right)\left(\frac{\delta}{\pi T}\right)^{1/2},$$

$$\delta\ll T;\quad D\approx\frac{1}{2}\left(\frac{\delta}{\pi T}\right)^{1/2}\exp\left(-\frac{\delta}{T}\right),\quad \delta\gg T.$$

A plot of $D(\delta)$ is shown in Fig. 5. It follows from these results that the IVC logarithmic derivatives, whose order of magnitude is $(U_0/T)^{1/2}\gg1$, has at $I=I_0$ a negative jump of order unity.

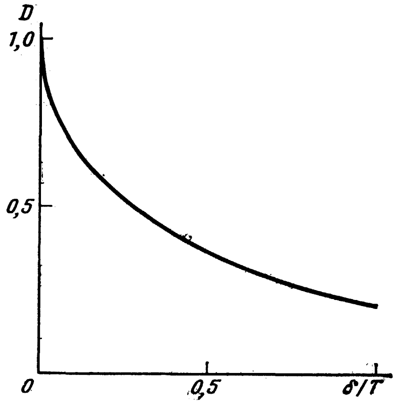

FIG. 5. Discontinuity of the IVC logarithmic derivative at the threshold.

7. ACTIVATION PHASE REVERSALS AND DESTRUCTION OF JUNCTION SUPERCONDUCTIVITY

So long as the current I through the junction is less than the critical I_c, the superconducting state of the junction corresponds to the minimum of its potential energy, and for the junction to go over to the resistive state or to one of the neighboring potential wells it is necessary to surmount a potential barrier. Accordingly, the lifetime τ of the junction superconductivity and the probabilities w_n of transition with a phase flip by $2\pi n$ are determined by activation processes.

Assume that at the initial instant the junction state corresponds to a phase distribution near one of the minima of the potential $U(\varphi)$. The activation processes flip the phase over to the neighboring minima, and destroy thereby the superconductivity. In this situation one of the potential minima is singled out, so that the translational symmetry used for the problem in the preceding section is violated. In place of the equations for the functions $f^{R,L}(\varepsilon)$ we must write therefore the following infinite system of equations:

$$f_n^R(\varepsilon)=\int_{-\infty}^{\infty}g(\varepsilon-\varepsilon'-U)\,[f_{n-1}^R(\varepsilon')\theta(\varepsilon')+f_n^L(\varepsilon')\theta(-\varepsilon')]\,d\varepsilon',$$

$$f_n^L(\varepsilon)=\int_{-\infty}^{\infty}g(\varepsilon-\varepsilon'+U)\,[f_n^R(\varepsilon')\theta(-\varepsilon')+f_{n+1}^L(\varepsilon')\theta(\varepsilon')]\,d\varepsilon', \tag{41}$$

where n is the serial number of the minimum. These equations must be solved subject to the boundary conditions

$$f_0^{R,L}(\varepsilon)\approx\frac{\omega}{2\pi T}\exp\left[-\frac{1}{T}\left(U_0+\varepsilon\mp\frac{U}{2}\right)\right],$$

$$f_n^{R,L}\approx\mathrm{const},\quad -\varepsilon\gg T,$$

assuming that the initial state corresponds to the minimum $n=0$.

Carrying out the transformations (21), we obtain from (41) the system

$$\varphi_n^{R+}+\varphi_n^{R-}=g_-(\varphi_{n-1}^{R+}+\varphi_n^{L-}),$$

$$\varphi_n^{L+}+\varphi_n^{L-}=g_+(\varphi_{n+1}^{L+}+\varphi_n^{R-}),$$

where the subscripts + and − were made into superscripts for convenience, and the argument λ was left out. We solve these equations for φ_n^{R-} and φ_n^{L-} (it is just these functions which contain no shift with respect to n), and taking into account the expression for the decay rate

$$\frac{1}{\tau}=\int_0^{\infty}(f_0^R-f_1^L+f_0^L-f_{-1}^R)\,d\varepsilon$$

we obtain the difference $\varphi_n=\varphi_n^R-\varphi_{n+1}^L$. The new functions satisfy the system of equations

$$(1-g_-g_+)\varphi_n^-=g_-\varphi_{n-1}^+-(1+g_+g_-)\varphi_n^++g_+\varphi_{n+1}^+, \tag{42}$$

with $1/\tau=\varphi_0^+(0)-\varphi_{-1}^+(0)$. The boundary conditions for $\varphi_n(\lambda)$ are

$$\varphi_0^-\approx-\frac{i\omega}{2\pi}\,\frac{\exp[-(U_0-U/2)T]}{\lambda+i},$$

$$\varphi_{-1}^-\approx\frac{i\omega}{2\pi}\,\frac{\exp[-(U_0+U/2)/T]}{\lambda+i},$$

$$|\lambda+i|\ll1, \tag{43}$$

while the remaining $\varphi_n^-(\lambda)$ have no pole at $\lambda = -i$.

To solve the system (42), we introduce the functions

$$\varphi_\pm(k,\lambda)=\sum_{n=-\infty}^{\infty} e^{-ik(n+1/2)}\varphi_n^\pm(\lambda),$$

which satisfy the equation

$$\varphi_+(k,\lambda)=-H(k,\lambda)\varphi_-(k,\lambda),\quad H(k,\lambda)=\frac{1-h_+h_-}{(1-h_+)(1-h_-)},\tag{44}$$

where

$$h_\pm(k,\lambda)=g_\pm(\lambda)\exp(\pm ik\lambda)$$

$$=\exp\left[-\frac{\delta}{T}\left(\lambda^2+i\lambda\left(1\pm\frac{U}{\delta}\right)\right)\pm ik\right].$$

From the conditions (43) we obtain for $\varphi(k,\lambda)$ the boundary condition

$$\varphi^-(k,\lambda)=-\frac{\omega}{\pi}\frac{\sin(k/2+iU/2T)\exp(-U_0/T)}{\lambda+i},\quad |\lambda+i|\ll 1,$$

where τ is connected with $\varphi(k,\lambda)$ by the relation

$$\frac{1}{\tau}=i\int_{-\pi}^{\pi}\frac{dk}{\pi}\sin\left(\frac{k}{2}\right)\varphi^+(k,0).$$

Expressing the kernel of (44) as a product of $H_+(k,\lambda)$ and $H_-(k,\lambda)$ in full analogy with expressions (23) and (26), we obtain the solution for $\varphi^+(k,\lambda)$ in the form

$$\varphi^+(k,\lambda)$$

$$=\frac{\omega\sin(k/2+iU/2T)H_+(k,\lambda)H_-(k,-i)\exp(-U_0/T)}{\pi(\lambda+i)}.$$

It is convenient to write the decay rate in the Arrhenius form

$$\frac{1}{\tau}=\frac{\omega}{2\pi}\mathscr{A}(\delta,U)\exp\left[-\frac{1}{T}\left(U_0-\frac{U}{2}\right)\right],$$

where the exponential contains the height of the lowest barrier $U_0-U/2$. We write for the factor $\mathscr{A}$ preceding the exponential

$$\mathscr{A}(\delta,U)=\int_{-\pi}^{\pi}\frac{dk}{2\pi}w(k,\delta,U)$$

and use the definition

$$w(k,\delta,U)=4\sin\left(\frac{k}{2}\right)\sin\left(\frac{k}{2}+\frac{iU}{2T}\right)$$

$$\times H_+(k,0)H_-(k,-i)\exp\left(-\frac{U}{2T}\right).\tag{45}$$

Just as the junction IVC, the function w differs in form at $U<8$ and at $U>\delta$. To find the corresponding expressions we write down the factors $H_\pm(k,\lambda)$ in different regions of λ:

$$H_+(k,\lambda)=\frac{\Phi(2\delta,1-2i\lambda,1)}{\Phi_1(k,\alpha-2i\lambda,\alpha)\Phi_1(-k,\beta-2i\lambda,\beta)},$$

$$\operatorname{Im}\lambda>-\frac{\beta}{2},\tag{46}$$

$$H_+(k,\lambda)=\frac{\Phi(2\delta,1-2i\lambda,1)\Phi_1(-k,\beta-2i\lambda,\beta)}{\Phi_1(k,\alpha-2i\lambda,\alpha)[1-h_-(k,\lambda)]},$$

$$-\frac{\beta}{2}>\operatorname{Im}\lambda>-\frac{1}{2},\tag{47}$$

$$H_-(k,\lambda)=\frac{\Phi(2\delta,1-2i\lambda,1)}{\Phi_1(k,\alpha-2i\lambda,\alpha)\Phi_1(-k,\beta-2i\lambda,\beta)},$$

$$\operatorname{Im}\lambda<-\frac{\alpha}{2},\tag{48}$$

$$H_-(k,\lambda)=\frac{\Phi(2\delta,1-2i\lambda,1)\Phi_1(k,\alpha-2i\lambda,\alpha)}{\Phi_1(-k,\beta-2i\lambda,\beta)[1-h_+(k,\lambda)]},$$

$$-\frac{1}{2}>\operatorname{Im}\lambda>-\frac{\alpha}{2},\tag{49}$$

where, as before, $\alpha=1+U/\delta\equiv 1+I/I_0$, $\beta=1-U/\delta\equiv 1-I/I_0$, and we have introduced the new function

$$\Phi_1(k,\mu,\nu)$$

$$=\exp\left\{\int_0^{\pi/2}\frac{dx}{\pi}\ln\left[1-\exp\left(-\frac{\delta}{4T}(\mu^2\operatorname{tg}^2x+\nu^2)+ik\right)\right]\right\}.$$

In accord with (45), we get

$$w(k,\delta,U)=\frac{4A(2\delta)\sin(k/2)\sin(k/2+iU/2T)\exp(-U/2T)}{\Phi_1(k,\alpha,\alpha)\Phi_1(-k,\beta,\beta)\Phi_1(k,\beta,\alpha)\Phi_1(-k,\alpha,\beta)},$$

$$U<\delta,\tag{50}$$

$$w(k,\delta,U)=A(2\delta)\frac{\Phi_1(k,\beta,\alpha)\Phi_1(-k,\beta,\beta)}{\Phi_1(k,\alpha,\alpha)\Phi_1(-k,\alpha,\beta)},\quad U>\delta,\tag{51}$$

where expressions (46) and (48) are used at $U<\delta$, and (47) and (49) at $U>\delta$.

The function $w(k,\delta,U)$ vanishes at $k=0$ if $U<\delta$ and differs from zero if $U>\delta$. The reason is that at $U<\delta$ a particle that leaves a potential well may turn out after some time to be again in one of the wells. At $U>\delta$, on the contrary, there is a finite probability of the particle going into a state of accelerated motion, in view of the neglect of the energy dependence of δ in the scheme employed here.

In certain cases it may be more convenient to use the expression

$$\Sigma=-\sum_{n=1}^{\infty}\frac{1}{n}\cos\left[n\left(k+\frac{iU}{2T}\right)\right]\left[e^{nU/2T}\operatorname{erf}\left(\frac{\alpha}{2}\left(\frac{\delta n}{T}\right)^{1/2}\right)\right.$$

$$\left.+e^{-nU/2T}\operatorname{erf}\left(\frac{\beta}{2}\left(\frac{\delta n}{T}\right)^{1/2}\right)\right],$$

which is valid for both $U<\delta$ and $U>\delta$, where erf(x) is the error integral. The existence of such an expression shows that, in contrast to the IVC, the activation probabilities have no singularities at the threshold $U=\delta$.

The probability of the particle transition to the n-th minimum, i.e., the probability of a phase flip by $2\pi n$, is given by

$$w_n(\delta,U)=-\int_{-\pi}^{\pi}\frac{dk}{2\pi}e^{ikn}w(k,\delta,U)\Big/\int_{-\pi}^{\pi}\frac{dk}{2\pi}w(k,\delta,U).\tag{52}$$

The sum of w_n over n is equal to unity at $U<\delta$, for in this

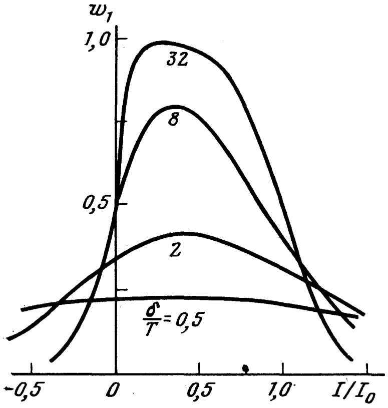

FIG. 6. Relative probability of 2π phase flip.

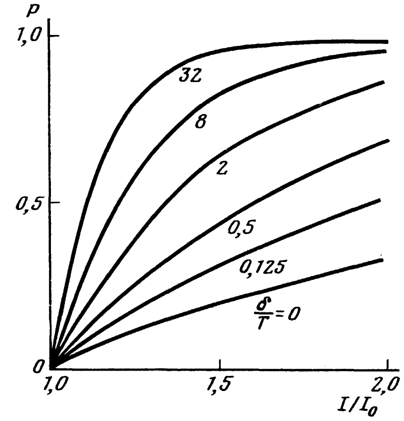

FIG. 8. Relative probability of activation of a junction into the resistive state.

case $w(0,\delta,U)=0$, and is less than unity at $U>\delta$, when a probability $P(\delta,U)$ appears that the particle will not be trapped in any of the potential wells,

$$P(\delta,U)=1-\sum_{n=-\infty}^{\infty}{}' w_n(\delta,U)=w(0,\delta,U)\Big/\int_{-\pi}^{\pi}\frac{dk}{2\pi}w(k,\delta,U). \tag{53}$$

The dependences of w_1, w_2, and P on the relative current I/I_0 at different δ/T are shown in Figs. 6–8. In limiting cases it is possible to obtain simpler expressions for these quantities.

The most interesting is the case $\delta \ll T$. To find $w(k,\delta,U)$ in this case we use the asymptotic relations

$$\Phi_1(k,\mu,\nu)=(\delta/4T)^{1/2}[\mu+(\nu^2-4ikT/\delta)^{1/2}],\quad \delta/T,\ k\ll 1, \tag{54}$$

$$\Phi_1(k,\mu,\nu)=(1-e^{ik})^{1/2}\left[1+\sum_{n=1}^{\infty}\left(\frac{\mu^2\delta}{4\pi nT}\right)^{1/2}e^{ink}\right],\quad \delta/T\ll k. \tag{55}$$

When calculating the denominators of (52) and (53) we can assume that $w=2\delta$. Taking (54) into account, we obtain for $P(\delta,U)$

$$P=\frac{U-\delta}{U+\delta}=\frac{I-I_0}{I+I_0},\quad \frac{\delta}{T}\ll 1,\quad I>I_0.$$

When $w_n(\delta,U)$ is calculated for $n\sim 1$, the entire interval $(-\pi,\pi)$ contributes to the integral (52), so that the use of the asymptotic form (55) yields

$$w_n=\left(\frac{\delta}{\pi nT}\right)^{1/2}+\frac{\delta}{\pi T}\ln\left(\frac{n\delta}{\pi T}\right),\quad \frac{T}{\delta}\gg n\geqslant 1, \tag{56}$$

where the second term takes into account the properties of the function $\mathscr{P}$ defined below. It can be seen that in this limit w_n is independent of the current through the junction.

If $n \gg 1$, the main contribution to the integral (52) is made by $k \ll 1$. The asymptotic form (54) must therefore be used and the integral must be extended over the entire k axis. Shifting the integration contour into the upper half complex-k plane in such a way that it passes along the imaginary axis and encloses the branch point $k=i\beta^2\delta/4T$, we obtain

$$w_n(\delta,U)=\frac{\delta}{T}\mathscr{P}\left(\frac{n\delta}{T},U\right),\quad \frac{\delta}{T}\ll 1, \tag{57}$$

$$\mathscr{P}(x,U)=e^{-x\beta^2/4}\int_0^{\infty}\frac{dz}{\pi}\frac{z^2\exp(-xz^2/4)}{[\alpha+(\alpha^2+\beta^2+z^2)^{1/2}][\beta+(\alpha^2+\beta^2+z^2)^{1/2}]}. \tag{58}$$

This expression is valid for both $U<\delta$ $(\beta>0)$ and $U>\delta$ $(\beta<0)$. To obtain $w_n(\delta,U)$ at $n<0$ it suffices to use the identity $w_{-n}(\delta,U)=w_n(\delta,-U)$. We note that reversal of the sign of U is equivalent to the interchange $\alpha\rightleftarrows\beta$ in (58). At small $n\delta/T$, Eq. (57) leads to the asymptotic form (56), so that (57) is valid for all n. The dependence of $\mathscr{P}$ on I/I_0 is

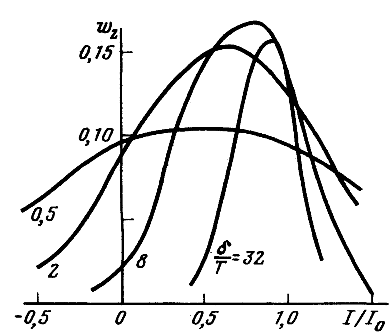

FIG. 7. Relative probability of 4π phase flip.

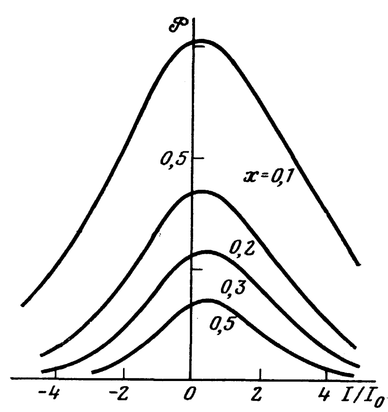

FIG. 9.

shown in Fig. 9 for $x = 0.1, 0.2, 0.3$, and 0.5. The relatively weak dependence of $\mathscr{P}$ on I/I_0 at $I \sim I_0$ agrees with the fact that the asymptotic form (56) does not depend on I at all. At $x \gtrsim 1$ the function $\mathscr{P}$ depends on I exponentially, as can be seen from the factor preceding the exponential.

In the high-dissipation limit $\delta \gg T$ below the threshold, when $\delta - U \gg (\delta T)^{1/2}$, only transitions to the nearest minima, with probabilities $w_{\pm 1} = [1 + \exp(\mp U/T)]^{-1}$, are possible. At the exact threshold, transitions with any $n > 0$ are possible, since

$$w(k, \delta, U) = (1 - e^{-ik})^{1/2}, \quad w_n = \frac{1}{2} \frac{\Gamma(n - 1/2)}{\Gamma(1/2)\Gamma(n+1)},$$

$$n > 0, \quad U = \delta \gg T.$$

For the first values of n we have $w_1 = 1/2$, $\omega_2 = 1/8$, $w_3 = 1/16$.

The probability of the junction becoming resistive at $\delta \gg T$ is $P(\delta, U) = A^{1/2}((U - \delta)^2/\delta T)$, whence it follows that at $U - \delta \gg (\delta T)^{1/2}$ each activation act makes the junction resistive, and the probability of finite phase flips is negligibly small.

8. NOISE CHARACTERISTICS OF JUNCTION

Consider the voltage correlator

$$S_V(\nu) = \int_0^t dt \cos \nu t (\overline{V(t) V(0)} - \overline{V}^2).$$

So long as $I < I_0$ the $V(t)$ plot has the form of random pulses of duration $\sim \omega^{-1}$, the area under which is a multiple of π/e in accord with the Josephson relation (2a). The function $S_V(\nu)$ is concentrated in this case in the high-frequency part of the spectrum at $\nu \sim \omega$. The procedure used in the present paper uses essentially averaging over times $t \gg \omega^{-1}$ and is therefore unsuitable for the calculation of $S_V(\nu)$ at $I < I_0$.

Above the threshold, at $I > I_0$, it can be assumed, as indicated in Sec. 2, that $V(t)$ is given by pulses of amplitude $V(I)$ with average duration τ_t, with an average time τ_a between them. In this case $S_V(\nu)$ takes the Lorentz form

$$S_V(\nu) = \overline{V}^2(I) \frac{\tau_a}{1 + \nu^2 \tau_t^2}, \quad I_1 > I > I_0,$$

where the inequality $\tau_t \ll \tau_a$ and the relation $\overline{V} = V(I)\tau_t/\tau_a$. are used explicity. At $I > I_0$, when $\tau_a \ll \tau_t$, we have

$$S_V(\nu) = \frac{V^2(I)\tau_a^2}{\tau_t(1 + \nu^2 \tau_a^2)}, \quad I > I_1.$$

Since $V(I)$ can be measured independently, and $V(I)$ can practically always be regarded as linear, measurement of $S_V(\nu)$ yields the activation time τ_a and, the trapping time τ_t. Note that the time of τ_a of activation into the resistive state exceeds the superconducting-state lifetime τ, since superconductivity destruction can be produced also by finite phase flips. The time τ_a is expressed in terms of the previously introduced functions by the relation

$$1/\tau_a(\delta, U) = P(\delta, U)/\tau(\delta, U).$$

We indicate also an expression for the trapping time

$$\tau_t = \tau_a \overline{V}(I)/V(I),$$

which is valid also at $I > I_1$, when $\overline{V}(I)$ given by (38) becomes larger than $V(I)$. Naturally, $\overline{V}(I)$ at $I > I_1$ is not related to the junction IVC, but is nevertheless useful for the calculation of τ_t. The difference between the last expression for τ_t and (10) is due to the fact that in the calculation of $\overline{V}(I)$ we have normalized to unity not the total number of particles but only the number of particles at the bottom of the potential well.

The integral noise intensity (the area under the $S_V(\nu)$ curve) is of the order of $V^2(I)\tau_t/\tau_a$ at $\tau_t \ll \tau_a (I < I_1)$ and of the order of $V^2(I)\tau_a/\tau_t$ at $\tau_t \gg \tau_a (I > I_1)$. This means that the integral intensity as a function of the current through the junction has an exponentially sharp peak at $I = I_1$, when the noise is comparable with the average current. The width of the $S_V(\nu)$ distribution is determined by the shorter of the times τ_a and τ_t, so that this distribution becomes rapidly narrower as the current is increased from I_0 to I_1, and changes relatively slowly at $I > I_1$.

9. CONCLUSION

The results above are valid if the inequalities $U_0 \gg T$ and $\omega \gg \gamma$ hold, with greatest interest attached to the case $U_0 \gamma \sim \omega T$. For the investigated activation transitions to have a sufficiently high probability, the excess of U_0 over T must not be particularly large. The foregoing conditions are apparently difficult to realize in the presently produced Josephson junctions. Thus, a large U_0 at zero current suppresses the superconducting-state decay probability so strongly that to observe this phenomenon one must use currents close to the critical value.[8] In this situation the junction is almost always activated into the resistive state, and no trapping of the phase in a neighboring minimum takes place. The entire process is thus analogous to activation of a particle from a solitary potential well. The previously developed theory[5] permits calculation of the factor preceding the exponential in the corresponding decay probability, but this factor can hardly be measured against the background of the exponential temperature and external-field dependences of the activation.

Our results are thus important for high-Q junctions, whose activation phase flips can be observed with sufficient probability even at zero current. The IVC of such junctions for the below-threshold region $I < I_0$ are shown in Fig. 5. The course of the IVC at $I > I_0$ is determined mainly by the exponential factor shown in Fig. 3.

The approach developed above has enabled us to calculate the probabilities of phase flipping between the potential minima. Figures 6 and 7 show the probabilities of 2π- and 4π-phase-flip probabilities vs the external field. It can be seen that if the dissipation is not too small, $\delta \gtrsim T$, the phase flip is mainly 2π. The 4π-phase-flip probability should nonetheless be likewise regarded as fully noticeable, since it reaches approximately 0.15 in a definite range of currents. In the case of low dissipation, $\delta \ll T$, or near the threshold current, other phase flips have also comparable probabilities. It must be emphasized here that the asymptotic expressions corresponding to the limit $\delta \ll T$ are valid only at very small δ/T, viz., at $\delta/T \lesssim 10^{-2}$, as follows from the form of the correction term in (56). We note that in contrast to the IVC the phase-flip probabilities have no singularities at the threshold current I_0.

Above the threshold, each activation of the junction from the minimum of its potential energy makes it resistive with a finite probability P. At low dissipation, $P = (I - I_0)/(I + I_0)$. The current dependence of P is shown for a number of dissipation values in Fig. 8. We regard an investigation of the relative probabilities of the phase flips and of transitions to the resistive state as more promising from the experimental viewpoint than the measurement of the IVC, inasmuch as the exponential dependence of the absolute probabilities on the current I is cancelled out in the recalculation to the indicated quantities.

Activation transitions from an equilibrium state of the junction take place when the fluctuations can be regarded as thermal. This makes likely the approach described above to the calculation of the activation probabilities, as well as to the calculation of the IVC at $I < I_0$, when the voltage is produced on the junction by infrequent phase flips. It appears that the notion that the fluctuations are in equilibrium can be used to calculate the activation probabilities also at $I > I_0$. On the contrary, to calculate the IVC in this region it may be necessary to take into account, in a self-consistent manner, the influence of the noise due to the nonequilibrium resistive state of the junction during the time τ_t of motion of this state.[13]

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