Quasienergy and optical spectra of a two-level system in a low-frequency field of arbitrary strength

I. Sh. Averbukh and N. F. Perel'man

Institute of Applied Physics, Moldavian Academy of Sciences
(Submitted 11 July 1984)
Zh. Eksp. Teor. Fiz. **88**, 1131–1146 (April 1985)

Uniform asymptotic expressions for the quasienergies and intensities of the quasienergy state satellites of two-level quantum systems are obtained on the basis of the theory of nonadiabatic transitions. The evolution of the optical spectra of a two-level system is described, in a unified manner, for fields ranging in strength from weak ones, where perturbation theory can be used, for fields that induce multiphoton resonances, and finally for ultrastrong fields in which the very concept of resonance becomes meaningless.

The most adequate description of the interaction of monochromatic electromagnetic radiation of sufficiently high intensity with quantum systems is in terms of quasienergy.[1] The concept of quasienergy states makes it possible to establish the most general laws of the behavior of quantum systems in a strong field and the structure of their optical spectra on the sole basis of symmetry considerations. Specific calculation for strong fields, on the other hand, with the exception of several exactly solvable cases, encounter considerable difficulties. Even in the simplest two-level model of a system in a periodic field, the available analytic results were obtained within the frameworks of various limiting cases. Use was made of the resonance approximation (the "rotating wave" approximation), wherein the antiresonant interaction terms are discarded (see, e.g., Ref. 2). The quasienergy and the transition probabilities in a low-frequency field were calculated successfully[3–8] in an adiabatic approximation based on the presence of the small parameter $\Delta^{-1}=\omega/\omega_0\ll 1$ (where ω_0 and ω are respectively the natural frequency of the quantum system and the field frequency). The intensities of the satellites of quasienergy states and the optical spectra for such a situation were considered in Refs. 8 and 9 by using adiabatic perturbation theory. The results of these references comprise a substantial advance beyond the framework of perturbation theory and of the pure resonance approximation, the use of which is limited by the condition $\Delta q^2\ll 1$ ($q=2d_{12}F_0/\hbar\omega_0$, d_{12} is the dipole moment of the transition and F_0 is the field intensity. The results of Refs. 3–8 are valid only for field strengths limited by the condition $q/\Delta\ll 1$. The quasienergy of a two-level system was obtained in Refs. 3 and 10 by expansion in the reciprocal constant of the coupling to the field, an expansion restricted by the condition $q/\Delta\gg 1$. Since the regions of applicability of these approximations do not overlap, numerical approximate-diagonalization methods, based on the Floquet-Lyapunov theory,[3,11–13] are used to obtain results that are valid in a wide range of field strengths. These results, however, are inevitably dependent on the specific parameters used in the calculations. Moreover, the difficulties of the numerical methods increase with increasing field strength and resonance order (with increasing q and Δ).

The question of the quasienergy and intensities of the satellites of the quasienergy states of a two-level system in a strong field has attracted additional interest in view of the successful development and perfection of the technique, proposed[17] in 1961 for the measurement of oscillating electric fields in a plasma by means of the satellites of forbidden lines of atoms and ions.[14–16] In this method one measures the intensities of the satellites of atomic emission lines corresponding to a dipole-forbidden transition between the atom levels 1 and 0. Radiation at the frequency $f=\omega_{10}+n\omega$ (ω_{10} is the frequency of the transition $1\to 0$, $n=\pm 1, \pm 3, \ldots,$) is observed because of the mixing of the state 1 with the other nearby excited state 2, which is dipole-coupled with 1. In relatively weak fields, only the first satellites with $n=\pm 1$ are observed, and it is the proportionality of their intensities to the squared field intensity which is used to measure the intensity in this diagnostics method. The most thoroughly investigated in experiment are the plasma satellites of the helium lines. A typical example is the pair of close-lying levels 4^1D, 4^1F ($\omega_0=5.43\ \mathrm{cm}^{-1}$). The 4^1D-2^1P transition corresponds to the allowed HeI line ($\lambda=4922$ Å), while the transition 4^1F-2^1P is forbidden in the absence of electric fields and is used to observe plasma satellites. Such a two-level scheme (the 4^1D and 4^1F levels intermixed by an alternating electric field) was used to calculate the intensities of the first two satellites both in weak fields by perturbation theory,[17] and in relatively strong low-frequency fields on the basis of the adiabatic approximation.[9]

Oscillating fields in a plasma reach in contemporary experiments quite high values and an appreciable number of satellites is observed. Their positions and intensities are essentially nonlinear functions of the field strength. It was proposed in Refs. 9, 12, and 18 to use these nonlinearities for quasilocal measurements of the fields. We note that in these experiments 9, 12, and 18 the electric-field frequencies were substantially lower than the distances between the levels mixed by the field, and this favored the use of the adiabatic approximation for the theoretical estimates.

The problem of calculating the intensities and positions of the emission lines (i.e., of the quasienergy states and of their satellites) becomes thus important also for the diagnos-

 0038-5646/85/040665-09$04.00

tics of oscillating electric fields in a plasma. These methods are possibly also of interest for estimating at a distance the electromagnetic fields in astronomy.[14,19,20]

In the present paper we obtain, by successive application of nonadiabatic-transition-theory methods, expressions for the quasienergies and intensities of satellites of quasienergy states of two-level systems in a low-frequency field ($\Delta \gg 1$) of arbitrary strength. The expressions derived cover as particular cases, within the range of their validity, the results obtained within the framework of all the approximations described above.[3–10] These expressions enable us to track in a unified manner the continuous evolution of the spectra from the weak-field region, where perturbation theory holds, through the region of fields that induce multiphonon resonance, to ultrastrong fields where the very concept of resonance becomes meaningless.

QUASIENERGY AND OPTICAL SPECTRA

We consider a two-level (states $|1\rangle$ and $|2\rangle$) quantum systems with level energies $\mp 1/2\hbar\omega_0$, acted upon by a low-frequency ($\omega \ll \omega_0$) electromagnetic field of intensity $F = F_0 \sin \omega t$. The probability amplitudes a_1 and a_2 of finding the system in the states $|1\rangle$ and $|2\rangle$ satisfy the system of equations

$$i\dot{a}_1 = -{}^1/_2\Delta a_1 + {}^1/_2\Delta q a_2 \sin\tau, \qquad i\dot{a}_2 = {}^1/_2\Delta a_2 + {}^1/_2\Delta q a_1 \sin\tau, \quad \tau = \omega t. \tag{1}$$

Since we consider a low-frequency field ($\Delta \gg 1$), it is natural to use an adiabatic basis, i.e., transform from the states $|1\rangle$ and $|2\rangle$ to the states

$$\varphi_1(\tau) = |1\rangle \cos(\chi/2) - |2\rangle \sin(\chi/2), \qquad \varphi_2(\tau) = |1\rangle \sin(\chi/2) + |2\rangle \cos(\chi/2), \quad \mathrm{tg}\,\chi = q \sin\tau, \tag{2}$$

which are at each instant of time instantaneous eigenvalues of the system Hamiltonian

$$H(\tau)\varphi_{1,2}(\tau) = \mp\Omega(\tau)\varphi_{1,2}(\tau), \tag{3}$$

$$\Omega(\tau) = \frac{\Delta}{2}(1 + q^2 \sin^2\tau)^{1/2}. \tag{4}$$

In matrix notation we have

$$H(\tau) = -{}^1/_2\Delta\sigma_z + {}^1/_2\Delta q \sigma_x \sin\tau, \tag{5}$$

where σ_x and σ_z are Pauli matrices. The wave function of the system can be represented in the form

$$\Psi(\tau) = b_1(\tau)\varphi_1(\tau)\exp\left\{i\int\Omega(\tau)\,d\tau\right\} + b_2(\tau)\varphi_2(\tau)\exp\left\{-i\int\Omega(\tau)\,d\tau\right\}, \tag{6}$$

where in the adiabatic approximation b_1 and b_2 are constant coefficients. In the general case these coefficients depend on the time in accordance with the following equations:

$$b_1(\tau) = a_1(\tau)\cos(\chi/2) - a_2(\tau)\sin(\chi/2), \qquad b_2(\tau) = a_1(\tau)\sin(\chi/2) + a_2(\tau)\cos(\chi/2). \tag{7}$$

The function $\Psi_\varepsilon(\tau)$ corresponding to a definite quasienergy satisfies the relation[1]

$$\Psi_\varepsilon(\tau + 2\pi) = \exp(-i\cdot 2\pi\varepsilon)\Psi_\varepsilon(\tau). \tag{8}$$

Besides being invariant to translation of τ by 2π, the Hamiltonian (5) is invariant to translation of τ by π with simultaneous reversal of the signs of the off-diagonal matrix element ("screw" symmetry[6,21]), which leads to the property

$$\Psi_\varepsilon(\tau + \pi) = \pm\sigma_z \exp(-i\pi\varepsilon)\Psi_\varepsilon(\tau), \tag{9}$$

which obviously agrees with (8).

According to (6) and (8), in the adiabatic approximation there are two values of the quasienergy

$$\varepsilon = \pm\left[\left(\frac{S}{2\pi} + \frac{1}{2}\right)(\mathrm{mod}\,1) - \frac{1}{2}\right], \qquad S = \frac{\Delta}{2}\int_0^{2\pi}(1 + q^2\sin^2\tau)^{1/2}\,d\tau, \tag{10}$$

corresponding to the values $b_1 = 0$, $b_2 = \mathrm{const}$ and $b_1 = \mathrm{const}$, $b_2 = 0$ in Eq. (6). The quasienergy in (10) is measured in units of $\hbar\omega$ and account is taken of the fact that it is determined accurate to an arbitrary integer. The adiabatic approximation is based on the slow variation of $\varphi_1(\tau)$, $\varphi_2(\tau)$, $\Omega(\tau)$. The condition $\Delta \gg 1$ is generally speaking not sufficient for this purpose. The adiabatic approximation is inapplicable not only in asymptotically strong fields $q \gg 1$ but also in relatively weak fields in the vicinities of multiphoton resonances (S equal to an odd multiple of π), when the values of ε [see (10)] become degenerate. In the general case the coefficients $b_1(\tau)$ and $b_2(\tau)$ become thus functions of the time, and to determine the quasienergy and the eigenvectors corresponding to it we must know the matrix that describes the coefficients $b_{1,2}(\tau + 2\pi)$ and $b_{1,2}(\tau)$ or, in accordance with the additional symmetry (9) the half-period matrix $\hat{F}$:

$$\begin{bmatrix} b_1(\tau+\pi) \\ b_2(\tau+\pi)\end{bmatrix} = \hat{F}\begin{bmatrix} b_1(\tau) \\ b_2(\tau)\end{bmatrix}. \tag{11}$$

According to (6), (9), and (11) the value of ε is completely determined by the condition that the system of equations

$$[\hat{F}_{11}b_1(\tau) + \hat{F}_{12}b_2(\tau)]\exp(i{}^1/_2S) = \pm\exp(-i\pi\varepsilon)b_1(\tau), \qquad [\hat{F}_{21}b_1(\tau) + \hat{F}_{22}b_2(\tau)]\exp(-i{}^1/_2S) = \mp\exp(-i\pi\varepsilon)b_2(\tau) \tag{12}$$

be solvable, meaning:

$$\sin\pi\varepsilon = \mp T\sin(S/2 - \varphi), \qquad T = |\hat{F}_{11}| = |\hat{F}_{22}|, \quad \varphi = -\arg\hat{F}_{11} = \arg\hat{F}_{22}. \tag{13}$$

The signs in (13) correspond to the signs in (9). In the derivation of (13) we used the fact that the matrix $\hat{F}$ is unimodular. The problem of determining ε was thus reduced to a calculation or estimate of the matrix elements of $\hat{F}$. To this end it is convenient to transform to the quantities

$$X_{1,2}(\tau) = \frac{1}{\sqrt{2}}(a_1 \pm a_2), \tag{14}$$

which satisfy the second-order equations

$$\ddot{X}_{1,2} + \frac{\Delta^2}{4}\left[(1 + q^2\sin^2\tau) \pm i\frac{2q}{\Delta}\cos\tau\right]X_{1,2} = 0. \tag{15}$$

Equations (15) contain the large parameter Δ which allows us to solve them in a quasiclassical approximation. The corresponding quasiclassical solutions are obtained in standard fashion by expanding the WKB phase in powers of Δ^{-1}. We obtain

$$X_1^{\rm WKB}=b_1\cos(\chi/2+\pi/4)\exp\left\{i\int\Omega(\tau)\,d\tau\right\}+b_2\sin(\chi/2+\pi/4)\exp\left\{-i\int\Omega(\tau)\,d\tau\right\},$$
$$X_2^{\rm WKB}=b_1\sin(\chi/2+\pi/4)\exp\left\{i\int\Omega(\tau)\,d\tau\right\}-b_2\cos(\chi/2+\pi/4)\exp\left\{-i\int\Omega(\tau)\,d\tau\right\}. \qquad (16)$$

These WKB solutions correspond to the functions obtained in the quasiclassical approximation [see (2)], and the solutions (16) are in the general case insufficient to determine the matrix $\hat{F}$. The WKB approximation for Eqs. (15) fails in the vicinities of the branch points τ_K of the quasiclassical "momentum" $\Omega(\tau)$ (of the zeros of the expression $1+q^2\sin 2\tau$) located in the complex τ plane:

$$\tau_k=\pm i\,\mathrm{Arsh}\,(q^{-1})+\kappa\pi,\quad \kappa=0,\ \pm1,\ \pm2\ldots. \qquad (17)$$

At $\Delta\gg1$, the regions where the WKB approximation does not hold are strongly localized (in accord with the value of Δ) in the vicinities of the points τ_K. The dimensions of these regions can be estimated as

$$|\tau-\tau_k|\sim\Delta^{-2/3}(1+q^2)^{-1/6},\quad \mathrm{Arsh}\,q^{-1}\gg\Delta^{-2/3}(1+q^2)^{-1/6}, \qquad (18)$$

and

$$|\tau-\tau_k|\sim(\Delta q)^{-1},\quad \Delta/q\ll1. \qquad (19)$$

In the case (18) the regions where the WKB approximation fails are isolated from one another, while in case (19) these regions contain pairs of complex-conjugate branch points that repeat with a period π. At $\Delta\gg1$ and at arbitrary q (at arbitrary field strength F_0) the regions where the WKB approximation fails, corresponding to half-periods of the field ($|\tau-\tau_k|\ll\pi$), never overlap. With increasing field strength (with increasing q) the localization of the regions where the WKB approximation fails improves. This feature (the not more than pairwise coalescence of the indicated regions) permits the use of the well known results of the theory of nonadiabatic transitions[22] and yields, by the adjoint-equation method[23] the matrix $\hat{F}$ by joining the WKB solutions (16) with the standard solutions of Eqs. (15) near τ_K. We can obtain by the same token for the matrix an asymptotic estimate that is valid for arbitrary q:

$$\hat{F}_{11}=\hat{F}_{22}^{*}=T(\delta)\exp(-i\varphi(\delta)),$$
$$\hat{F}_{12}=-\hat{F}_{21}^{*}=\exp(-\delta-iS),\quad T(\delta)=(1-e^{-2\delta})^{1/2}; \qquad (20)$$

$$\varphi(\delta)=\frac{\delta}{\pi}-\frac{\delta}{\pi}\ln\frac{\delta}{\pi}+\arg\Gamma\left(i\frac{\delta}{\pi}\right)+\frac{\pi}{4},$$
$$\delta=-i\int_{\tau_K^*}^{\tau_K}\Omega(\tau)\,d\tau=\frac{\Delta}{(1+q^2)^{1/2}}D\left(\frac{\pi}{2},\frac{1}{(1+q^2)^{1/2}}\right),$$
$$S=2\Delta(1+q^2)^{1/2}E\left(\frac{\pi}{2},\frac{q}{(1+q^2)^{1/2}}\right). \qquad (21)$$

Here $\Gamma(x)$ is the gamma function,

$$D\left(\frac{\pi}{2},k\right)=\frac{1}{k^2}\left[F\left(\frac{\pi}{2},k\right)-E\left(\frac{\pi}{2},k\right)\right],$$

and $F(\pi/2,k)$, $E(\pi/2,k)$ are complete elliptic integrals of first and second order.

Equations (12) and (20) yield for ε an expression valid, at the accuracy $\sim\Delta^{-1}$ assumed and for arbitrary field strength

$$\sin\pi\varepsilon=\mp(1-e^{-2\delta})^{1/2}\sin[S/2-\varphi(\delta)]. \qquad (22)$$

In relatively weak fields ($\delta\gg1$) the function $\varphi(\delta)\approx-\pi/12\delta\to0$ and expression (22) becomes equal to the result of Ref. 6, which is devoted to this case. Far from the resonances, where $S=(2K+1)\pi$, the difference between $T(\delta)$ and unity can be neglected and (22) becomes equal to expression (10) obtained in the adiabatic approximation. Near the resonances it is necessary to take into account not only the difference between $T(\delta)$ and unity (see Ref. 6), but also the fact that $\varphi(\delta)\neq0$. The former leads to a splitting of the quasienergy in the resonance region,[6] and the latter to a shift of the resonance position relative to that calculated in the adiabatic approximation. Allowance for the small quantity $\varphi(\delta)$ in the vicinity of the resonance at $\delta\gg1$ is essential not only for its correct description, since $\varphi(\delta)$ has only a power-law smallness (in the parameter $1/\delta$) whereas the splitting is exponentially small [$\propto\exp(-\delta)$].

In very strong fields ($\delta\ll1$) the function $\varphi(\delta)\to-\pi/4$ like

$$\varphi(\delta)\approx\frac{\delta}{\pi}-\frac{\delta}{\pi}\ln\frac{\delta}{\pi}-\psi(1)\frac{\delta}{\pi}-\frac{\pi}{4}. \qquad (23)$$

Here $\psi(z)$ is the digamma function [25] and $\psi(1)\approx0.5772$. In this case $\delta\approx\pi\Delta/4q$, $S\approx2\Delta q$, i.e.,

$$\varepsilon\approx\mp\left(\frac{\Delta}{2\pi q}\right)^{1/2}\sin\left(\Delta q+\frac{\pi}{4}\right). \qquad (24)$$

As the field increases, ε oscillates thus and tends to zero as $q\to\infty$. The vanishing of ε in asymptotically strong fields is natural, for in this case the dynamics of the system is dictated by an external force and turns out to be periodic with a period $2\pi/\omega$. The result (24) can be obtained directly from (1) by expansion in the small parameter Δ/q (see, e.g., Refs. 3 and 10).

Figure 1 shows the field dependence of ε in a wide range of the field parameters. At $q\ll1$ the $\varepsilon(q)$ dependence is expressed by the quadratic Stark effect. At $q\sim1$, field-induced multiphonon resonances are observed, and their exact positions and splittings are determined by the parameter δ [see Eq. (22)]. With increasing q the resonances broaden and $\varepsilon(q)$ takes the asymptotic form (24). Equation (22) gives thus a unified, uniform asymptotic estimate of the quasienergy at an arbitrary low-frequency field strength. We note that nonuniform asymptotic estimates of the quasienergy (with overlapping validity regions) were obtained in Ref. 26 for the cases of relatively weak and of sufficiently strong fields (see also Ref. 21).

The wave function of a system with given ε can be represented as a sum of the contributions from the satellites:

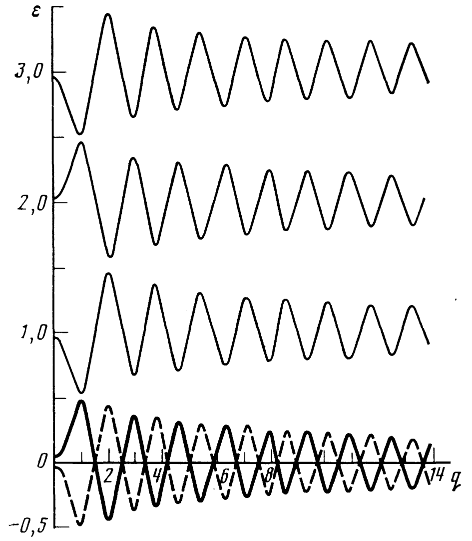

FIG. 1. Dependence of the quasienergy of a two-level system on the field intensity at $\Delta = 4.098$. The thick lines (solid and dashed) show the quasienergy branches referred to the first Brillouin zone.

$$\Psi_\varepsilon(\tau)=e^{-i\varepsilon\tau}\sum_{n=-\infty}^{+\infty}(P_n^\varepsilon|1\rangle+Q_n^\varepsilon|2\rangle)e^{-in\tau}, \tag{25}$$

where the Fourier components of the periodic part of $\Psi_\varepsilon(\tau)$ are

$$P_n^\varepsilon=\frac{1}{2\pi}\int_0^{2\pi}d\tau\langle 1|\Psi_\varepsilon(\tau)\rangle\exp\{i\varepsilon\tau+in\tau\}, \tag{26}$$

$$Q_n^\varepsilon=\frac{1}{2\pi}\int_0^{2\pi}d\tau\langle 2|\Psi_\varepsilon(\tau)\rangle\exp\{i\varepsilon\tau+in\tau\}. \tag{27}$$

Using (9), we obtain

$$P_n^\varepsilon=\frac{1}{2\pi}(1\pm\cos n\pi)\int_0^{\pi}\langle 1|\Psi_\varepsilon(\tau)\rangle\exp\{i\varepsilon\tau+in\tau\}d\tau, \tag{28}$$

$$Q_n^\varepsilon=\frac{1}{2\pi}(1\mp\cos n\pi)\int_0^{\pi}\langle 2|\Psi_\varepsilon(\tau)\rangle\exp\{i\varepsilon\tau+in\tau\}d\tau. \tag{29}$$

The signs $\pm$ ($\mp$) in (28) and (29) correspond to the signs in (22). According to (28) and (29), the projection of the quasienergy wave function (25) on the state 1 or 2 contain either only even or only odd satellites.

The quasienergy and the satellites of quasienergy states manifest themselves directly in the optical spectra of a quantum system. Consider, in particular, the emission spectra of a two-level (levels 1 and 2) system in a low-frequency field, due to photon emission and to a transition to the ground state 0. We assume the transitions $1\leftrightarrow 2$ and $2\leftrightarrow 0$ to be dipole-allowed, and the transition $1\leftrightarrow 0$ forbidden. This situation is in accord with the technique of determining the alternating low-frequency electric field in a plasma by measuring the satellites of forbidden helium lines.[9,18] The wave function of an atom situated in a strong low-frequency field that mixes the states 1 and 2 can be represented in the form

$$\Phi(\tau)=c_+\Psi_+(\tau)+c_-\Psi_-(\tau), \tag{30}$$

where $\Psi_\pm(\tau)$ are the quasienergetic functions (25) corresponding to the two values $\varepsilon=\pm\varepsilon_0$ in accord with (22), and

$$\varepsilon_0=\frac{1}{\pi}\arcsin\left\{T(\delta)\sin\left[\frac{S}{2}-\varphi(\delta)\right]\right\}.$$

The constants $c_\pm$ are determined essentially by the manner in which the interaction of the atom with the field is initiated. Thus, for example, if the strong field is produced in the course of the experiment in a spatially bounded region, the rate of application of the field to each atom that enters the interaction region at a definite velocity depends on the spatial gradient of the field. If the characteristic field-growth time t^* exceeds appreciably $(\delta\omega_0)^{-1}$, (where $\delta\omega_0$ is the characteristic detuning from resonance in the transition region), the field can be regarded as turned-on adiabatically. The system is then in one of the quasienergetic states, which goes over into the specified initial state (1 or 2) when the field is turned off. If, however, $t_*\ll(\delta\omega_0)^{-1}$, the switching regimes should be regarded as instantaneous and $\Phi(\tau)$ is a superposition of $\Psi_\pm(\tau)$. The coefficients $c_\pm$ are then determined by the condition that at the instant τ_0 of the start of the interaction the atom be in the state 1 or 2. We note that in sufficiently strong fields the distinction between the turning-on regimes is not trivial, since multiphoton effects of different orders can be induced as the atom enters into the region of a growing field (see Fig. 1). The turning-on can be regarded as adiabatic only if $\hbar/t_*$ is considerably smaller than all the quasienergetic splittings in the resonance, and as instantaneous in the opposite case. In the general case either condition can be satisfied only for some of the resonances, and the determination of the coefficients $c_\pm$ calls for a separate analysis for each specific manner of field growth.

We proceed now to calculate the line intensities $I(f)$ of the emission by atoms in a state $\Phi(\tau)$. The intensity of emission with transition to the level 0 is obtained by using standard perturbation theory

$$I(f)\propto|c_+|^2\sum_n|Q_n^{+\varepsilon_0}|^2\delta\left(\varepsilon_0+n+\frac{|E_0|-\hbar f}{\hbar\omega}\right)$$
$$+|c_-|^2\sum_m|Q_m^{-\varepsilon_0}|^2\delta\left(-\varepsilon_0+m+\frac{|E_0|-\hbar f}{\hbar\omega}\right), \tag{31}$$

where E_0 is the energy corresponding to the state 0. It follows from (31) that the system emission spectrum consists of two overlapping sets of equivalent lines corresponding to different quasienergies. If the interaction is applied adiabatically, one of the sets ($c_-=0$ or $c_+=0$) is observed. Near multiphoton resonance ($|\varepsilon_0|\approx 0.5$), the conditions for sudden switching-on $(1-2|\varepsilon_0|)\omega t_*\ll 1$ can be realized. In this case the optical spectra of the system constitutes a set of equidistant lines each split into two components. The component intensities are comparable, and the distance between them is $(1-2|\varepsilon_0|)\omega$.

Indeed, by using, e.g., the initial condition $\Phi(\tau_0)=|1\rangle$ and Eqs. (6) and (31), we get

$$c_\pm(\tau_0)=\frac{(1+\alpha_\pm^2)^{1/2}}{\alpha_\pm-\alpha_\mp}\left[\cos\frac{\chi(\tau_0)}{2}\exp\left\{-i\int_0^{\tau_0}\Omega(\tau)\,d\tau\right\}\right.$$

$$\left.-\alpha_\mp\sin\frac{\chi(\tau_0)}{2}\exp\left\{i\int_0^{\tau_0}\Omega(\tau)\,d\tau\right\}\right],\quad \alpha_\pm=\frac{b_1(\pm\varepsilon_0)}{b_2(\pm\varepsilon_0)},\quad (32)$$

where according to (12), (20), and (22)

$$b_1=\{1+e^{-2\delta}[T(\delta)\cos(S/2-\varphi)\pm R]^{-2}\}^{-1/2},$$
$$b_2=\pm e^{iS/2}\{1+e^{-2\delta}[T(\delta)\cos(S/2-\varphi)\mp R]^{-2}\}^{-1/2},\quad (33)$$
$$R=[\cos^2(S/2-\varphi)+e^{-2\delta}\sin^2(S/2-\varphi)]^{1/2}.$$

In accordance with (33), the line intensities in the regime considered depend generally speaking on τ_0 (on the phase of the strong field at the switching instant). For not too strong fields ($q\ll1$), however, $|c_\pm|$ ceases to depend on τ_0, as can be seen from (32). Near the multiphoton resonance ($\varepsilon_0\approx\pm0.5$), it follows from (32) and (33) that the coefficients are $|b_1(\pm\varepsilon_0)|=|b_2(\pm\varepsilon_0)|=1/\sqrt{2}$ and $|c_\pm|=1/\sqrt{2}$. According to (31), near each line with fixed n there is located a line separated from it by a distance $(1-2|\varepsilon_0|)$ and corresponding to $m=n+\operatorname{sign}\varepsilon_0$. It is easy to verify that $|Q_n^{+\varepsilon0}|\approx|Q_{n+\operatorname{sign}\varepsilon0}^{-\varepsilon0}|$, so that at resonance the lines of the nth doublet are equal in intensity. Away from resonance (on account of a change of ω or of F_0), one of the doublet lines is much stronger than the other. These features constitute a generalization of the Autler-Townes effect (in optical double resonance), known in nonlinear spectroscopy, to include the case of multiphoton resonances in strong fields. The picture of the spectrum is now obtained (asymptotic estimates of Q_n^ε are given below) outside the framework of the resonance approximation (with allowance for "antiresonant" reradiations of photons in strong fields) and describes the doublet structure for the entire aggregate of the quasienergetic satellites.

ASYMPTOTIC ESTIMATES OF THE QUASIENERGETIC-STATE SATELLITE INTENSITIES

Using Eqs. (6), (28), and (29) we obtain

$$P_n^\varepsilon=\tfrac{1}{2}(1\pm\cos n\pi)\,[A(p)+B(p)],$$
$$Q_n^\varepsilon=\tfrac{1}{2}(1\mp\cos n\pi)\,[-C(p)+\mathscr{D}(p)],\quad (34)$$

where $p=\varepsilon+n$ and

$$A(p)=\frac{1}{\pi}\int_0^\pi d\tau\, b_1(\tau)\cos\frac{\chi(\tau)}{2}\exp\left[i\int_0^\tau\Omega(\tau)\,d\tau+ip\tau\right],\quad (35)$$

$$B(p)=\frac{1}{\pi}\int_0^\pi d\tau\, b_2(\tau)\sin\frac{\chi(\tau)}{2}\exp\left[-i\int_0^\tau\Omega(\tau)\,d\tau+ip\tau\right],\quad (36)$$

$$C(p)=\frac{1}{\pi}\int_0^\pi d\tau\, b_1(\tau)\sin\frac{\chi(\tau)}{2}\exp\left[i\int_0^\tau\Omega(\tau)\,d\tau+ip\tau\right],\quad (37)$$

$$\mathscr{D}(p)=\frac{1}{\pi}\int_0^\pi d\tau\, b_2(\tau)\cos\frac{\chi(\tau)}{2}\exp\left[-i\int_0^\tau\Omega(\tau)\,d\tau+ip\tau\right].\quad (38)$$

Since $\Omega(\tau)$ contains the large parameter Δ, it is natural to estimate the integrals (35)–(38) by the saddle-point method. By way of example we consider the asymptotic estimate of $\mathscr{D}$ (estimates of A, B, and C are obtained similarly). The saddle points in the integral (38) are given by the equation

$$\tfrac{1}{2}\Delta(1+q^2\sin^2\tau)^{1/2}=p.\quad (39)$$

For a unique determination of the branch of the square root in (39), we draw cuts from the branch points τ_K and τ_K^*, as shown in Fig. 2, and assume the square root to be positive on the real axis. The solutions of (39) are

$$\tau_r=\pm i\operatorname{Arch}\rho+\pi/2+\pi r,\quad p>\tfrac{1}{2}\Delta(1+q^2)^{1/2},\quad (40)$$

$$\tau_r=\pm\arcsin\rho+\pi r,\quad \tfrac{1}{2}\Delta<p<\tfrac{1}{2}\Delta(1+q^2)^{1/2},\quad (41)$$

$$\tau_r=\pm i\operatorname{Arsh}\rho+\pi r,\quad 0<p<\tfrac{1}{2}\Delta,\quad (42)$$

$$\rho=\left|\frac{p^2-\Delta^2/4}{\Delta^2q^2/4}\right|^{1/2},\quad r=0,\pm1,\pm2\ldots.$$

At $p<0$ the saddle points lie on the second sheet of the Riemann surface (on which the square root in (39) is negative on the real axis) at points determined by Eqs. (40)–(42), respectively. The disposition of the saddle points corresponding to cases (40)–(42) is shown in Fig. 1. As p is decreased right down to $p=1/2\Delta(1+q^2)^{1/2}$ the saddle points (see Fig. 2a) approach the real axis, merge at the points $\tau=\pm\pi/2+\pi r$, and then diverge from these positions along the real axis (see Fig. 2b). With further decrease of p the saddle points approach the values $\tau=\pi r$ and again coalesce pairwise at $p=\Delta/2$. At $p<\Delta/2$ the saddle points diverge from the

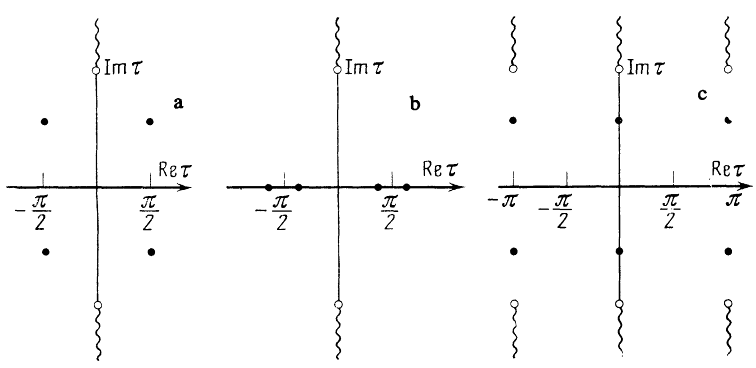

FIG. 2. Arrangement of the branch points of $\Omega(\tau)$ and of the saddle points τ_r on the complex τ plane. The wavy lines indicate the cuts.

points $\tau=\pi r$ along the imaginary axis (see Fig. 2c). If the saddle points are far enough from one another, the satellite amplitudes can be estimated by summing the independent contributions of each of the isolated saddle points, as was done in Ref. 8. Using, however, the fact that not more than two saddle points coalesce simultaneously, a uniform asymptotic estimate valid for a continuous variation of p (for an arbitrary number n of satellites) can be obtained. It is convenient to use for this purpose the cubic-transformation method.[27] Consider, for example the case when saddle points coalesce at $\tau=\pi/2$ as they approach each other along the imaginary axis. We introduce in the integrand a new variable w that maps $\tau=\tau(\omega)$:

$$p\tau-\frac{\Delta}{2}\int_0^\tau d\tau(1+q^2\sin^2\tau)^{1/2}=\frac{1}{3}w^3+Q_1^2w+C_1, \qquad (43)$$

$$\frac{2}{3}Q_1^3=pz_1-\frac{\Delta}{2}\int_0^{z_1}dz(1+q^2\operatorname{ch}^2 z)^{1/2}, \quad z_1=\operatorname{Arch}\rho,$$

$$C_1=p\frac{\pi}{2}-\frac{\Delta}{2}\int_0^{\pi/2}d\tau(1+q^2\sin^2\tau)^{1/2}=p\frac{\pi}{2}-\frac{S}{4}. \qquad (44)$$

The mapping (43) sets in correspondence the points $\tau=\pi/2\pm i\operatorname{arccosh}\rho$ with the points $\pm iQ_1$, and is mutually single-valued. The integral (38) is transformed into

$$\mathscr{D}(p)=\frac{1}{\pi}\int_{w(0)}^{w(\pi)}dw\,b_2[\tau(w)]\frac{d\tau}{dw}\cos\left\{\frac{1}{2}\chi[\tau(w)]\right\}$$

$$\times\exp\left[i\frac{1}{3}w^3+iQ_1^2w\right]\exp\left[ip\frac{\pi}{2}-i\frac{S}{4}\right]. \qquad (45)$$

If the correspondence (43) is correctly chosen, $d\tau/dw$ is finite everywhere in the region that makes the principal contribution to (38), including at the saddle points themselves, where

$$\left.\frac{d\tau}{dw}\right|_{w=iQ_1}=\left.\frac{d\tau}{dw}\right|_{w=-iQ_1}=(2pQ_1)^{1/2}K(\rho),$$
$$K(\rho)=2[\Delta q(\rho)^{1/2}(|\rho^2-1|)^{1/4}]^{-1}. \qquad (46)$$

Recognizing that the cosine in (45) also takes on equal values at both saddle points and that $b_2[\tau(\omega)]$ is a constant, since the saddle points essential for the calculation of the integral are far from those branch points of Ω near which the WKB approximation fails, we easily obtain

$$\mathscr{D}(p)\approx b_2[2Q_1(2p+\Delta)]^{1/2}K(\rho)\operatorname{Ai}(Q_1^2)\exp(ip\pi/2-iS/4). \qquad (47)$$

Here $\operatorname{Ai}(Q_1^2)$ is an Airy function. For Q_1 we get

$$^2/_3Q_1^3=p\operatorname{Arch}\rho-{}^1/_2\Delta(1+q^2)^{1/2}\{F(\psi_1,k)-E(\psi_1,k)\}$$
$$-p(\rho)^{-1}(\rho^2-1)^{1/2}, \qquad (48)$$
$$\psi_1=\arccos(\rho)^{-1}, \quad k=(1+q^2)^{-1/2}.$$

Here $F(\psi_1,k)$, $E(\psi_1,k)$ are incomplete elliptic integrals of the first and second kind, respectively. A similar estimate yields

$$B(p)\approx-ib_2[2Q_1^{-1}(2p-\Delta)]^{1/2}K(\rho)\operatorname{Ai}'(Q_1^2)$$
$$\times\exp(ip\pi/2-iS/4). \qquad (49)$$

The values of A and C are (exponentially) much smaller than those of B and $\mathscr{D}$, since at $\Delta\gg 1$ the respective integrands oscillate very strongly on the real axis and have no saddle points near this axis. Estimates of B and $\mathscr{D}$ obtained by a similar method for other arrangements of the saddle points [see (41), (42)] are given in the Appendix.

We have thus obtained for the satellite amplitudes a uniform asymptotic estimate valid as the parameter p varies over a wide range. At the points $p=1/2\Delta(1+q^2)^{1/2}$ and $p=\Delta/2$ the corresponding estimates on the left and right, naturally, coalesce. Expressions (47), (49), (A.1), and (A.2) yield an estimate in fields of any strength. On the other hand expressions (A.4), (A.5), (A.8), and (A.9) may be wrong in sufficiently strong fields, when the branch points of $\Omega(\tau)$, near which the WKB approximation fails, approach the real axis and "crawl" on the saddle points clamped between them. In this case the main contribution to the integrals in (36) and (38) is made by the regions near the points 0 and π. Since the dimensions of these regions are $\sim 1/\Delta q\ll 1$, to determine $X_{1,2}$ in this case we can use in Eqs. (1) an approximation linear in τ, analogous to the linear-terms approximation in the Landau-Zener theory [22], Taking the Fourier transforms of the resultant equations, we obtain equations for the quantities of direct interest to us:

$$X_{1,2}(p)=\frac{1}{2\pi}\int_0^{2\pi}X_{1,2}(\tau)\exp(ip\tau)\,d\tau, \qquad (50)$$

$$\frac{\Delta^2q^2}{4}\frac{d^2}{dp^2}X_{1,2}(p)+\left[p^2-\frac{\Delta^2}{4}\mp i\frac{\Delta q}{2}\right]X_{1,2}(p)=0. \qquad (51)$$

The general solution of Eqs. (51) is expressed in terms of parabolic-cylinder functions[24]:

$$X_1(p)=G_1D_{i\nu}(z)+G_2D_{i\nu-1}(-iz), \qquad (52)$$

$$X_2(p)=G_3D_{-i\nu-1}(z)+G_4D_{i\nu}(-iz), \qquad (53)$$

$$z=2p(\Delta q)^{-1/2}\exp\left(-i\frac{\pi}{4}\right), \quad \nu=\frac{\Delta}{4q}.$$

Only two of the four coefficients G_1, G_2, G_3, and G_4 are linearly independent, in view of the relation

$$i\frac{\Delta q}{2}\frac{d}{dp}X_1(p)+pX_1(p)=-\frac{\Delta}{2}X_2(p), \qquad (54)$$

obtainable from (1) by a linear approximation and a Fourier transformation. Choosing G_1 and G_2 to be the independent ones and using the relations

$$D_{n+1}(z)-zD_n(z)+nD_{n-1}(z)=0,$$
$$(d/dz)D_n(z)+{}^1/_2zD_n(z)-nD_{n-1}(z)=0,$$

we get

$$G_3=-(\nu)^{1/2}e^{-i\pi/4}G_1, \quad G_4=(\nu)^{-1/2}e^{-i\pi/4}G_2.$$

We obtain the coefficients G_1 and G_2 from the condition that $X_{1,2}(p)$ coincide with the corresponding quantities calculated below by the saddle-point method using the asymptotic expressions (16). This is possible, since the region $|\tau|\ll 1$ where the linear approximation is valid is considerably larger than the region $|\tau|\sim 1/\Delta q\ll 1$ where the WKB approximation fails and the asymptotic expression (16) cannot be used. Within the framework of the linear approximation, when $X_{1,2}$ are calculated with the aid of (16), the main contribution is made by the saddle points $\tau_\pm = \pm 2p/\Delta q$. We obtain

$$X_1=\left(\frac{2}{\pi\Delta q}\right)^{1/2}\left[\frac{\Delta}{4p}b_2(\tau_-)\exp\left(i\frac{\pi}{4}-i\eta\right)\right.\qquad(55)$$
$$\left.+b_2(\tau_+)\exp\left(-i\frac{\pi}{4}+i\eta\right)\right],$$

$$\eta=-\frac{\Delta}{2}\int_0^{\tau_+}d\tau(1+q^2\tau^2)^{1/2}+p\tau_+.$$

The coefficient $b_2(\tau_-)$ is expressed in terms of $b_1(\tau_+)$ and $b_2(\tau_+)$ with the aid of the matrix $\hat{F}_0^{-1}$, where $\hat{F}_0$ is a matrix similar to (20), from which it differs only in that the signs of the off-diagonal elements and of the corresponding factors $e^{\pm iS}$ are reversed. At $p\gg\Delta/2$ (i.e., $z^2/4\gg\nu$) but $p\ll 1/2\Delta(1+q^2)^{1/2}$, we identify (52) with (55) by using the asymptotic forms of the parabolic-cylinder functions.[24] We obtain

$$G_1=\pm\left(\frac{2}{\pi\Delta q}\right)^{1/2}\times\exp\left\{-i\left(\frac{\delta}{2\pi}-\frac{\delta}{2\pi}\ln\frac{\delta}{\pi}+\frac{\pi}{4}-\pi\varepsilon-\frac{S}{2}\right)\right\}e^{-3\delta/4}b_1,$$
$$G_2=\pm\left(\frac{2\nu}{\pi\Delta q}\right)^{1/2}\times\exp\left\{i\left(\frac{\delta}{2\pi}-\frac{\delta}{2\pi}\ln\frac{\delta}{\pi}+\frac{\pi}{2}+\pi\varepsilon-\frac{S}{2}\right)\right\}e^{-3\delta/4}b_2.\qquad(56)$$

The sought quantity Q_n^ε is thus equal to

$$Q_n^\varepsilon=1/2(1\mp\cos n\pi)[G_1D_{-i\nu}(z)+G_2D_{-i\nu-1}(-iz)-i(\nu)^{1/2}e^{i\pi/4}G_1D_{-i\nu-1}(z)+i(\nu)^{-1/2}e^{i\pi/4}G_2D_{i\nu}(-iz)].\qquad(57)$$

When the adiabatic approximation is valid, $\nu=\delta/\pi\gg 1$, Eqs. (47)–(49), (A.1)–(A.10), and (57), whose validity regions overlap considerably, provide an asymptotic estimate of the satellite intensities for fields of arbitrary strength.

DISTRIBUTION OF SATELLITE INTENSITIES AND QUALITATIVE STRUCTURE OF THE OPTICAL SPECTRA IN VARIOUS LIMITING CASES

We consider some limiting cases that follow from the expressions derived above.

At $p\sim 1/2\Delta(1+q^2)^{1/2}$ $(Q_{1,2}\lesssim 1)$ and $p\sim 1/2\Delta(Q_{3,4}\lesssim 1)$ Eqs. (47), (A.1), (A.4), and (A.8) describe the intensity distributions of the satellites that appear in the optical spectra of the system considered in the region of their maximum values. As p moves farther away from values corresponding to the optical-band maxima, the asymptotic expressions for Airy functions can be used. Thus, for example at $Q_{1,2}\gg 1$ and $p>1/2\Delta(1+q^2)^{1/2}$ we have

$$\mathscr{D}(p)\approx\left(\frac{2p+\Delta}{2\pi}\right)^{1/2}K(\rho)\exp\left(ip\frac{\pi}{2}-i\frac{S}{4}-\frac{2}{3}Q_1^3\right),\qquad(58)$$

and at $1/2\Delta<p<1/2\Delta(1+q^2)^{1/2}$

$$\mathscr{D}(p)\approx\left(\frac{2p+\Delta}{\pi}\right)^{1/2}\times K(\rho)\exp\left(ip\frac{\pi}{2}-i\frac{S}{4}\right)\sin\left(\frac{2}{3}Q_2^3+\frac{\pi}{4}\right).\qquad(59)$$

Similar estimates for the intensities of the satellites of a two-level system were obtained in Ref. 8. At $0<p<\Delta/2$ we get the estimate

$$\mathscr{D}(p)\approx\left(\frac{2p+\Delta}{2\pi}\right)^{1/2}K(\rho)\exp\left(-\frac{2}{3}Q_4^3\right).\qquad(60)$$

Expressions such as (58) or (60) go over to the perturbation-theory result as $q\to 0$. For example, using at $q\ll 1$ the asymptotic values of the elliptic integrals contained in the definition (48), we readily obtain from (58)

$$|\mathscr{D}(p)|\approx\frac{1}{(2\pi)^{1/2}}\left(\frac{2p+\Delta}{p^2-\Delta^2/4}\right)^{1/2}\times\left[\frac{eq}{4}\left(\frac{2p+\Delta}{2p-\Delta}\right)^{1/2}\right]^{p-\Delta/2}\left(\frac{2\Delta}{2p+\Delta}\right)^{p}.\qquad(61)$$

Thus, as $q\to 0$ expression (61) yields the sought power-law dependence of the satellite intensity on the field strength. The pulsating dependence of these intensities on q and p at $1/2\Delta<p<1/2\Delta(1+q^2)^{1/2}$, which follows from (59), cannot be obtained by perturbation theory.

The physical cause of the pulsations is the following. Consider the instantaneous arrangement of the system levels in the adiabatic approximation (see Fig. 3). In accord with the foregoing, the system radiates at the frequencies $f=|E_0|/\hbar+(\varepsilon+n)\omega$ (n is an integer) shown by the horizontal lines in Fig. 3. If the swing $(1/2\Delta[(1+q^2)^{1/2}-1]\hbar\omega)$ of the instantaneous position of the adiabatic energy greatly exceeds $\hbar\omega$, emission at the frequency of a satellite with a specified number n takes place mainly at those instants of time when the instantaneous position of the adiabatic-energy level coincides with the satellite position, as marked by the intersection points in Fig. 3, which coincide with the saddle points. Since this situation takes place twice during the half-cycle of the field for the considered range of p $(1/2\Delta<p<1/2\Delta(1+q^2)^{1/2})$, the corresponding contributions to the emission intensity interfere. The phase factor responsible for this interference is, obviously,

$$\exp\left\{i\left[\frac{\Delta}{2}\int_{\tau_1}^{\tau_2}d\tau(1+q^2\sin^2\tau)^{1/2}-p(\tau_2-\tau_1)\right]\right\},$$

and this leads directly to pulsating dependences of the type (59). The contribution from each region near the points τ_1

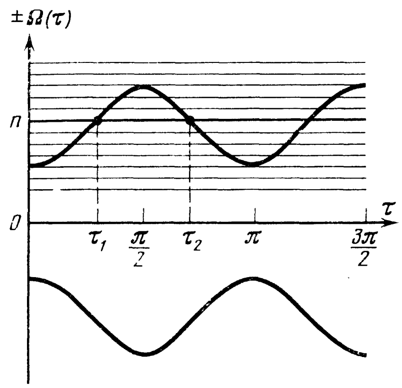

FIG. 3. Time dependence of the adiabatic level of the system. The horizontal lines show the positions of the satellites.

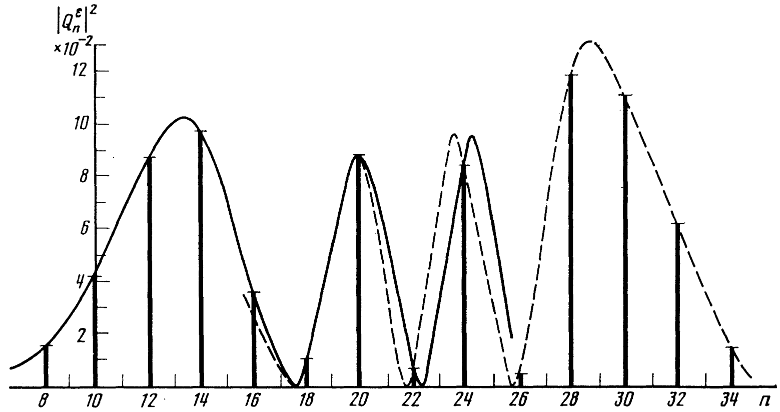

FIG. 4. Satellite intensities ($\Delta = 20$, $q = 3$, $\varepsilon = 0.242$, $n > 0$). The figure shows the envelopes (solid and dashed curves) calculated from relations (47), (A.1) and (A.4), (A.8), respectively.

and τ_2 is proportional to the time that the system stays in the vicinity of these points. Since the system stays longest near the adiabatic-energy extrema $1/2\Delta\hbar\omega$ and $1/2\,\Delta(1+q^2)^{1/2}\hbar\omega$, maximum emission intensities should be observed at the frequencies of the satellites corresponding to these energies, in agreement with the results above. At $p < 1/2\Delta$ and $p > 1/2\Delta(1+q^2)^{1/2}$, the instantaneous adiabatic energy coincides nowhere with the position of a satellite, and this causes the intensities of the corresponding satellites to be exponentially small.

We note that the foregoing qualitative picture of the spectrum is deduced essentially only from the periodicity and smoothness of the perturbation, so that the mechanism for the onset of a pulsating and exponentially decreasing dependence of the satellites on n and q is universal for arbitrary (including degenerate) two-level systems in strong periodic fields of arbitrary form.

Figure 4 shows for $p > 0$ the numerically calculated $|Q_n^\varepsilon|^2$, which is in full agreement with the qualitative picture described above. The intensities of the satellites near the right-hand and left-hand maxima were calculated from (47), (A.1) and from (A.4), (A.8) respectively. In the central part of the band, at the chosen parameters, the results of different formulas, which have overlapping validity ranges, agree within not worse than a few percent.

We consider now the asymptotic forms of the obtained expressions in very strong ($q \to \infty$) fields. In the region of the maximum at $p \sim 1/2\Delta(1+q^2)^{1/2} \approx \Delta q/2$ we easily obtain with the aid of (47) and (A.1)

$$|Q_n^\varepsilon|^2 \approx 2^{4/3}(\Delta q)^{-2/3}\mathrm{Ai}^2[2^{2/3}(\Delta q)^{-1/3}(p-\Delta q/2)]. \tag{62}$$

As expected, (62) agrees with the asymptotic form ($n \sim \Delta q/2 \gg 1$) of the exact solutions $|Q_n^\varepsilon|^2 \sim J_n^2(\Delta q/2)$ (where $J_n(x)$ is a Bessel function) for the satellite intensities of a doubly degenerate level in a strong external field.[28–30] At $p \sim \Delta/2$ in sufficiently strong fields it is necessary to use Eq. (57), from which follows as $q \to \infty$ (i.e., at $\nu = \Delta/4q \ll 1$) that

$$|Q_n^\varepsilon|^2 = \frac{1}{\pi\Delta q}(1\mp\cos n\pi)^2\cos^2\left(\frac{\Delta q}{2}+\frac{n\pi}{2}-\frac{\pi}{4}\right), \tag{63}$$

which also agrees, naturally, with the asymptotic form ($\Delta q/2 \gg n$, $\Delta q/2 \gg 1$) of the exact solution for the case of a degenerate level.

The results of this paper provide thus a rather complete description of the quasienergetic states and optical spectra of two-level quantum system in low-frequency fields of arbitrary strength. When using these results one must, however, recognize that the separation of the two-level system from the complete spectrum is itself valid only so long as the matrix elements of the interaction between the separated pair and the other levels of the atoms are substantially smaller than the energy distances to these levels. Thus, in cited example of the He I line ($\lambda = 4922$ Å) the separation of the level pair 4^1D, 4^1F is based on the fact that the energy distance to level 4^1P closest to them exceeds $\hbar\omega_0$ by an order of magnitude. In fields on the order of several times ten kV/cm and higher, however, it becomes important to take the 4^1P level into account[31] and the two-level approximation no longer holds.

We note in conclusion that the results above can be used also to investigate the behavior of two-level systems in quasiresonant bichromatic fields, as well as to consider multiphoton interband transitions in crystals.

The authors thank V. A. Kovarskiĭ and V. N. Kraĭnov for interest in the work and for a valuable discussion of its results.

APPENDIX

The case $1/2\Delta < p < 1/2\Delta(1+q^2)^{1/2}$. At

$$\left|\frac{\Delta}{2}\int_0^{\arcsin\rho} d\tau\,(1+q^2\sin^2\tau)^{1/2} - p\arcsin\rho\right| \gg 1$$

we have

$$\mathscr{D}(p) \approx b_2[2Q_2(2p+\Delta)]^{1/2}K(\rho)\exp\left(ip\frac{\pi}{2}-i\frac{S}{4}\right)\mathrm{Ai}(-Q_2^2), \tag{A.1}$$

$$B(p) \approx -ib_2[2Q_2^{-1}(2p-\Delta)]^{1/2}K(\rho) \times \exp\left(ip\frac{\pi}{2}-i\frac{S}{4}\right)\mathrm{Ai}'(-Q_2^2). \tag{A.2}$$

Here

$$\frac{2}{3}Q_2^3 = \frac{\Delta}{2}\int_0^{\psi_2} dz\,(1+q^2\cos^2 z)^{1/2} - p\psi_2 = \frac{\Delta}{2}(1+q^2)^{1/2}E(\psi_2, k') - p\psi_2,$$

$$\psi_2 = \arccos\rho, \quad k' = q(1+q^2)^{-1/2}. \tag{A.3}$$

If, however,

$$\left| \frac{\Delta}{2} \int\limits_{\arcsin\rho}^{\pi/2} d\tau (1+q^2\sin^2\tau)^{1/2} - p\left(\frac{\pi}{2} - \arcsin\rho\right) \right| \gg 1,$$

we have

$$\mathscr{D}(p) \approx b_2 [2Q_3(2p+\Delta)]^{1/2} K(\rho) \,\mathrm{Ai}(-Q_3^2), \tag{A.4}$$

$$B(p) \approx -ib_2 [2Q_3^{-1}(2p-\Delta)]^{1/2} K(\rho) \,\mathrm{Ai}'(-Q_3^2), \tag{A.5}$$

where

$$\frac{2}{3} Q_3^3 = -\frac{\Delta}{2} \int\limits_0^{\arcsin\rho} d\tau (1+q^2\sin^2\tau)^{1/2} + p\arcsin\rho$$

$$= -\frac{\Delta}{2}(1+q^2)^{1/2} E(\psi_3, k') + p\arcsin\rho + \frac{1}{p}[K(\rho)]^2,$$

$$\psi_3 = \arcsin\frac{1}{pq}\left[\left(p^2 - \frac{\Delta^2}{4}\right)(1+q^2)\right]^{1/2}. \tag{A.6}$$

The regions where (A.1), (A.2) and (A.4), (A.5) are valid overlap considerably. The results of these equations agree when the saddle points can be considered separately ($Q_2 \gg 1$, $Q_3 \gg 1$).[8]

The case $0 < p < 1/2\Delta$. At

$$\left| \frac{\Delta}{2} \int\limits_{\mathrm{Arsh}\,\rho}^{\mathrm{Arsh}(q^{-1})} dz (1-q^2\,\mathrm{sh}^2 z)^{1/2} - p[\mathrm{Arsh}\,\rho - \mathrm{Arsh}(q^{-1})] \right| \gg 1, \tag{A.7}$$

we have

$$\mathscr{D}(p) \sim b_2 [2Q_4(2p+\Delta)]^{1/2} K(\rho) \,\mathrm{Ai}(Q_4^2), \tag{A.8}$$

$$B(p) \sim b_2 [2Q_4^{-1}(\Delta-2p)]^{1/2} K(\rho) \,\mathrm{Ai}'(Q_4^2), \tag{A.9}$$

$$\frac{2}{3} Q_4^3 = \frac{\Delta}{2} \int\limits_0^{\mathrm{Arsh}\,\rho} dz (1-q^2\,\mathrm{sh}^2 z)^{1/2} - p\,\mathrm{Arsh}\,\rho$$

$$= \frac{\Delta}{2}(1+q^2)^{1/2}[F(\psi_4, k) - E(\psi_4, k)] - p\,\mathrm{Arsh}\,\rho$$

$$+ p\left[\frac{\Delta^2/4 - p^2}{1/4\Delta^2(1+q^2) - p^2}\right]^{1/2}, \tag{A.10}$$

$$\psi_4 = \arcsin\left[\frac{(\Delta^2/4 - p^2)(1+q^2)}{1/4\Delta^2(1+q^2) - p^2}\right]^{1/2}.$$

At $p < 0$ similar estimates hold also for A and C. The values of B and $\mathscr{D}$ are in this case negligibly small compared with A and C.

[1]Ya. B. Zel'dovich, Usp. Fiz. Nauk **110**, 139 (1973) [Sov. Phys. Usp. **16**, 427 (1973)].
[2]N. B. Delone and V. N. Kraĭnov, Atom v sil'nom svetovom pole (Atom in a Strong Light Field), Atomizdat, 1978.
[3]J. H. Shirley, Phys. Rev. **B138**, 979 (1965).
[4]D. F. Zaretskiĭ and V. P. Kraĭnov, Zh. Eksp. Teor. Fiz. **66**, 537 (1974) [Sov. Phys. JETP **39**, 257 (1974)].
[5]V. P. Kraĭnov, *ibid.* **70**, 1197 (1976) [**43**, 622 (1976)].
[6]V. P. Kraĭnov and V. P. Yakovlev, *ibid.* **78**, 2204 (1980) [**51**, 1104 (1980)].
[7]V. A. Kovarskiĭ and S. A Baranov, *ibid.* **71**, 2033 (1976) [**44**, 1067 (1976)].
[8]N. F. Perel'man, *ibid.* **68**, 1640 (1975) [**41**, 822 (1975)].
[9]M. P. Brizhenev, V. P. Gavrilenko, S. V. Egorov, B. G. Eremin, A. V. Kostrov, E. A. Oks, and Yu. M. Shagiev, *ibid.* **85**, 893 (1983) [**58**, 517 (1983)].
[10]O. A. Melikyan, Optica Acta **28**, 997 (1983).
[11]J. V. Maloney and W. J. Meath, Mol. Phys. **31**, 1537 (1976).
[12]W. W. Hicks, R. A. Hess, and W. S. Cooper, Phys. Rev. **A5**, 490 (1972).
[13]A. G. Fainshtein, N. L. Manakov, and L. P. Rapoport, J. Phys. **B11**, 2561 (1978).
[14]H. R. Griem, Spectral Line Broadening by Plasmas, Academic, 1974.
[15]E. I. Kuznetsov and D. A. Shcheglov, Metody diagonstiki vyskokotemperaturnoĭ plazmy (Methods of High-Temperature-Plasma Diagnostics), Atomizdat, 1980.
[16]A. V. Vinogradov, I. Yu. Skobelev, and E. A. Yukov, Usp. Fiz. Nauk **129**, 177 (1979) [Sov. Phys. Usp. **22**, 771 (1979).
[17]M. Baranger and B. Mozer, Phys. Rev. **123**, 25 (1961).
[18]N. J. Wiegart, Phys. Rev. **A27**, 2114 (1974).
[19]V. S. Strel'nitskiĭ, Sov. Phys. Usp. **113**, 463 (1974) [Sov. Phys. Usp. **17**, 507 (1975).
[20]M. Elitzur, Rev. Mod. Phys. **54**, 1225 (1982).
[21]P. A. Braun and G. P. Miroshnichenko, Opt. Spektrosk. **49**, 1024 (1980)].
[22]E. E. Nikitin and S. Ya. Umanskiĭ, Neadiabaticheskie perekhody pri medlennykh atomnykh stolknoveniyakh (Nonadiabatic Transitions in Slow Atomic Collisions), Atomizdat, 1979.
[23]V. A. Kovarskiĭ, N. F. Perel'man, I. Sh. Averbukh, S. A. Baranov, and S. S. Todirashku, Neadiabaticheskie perekhody v sil'nom elektromagnitnom pole (Nonadiabatic Transitions in a Strong Electromagnetic Field). Kishinev: Shtinitsa, 1980.
[24]I. S. Gradshtein and I. M. Ryzhik, Tables of Integrals, Sums, Series, and Products, Academic, 1965.
[25]M. Abramowitz and I. A. Stegun, eds., Handbook of Mathematical Functions, Dover, 1964.
[26]V. M. Dubovik, V. D. Popov, and V. P. Yakovlev in: Interaction of Laser Radiation with Resonant Media [in Russian], Energoizdat, 1982, p. 108.
[27]M. V. Fedoruk, Metod perevala (The Saddle-Point Method), Nauka, 1977.
[28]D. I. Blochinzev, Phys. Zs. Sow. Union **4**, 501 (1933).
[29]L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47**, 1945 (1964) [Sov. Phys. JETP **20**, 1307 (1946)].
[30]V. A. Kovarskiĭ, Mnogokvantovye perekhody (Multiquantum Transitions), Kishinev; Shtinitsa, 1974.
[31]H. A. Bethe and E. E. Salpeter, Quantum Mechanics of One- and Two-Electron Systems, Springer, 1958.

Translated by J. G. Adashko