Potential scattering of electrons in the presence of an electromagnetic wave

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The sufficient conditions for the generalization of the known Bunkin-Fedorov equation beyond the scope of the Born approximation are considered. These conditions are obtained directly by calculating the scattering amplitude in the second Born approximation. An eikonal approximation is developed for scattering in a wave field. The amplitude obtained satisfies the optical theorem and permits solving the problem of scattering by two centers in a bichromatic field. Scattering by a truncated Coulomb potential is specifically considered.

INTRODUCTION

The problem of potential scattering in the presence of a strong electromagnetic wave arises when multiphoton stimulated bremsstrahlung (SB) is considered in the analysis of the heating of the electron component of a weakly ionized plasma by absorption of laser radiation.[1] Interest attaches to the dependence of the cross section of the process on the intensity **F**, frequency ω, and polarization vector λ of the wave.

To solve this problem, use was made of the Born approximation in terms of the scattering potential with exact wave functions of the electron in the wave,[1] and of a low-frequency approximation.[2] The differential cross section for scattering with emission (absorption) of s photons is of e form

$$\frac{d\sigma}{d\Omega}(\mathbf{q}_i,\mathbf{q}_f)=\frac{q_f}{q_i}J_s^2(X_{fi})\left(\frac{d\sigma}{d\Omega}\right)_0,$$
$$X_{fi}=\alpha\mathbf{q}_s=e\mathbf{F}\mathbf{q}_s/m_e\omega^2,\quad \mathbf{q}_s=\mathbf{q}_f-\mathbf{q}_i,\quad \varepsilon_f=\varepsilon_i-s\hbar\omega. \tag{I.1}$$

Here $(d\sigma/d\Omega)_0$ is the differential cross section for scattering by a potential in the absence of a field, and J_s is a Bessel function. Equation (I.1) is exact in the Born approximation, while the attempts made in Ref. 2 to extend the region of validity of (I.1) are not convincing.

It is shown in the present paper that the condition $\hbar\omega/\varepsilon_i\ll1$ for validity of the low-temperature approximation is necessary but not sufficient to be able to use (I.1) beyond the validity region of the Born approximation. We note that the method used in Refs. 3 and 4 to derive Eq. (I.1), wherein the field is taken into account exactly in some parts of the equation and by perturbation theory in others, is untenable. The question of finding for (I.1) a validity region other than that for the Born approximation will be considered in Sec. 1.

It follows from the analysis that for (I.1) to be valid in the second Born approximation it suffices to require satisfaction of the conditions

$$\omega\tau\ll1,\quad s\gamma=(s/s_{max})(v_F/v)\ll1,\quad eF\tau^2/2m_e\ll R, \tag{I.2}$$

where

$$\gamma=v_F/s_{max}v,\quad v_F=eF/m_e\omega,$$
$$s_{max}\sim(eF/m_e\omega^2)2q_i\sin(\theta/2),$$

R is the effective radius of the potential, τ is the characteristic time determined by the form of the potential and by the scattering condition, and θ is the scattering angle. At $s\ll s_{max}$ the restrictions on the field intensity and on the frequency are independent. The condition on the frequency is equivalent to a requirement that the collision be instantaneous, and the restriction on the field intensity means smallness of the displacement produced by the field during the characteristic scattering time τ compared with the effective radius of the potential.

In the case $s\sim s_{max}$ the conditions take the form

$$\omega\tau\ll1,\quad eF\tau^2/2m_e\ll R,$$
$$v_F=eF/m_e\omega\ll R/\tau=v. \tag{I.2'}$$

It follows from (I.2') that in this case there can not be an arbitrarily low frequency. The reason is the need to take into account the quantum character of the SB, due to absorption (emission) of a fixed number of photons of the order of s_{max}.

If the conditions for the validity of the Born approximation are violated, but the electron energy is high enough, the eikonal approximation can be used. A formal solution for this case is given in Ref. 5, but this result is difficult to use directly (see Sec. 2). In Sec. 2 we obtain another expression for the scattering amplitude, in which a limiting transition is possible to known approximations, and these can be compared and analyzed (Sec. 3). We note that in the case of scattering in a field the eikonal approximation does not reduce to the first Born approximation.

In Sec. 4 we consider in detail the case of a truncated Coulomb potential. The scattering amplitudes obtained in Secs. 1 and 2 satisfy the optical theorem, and permit also the solution to be generalized to include scattering, in the presence of a laser wave, by two scattering centers and to include scattering in a bichromatic field. This is done in the eikonal approximation in Sec. 5.

1. POTENTIAL SCATTERING IN A POTENTIAL FIELD

The Schrödinger equation for the scattering of an electron $-e$ by a potential $U(\mathbf{r})$ in the presence of a monochromatic field with a Hamiltonian H_F is of the form

$$\left[\frac{1}{2m_e}\hat{\mathbf{p}}^2+\frac{e}{m_ec}\hat{\mathbf{A}}\hat{\mathbf{p}}+U(\mathbf{r})+H_F-E\right]\Psi=0. \tag{1.1}$$

The field is described in the second-quantization representation

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$$H_F=\hbar\omega(a^+a+{}^1/_2),\quad H_F|n\rangle=\hbar\omega(n+{}^1/_2)|n\rangle,$$

where a and a^+ are the operators of creation and annihilation of photons with a given frequency ω and with a polarization vector λ. The vector-potential operator is

$$\hat{\mathbf{A}}=\left(\frac{2\pi\hbar c^2}{V\omega}\right)^{1/2}(\lambda a+\lambda^* a^+),$$

V is the normalization volume, and $|n\rangle$ is the state vector of a field with n phonons.

It is known that at $U=0$ Eq. (1.1) has an exact solution that can be written in the form

$$\Psi_{\mathbf{q},n}(\mathbf{r})=(2\pi)^{-3/2}\exp(i\mathbf{qr}-|\rho_q|^2/2)\hat{D}(\rho_q)|n\rangle,$$
$$\hat{D}(\rho)=\exp(-\rho^* a+\rho a^+),\quad \rho_q=-(\lambda^*\mathbf{q})\frac{e}{m_e\omega^2}\left(\frac{2\pi\hbar\omega}{V}\right)^{1/2}. \tag{1.2}$$

The operator $\hat{D}(\rho)$ is the shift operator and its properties are known (Ref. 6, Chaps. I and VI).

The states of an electron in a quantized field are characterized by a momentum $\hbar\mathbf{q}$. The subscript n means that the field had contained n photons in the absence of the interaction. It is easy to prove that the functions (1.2) are orthonormal:

$$\int d\mathbf{r}\Psi^*_{\mathbf{q},n}(\mathbf{r})\Psi_{\mathbf{q}',n'}(\mathbf{r})=\delta(\mathbf{q}'-\mathbf{q})\delta_{nn'} \tag{1.3}$$

and constitute a complete system, as follows from the equality

$$\sum_n\int d\mathbf{q}\Psi^*_{\mathbf{q},n}(\mathbf{r})\Psi_{\mathbf{q},n}(\mathbf{r}')=\delta(\mathbf{r}'-\mathbf{r}). \tag{1.4}$$

Equation (1.1) is the stationary Schrödinger equation, therefore continuous-spectrum problems can be analyzed by using the formalism of the stationary theory of scattering. The amplitude of the transition from the state $|i\rangle$, characterized by the numbers $(\mathbf{q}_i,n)$ into a state $|f\rangle$ with $(\mathbf{q}_f,n+s)$, i.e., the scattering process $\mathbf{q}_i\to\mathbf{q}_f$ with absorption or emission of s electromagnetic-field quanta, is written in the form

$$f_s(\mathbf{q}_i,\mathbf{q}_f)=-\frac{(2\pi)^3}{4\pi}\frac{2m_e}{\hbar^2}\langle f|\hat{T}(E_i+i0)|i\rangle, \tag{1.5}$$

where

$$\hat{T}(E+i0)=U+U(E-H+i0)^{-1}U,$$
$$E_i=\varepsilon_i+n\hbar\omega+\Delta\varepsilon=\varepsilon_{in}=\hbar^2q_f^2/2m_e+(n+s)\hbar\omega+\Delta\varepsilon,$$
$$\Delta\varepsilon=-\frac{2\pi\hbar\omega}{V}\frac{e^2|\mathbf{q}\lambda^*|^2}{m_e^2\omega^4}\hbar\omega+\frac{\hbar\omega}{2},$$

and H is the total Hamiltonian from (1.1).

We consider first the Born approximation, and analyze next the general case. The scattering amplitude in the first Born approximation is defined by the expression

$$f_s^{(1)}(\mathbf{q}_i,\mathbf{q}_f)=-\frac{m_e}{2\pi\hbar^2}\int d\mathbf{r}\exp\{-i(\mathbf{q}_f-\mathbf{q}_i)\mathbf{r}\}U(\mathbf{r})M^n_{n+s},$$
$$M^n_{n+s}=\langle n+s|\hat{D}^+(\rho_f)\hat{D}(\rho_i)|n\rangle \tag{1.6}$$
$$=\left[\frac{(n+s)!}{n!}\right]^{1/2}e^{-\nu/2}\exp\left(i\gamma s+\frac{\rho_i\rho_f^*-\rho_f\rho_i^*}{2}\right)$$
$$\times(-\nu^{1/2})^{-s}L^{-s}_{n+s}(\nu),$$

where $\nu^{1/2}e^{i\gamma}=\rho_i-\rho_f$, $\rho_i\equiv\rho_{q_i}$, $\rho_f\equiv\rho_{q_f}$.

In the derivation we used the known expression for the shift operator and its matrix element (Ref. 6, Chap. VI, as well as the Appendix). In a quantized field, the SB is determined by Laguerre polynomials. In the classical-field limit ($n\to\infty$, $V\to\infty$, n/V finite), using the asymptotic expression for the Laguerre polynomial

$$L_n^\alpha(z)e^{-z/2}z^{\alpha/2}\approx J_\alpha[2(nz)^{1/2}][(n+\alpha)!/n!]^{1/2}. \tag{1.7}$$

Eq. (1.6) is reduced to the known Bunkin-Fedorov formula[1]:

$$|f_s^{(1)}(\mathbf{q}_i,\mathbf{q}_f)|^2=|f_0^B(\mathbf{q}_i,\mathbf{q}_f)|^2J_s^2(X_{fi}),$$
$$X_{fi}=(\mathbf{q}_f-\mathbf{q}_i)\lambda\frac{e}{m_e\omega^2}\left(\frac{8\pi n\hbar\omega}{V}\right)^{1/2}=(\mathbf{q}_f-\mathbf{q}_i)\frac{e\mathbf{F}}{m_e\omega^2}, \tag{1.8}$$

where $\mathbf{F}$ is the field-strength vector. We obtain for the differential cross section the expression (I.1), in which $(d\sigma/d\Omega)_0$ is determined in the Born approximation.

We consider now the scattering amplitude in the second Born approximation.

$$f_s^{(2)}(\mathbf{q}_i,\mathbf{q}_f)=-\frac{m_e}{2\pi\hbar^2}(2\pi)^3\langle f|UG_0U|i\rangle, \tag{1.9}$$

where G_0 is defined by the spectral representation

$$G_0(\mathbf{r},\mathbf{r}';E)=\sum_m\int\frac{d\mathbf{q}}{(2\pi)^3}\exp\{-|\rho_q|^2+i\mathbf{q}(\mathbf{r}-\mathbf{r}')\}$$
$$\times\frac{\hat{D}(\rho_q)|m\rangle\langle m|\hat{D}^+(\rho_q)}{E-\varepsilon_{mq}+i0}. \tag{1.10}$$

Substituting (1.10) in (1.9), summing over m, and going to the limit of the classical field we obtained (the calculation details are given in the Appendix)

$$f_s^{(2)}(\mathbf{q}_i,\mathbf{q}_f)$$
$$=-\frac{m_e}{2\pi\hbar^2}\sum_{k=-\infty}^{\infty}\int\frac{d\mathbf{q}}{(2\pi)^3}\frac{U(\mathbf{q}_f-\mathbf{q})J_k(X_{fq})U(\mathbf{q}-\mathbf{q}_i)J_{s+k}(X_{iq})}{\varepsilon_i-\varepsilon_q+k\hbar\omega+i0}. \tag{1.11}$$

The interpretation of the result is the following: the incident electron, with momentum $\hbar\mathbf{q}_i$ acquires as a result of the first interaction with the potential a momentum $\hbar\mathbf{q}$; in this case an $(s+k)$-photon SB is possible due to the momentum transfer $\hbar(\mathbf{q}-\mathbf{q}_i)$, and is described by a Bessel function. This is followed by free motion, with this momentum, until a second interaction causes the electron to acquire the final momentum $\hbar\mathbf{q}_f$; this is accompanied by a k-photon SB due to the momentum change $\hbar(\mathbf{q}_f-\mathbf{q})$. Integration with respect to $\mathbf{q}$ and summation over k mean that in the interval between the interactions the electron can have an arbitrary momentum, and when momentum is transferred an arbitrary k-photon SB is possible.

The scattering amplitude with allowance for the second Born approximation satisfies the optical theorem: the total cross section is determined by the imaginary part of the scattering amplitude through zero angle

$$\operatorname{Im}f_0(0)=(q_i/4\pi)\sigma_t. \tag{1.12}$$

The result follows from (1.11) and (1.8):

$$\mathrm{Im}\, f_0(0)=\mathrm{Im}\, f_0^{(2)}(0)=-\frac{m_e^2}{\pi\hbar^2}\int\frac{d\Omega_{\mathbf{q}}}{(2\pi)^3}$$

$$\times\sum_{k=-\infty}^{\infty}|U(\mathbf{q}_k-\mathbf{q}_i)|^2 J_k^2(X_{qi})\left(-\frac{\pi q_k}{2}\right)\Bigg|_{q_k=(q_i^2+2m_e\hbar^{-1}\omega k)^{1/2}}$$

$$=\frac{q_i}{4\pi}\int d\Omega_{\mathbf{q}}\sum_{k=-\infty}^{\infty}\frac{q_k}{q_i}|f_k^{(1)}(\mathbf{q}_i,\mathbf{q}_k)|^2=\frac{q_i}{4\pi}\sigma_t.$$

We consider now those approximations that enable us to obtain Eq. (I.1) in the second Born approximation. We note here that in (1.11) the integrand is singular. It is more convenient to start with Eq. (A.2) of the Appendix, in which there is no singularity:

$$f_s^{(2)}=-\frac{m_e}{2\pi\hbar^2}\int\frac{d\mathbf{q}}{(2\pi)^3}U(\mathbf{q}_f-\mathbf{q})U(\mathbf{q}-\mathbf{q}_i)\left(-\frac{i}{\hbar}\right)\int_0^\infty dt$$
$$\times\exp\left\{\frac{i}{\hbar}(\varepsilon_i-\varepsilon_q+i0)t\right\}e^{i\gamma(t)s}J_s(f^{1/2}(t)),\qquad(1.13)$$

where

$$(f(t))^{1/2}e^{i\gamma(t)}=\frac{e\mathbf{F}}{m_e\omega^2}[\mathbf{q}-\mathbf{q}_f+e^{-i\omega t}(\mathbf{q}_i-\mathbf{q})].$$

The characteristic interval of integration with respect to t in (1.13) is estimated at

$$\bar{\varepsilon}\sim\hbar^2/2m_eR^2,\quad q_iR\ll1;$$
$$\bar{\varepsilon}\sim\hbar v/R,\quad q_iR\gg1.$$

The argument of the Bessel function determines the maximum possible number of absorbed or emitted photons:

$$s_{max}\sim(eF/m_e\omega^2)2q_i\sin(\theta/2),\quad q_iR\ll1,$$
$$s_{max}\sim(eF/m_e\omega^2)R^{-1},\quad q_iR\gg1,$$

expanding $(f(t))^{1/2}$ and $\gamma(t)$ in powers of the small parameter $\omega\tau\ll1$ we can verify that to go from (1.13) to (I.1) it suffices to require satisfaction of the conditions

$$\omega\tau\ll1,\quad s\gamma\ll\bar{\varepsilon}\tau/\hbar=1,\quad EF\tau^2/m_e\ll R.\qquad(\mathrm{I.2})$$

The conditions (I.2) were analyzed in the Introduction. It is of interest to compare the conditions for the generalization of the Bunkin-Fedorov equation with the conditions for validity of perturbation theory in the field, the latter conditions defined by the inequalities

$$(eF/m_e\omega^2)2q_i\sin(\theta/2)\ll1,\quad q_iR\ll1,$$
$$(eF/m_e\omega^2)R^{-1}\ll1,\quad q_iR\gg1.\qquad(1.14)$$

The conditions (1.14) are more stringent than (I.2). In the case of scattering by atomic potentials, perturbation theory cannot be used at $F\sim\omega^2$, whereas the condition (I.2) is violated at $F\sim\omega v$, i.e., the Bunkin-Fedorov formula can be generalized at a field intensity $\omega^2<F<\omega v$ outside the framework of the validity of perturbation theory in the potential (all the quantities here are in atomic units).

Let us analyze now the general case. The Green's function of the Hamiltonian H from (1.1) can be sought in the form of a generalized expansion:

$$G(\mathbf{r},\mathbf{r}';E)=\sum_{m,m'}\int d\mathbf{q}\,d\mathbf{q}'\Psi_{\mathbf{q},m}(\mathbf{r})g_{mm'}(\mathbf{q},\mathbf{q}';E)\Psi^*_{\mathbf{q}',m'}(\mathbf{r}').\qquad(1.15)$$

The Green's function satisfies the equation

$$(E-H)G(\mathbf{r},\mathbf{r}';E)=\delta(\mathbf{r}-\mathbf{r}').\qquad(1.16)$$

Substituting (1.15) in (1.16) and using the orthonormality of the functions $\Psi_{\mathbf{q},n}$, we can obtain the following system of equations for the functions $g_{m,m'}$:

$$(E-\varepsilon_{mq})g_{mm'}(\mathbf{q},\mathbf{q}';E)-\sum_{m''=0}^{\infty}\int d\mathbf{q}''U(\mathbf{q}-\mathbf{q}'')$$
$$\times g_{m''m'}(\mathbf{q}'',\mathbf{q}';E)A_{mm''}(\mathbf{q},\mathbf{q}'')=\delta(\mathbf{q}-\mathbf{q}')\delta_{mm'},$$
$$A_{mm''}(\mathbf{q},\mathbf{q}'')=\langle m|\hat{D}^+(\rho_q)\hat{D}(\rho_{q''})|m''\rangle.\qquad(1.17)$$

Note that in a formal integration of the equations with respect to the potential U when the conditions (I.2) are satisfied we can neglect the virtual SB due to momentum transfer in multiple scattering in any nth order of the expansion. This enables us to sum in (1.17) over m'', assuming completeness of the system. In other words, in place of the matrix equation in $g_{mm'}$ we have a system of unrelated equations:

$$[E-\hbar^2q^2/2m_e-(m+1/2)\hbar\omega]g_{mm'}(\mathbf{q},\mathbf{q}';E)$$
$$-\int d\mathbf{q}''U(\mathbf{q}-\mathbf{q}'')g_{mm'}(\mathbf{q}'',\mathbf{q}';E)=\delta(\mathbf{q}-\mathbf{q}')\delta_{mm'}.$$

The solution can be expressed in terms of the Green's function $g(\mathbf{q},\mathbf{q}';E)$ of an electron in a potential U in the absence of a field:

$$g_{mm'}(\mathbf{q},\mathbf{q}';E)=\delta_{mm'}g_m(\mathbf{q},\mathbf{q}';E)$$
$$=\delta_{mm'}g(\mathbf{q},\mathbf{q}';E-(m+1/2)\hbar\omega).$$

Substituting this result in (1.5), and carrying out calculations similar to those in the Appendix, we can obtain in the classical-field limit the following amplitude:

$$f_s(\mathbf{q}_i,\mathbf{q}_f)=J_s(X_{fi})\langle\mathbf{q}_f|t(\varepsilon_i+i0)|\mathbf{q}_i\rangle,\qquad(1.18)$$

where $t(E+i0)$ is the matrix for scattering by the potential U in the absence of a field. For the differential cross section we obtain expression (I.1). Note that "field ejection in the intermediate states"[3,4] is neglect of the virtual SB due to virtual momentum transfer in scattering.

Let us summarize the conditions whose satisfaction is sufficient for (I.1) to be valid in the general case:

1) conditions (I.2), which restricts the possible frequency and field strength at the specified scattering conditions, must be satisfied;

2) the field-free scattering amplitude can be represented as a convergent Born series in powers of the potential.

2. SCATTERING AMPLITUDE IN THE EIKONAL APPROXIMATION

The starting point in the determination of the scattering amplitude in the eikonal approximation is the Schrödinger equation (1.1). Since practical interest attaches to the oscillator-field states with large occupation numbers, it is convenient to use for the operators a and a^+ the representation[7]

$$a^+a=N+i^{-1}\frac{\partial}{\partial\varphi},\quad a^+\approx e^{i\varphi}N^{1/2},\quad a\approx e^{-i\varphi}N^{1/2},\quad 0\leqslant\varphi\leqslant2\pi,$$

which is valid if the number N of photons is large; N is here a

c-number. In this representation, Eq. (1.1) takes the form

$$\left[-\frac{\hbar^2}{2m_e}\Delta+U(\mathbf{r})-\frac{ie\hbar(\mathbf{F}\nabla)}{m_e\omega}\cos\varphi+\frac{\hbar\omega}{i}\frac{\partial}{\partial\varphi}-E\right]\Psi=0. \qquad (2.1)$$

The scalar product for the functions $\Psi(\mathbf{r},\varphi)$ is defined as follows:

$$(\Psi_1,\Psi_2)=\int_0^{2\pi}\frac{d\varphi}{2\pi}\langle\Psi_1|\Psi_2\rangle=\int_0^{2\pi}\frac{d\varphi}{2\pi}\int d\mathbf{r}\Psi_1^*(\mathbf{r},\varphi)\Psi_2(\mathbf{r},\varphi). \qquad (2.2)$$

E has the meaning of the average energy in the $\Psi(\mathbf{r},\varphi)$ state.

We can seek the solution of (2.1) in the form

$$\Psi_{\mathbf{q}}(\mathbf{r},\varphi)=(2\pi)^{-3/2}\exp\left\{i\mathbf{q}\mathbf{r}-i\frac{\hbar q^2}{2m_e\omega}\varphi-\frac{ie(\mathbf{F}\mathbf{q})}{m_e\omega^2}\sin\varphi\right\}F(\mathbf{r},\varphi).$$

A partial differential equation is obtained for the correction function. This equation can be solved by discarding, in the eikonal approximation, the second derivatives with respect to the spatial coordinates:

$$F(\mathbf{r},\varphi)=\exp\left\{-\frac{i}{\hbar\omega}\int_0^{\varphi}U\left[\mathbf{r}-\frac{1}{m_e}\int_{\varphi'}^{\varphi}\left(\hbar\mathbf{q}-\frac{e\mathbf{F}}{\omega}\cos\varphi''\right)d\varphi''\right]d\varphi'\right\}. \qquad (2.3)$$

The wave function obtained[5] does not reflect the periodicity of the Hamiltonian as a function of φ. In addition, the solution of the partial differential equation is accurate only to within the form of an arbitrary function, so that it is desirable to obtain directly an expression that reflects the symmetry of the Hamiltonian and makes a transition to the limit possible not only in the field-free case (as in Ref. 5) but also to certain approximations (Born, low-frequency) for scattering in a field.

We seek the solution of (2.1) in the form

$$\Psi(\mathbf{r},\varphi)=\sum_{n=-\infty}^{\infty}e^{-in\varphi}\Psi_n(\mathbf{r}-\boldsymbol{\alpha}\sin\varphi),\qquad \boldsymbol{\alpha}=e\mathbf{F}/m_e\omega^2. \qquad (2.4)$$

We note the following equality, which is valid for any dependence of Ψ_n on the argument:

$$\left[\frac{\hbar\omega}{i}\frac{\partial}{\partial\varphi}-\frac{ie\hbar(\mathbf{F}\nabla)}{m_e\omega}\cos\varphi\right]\Psi_n(\mathbf{r}-\boldsymbol{\alpha}\sin\varphi)=0. \qquad (2.5)$$

Substituting (2.4) in (2.1) and using (2.5) we get

$$\sum_{n=-\infty}^{\infty}e^{-in\varphi}\left[-\frac{\hbar^2}{2m_e}\Delta+U(\mathbf{r})-E-n\hbar\omega\right]\hat{T}_\nabla\Psi_n(\mathbf{r})=0,$$
$$\hat{T}_\nabla=\exp(-\sin\varphi\boldsymbol{\alpha}\nabla); \qquad (2.6)$$

T_∇ is the shift operator. Equation (2.6) can be satisfied by stipulating that

$$\left[-\frac{\hbar^2}{2m_e}\Delta+U(\mathbf{r})-E-n\hbar\omega\right]\hat{T}_\nabla\Psi_n(\mathbf{r})=0. \qquad (2.7)$$

Knowing the solution of (2.7) we can write the general solution by using (2.4). We note the following identity:

$$e^{G(\mathbf{r})}\exp\left\{\sum_i\tau_i\frac{d}{dx_i}\right\}=\exp\left\{\sum_i\tau_i\frac{d}{dx_i}\right\}e^{G(\mathbf{r})}e^{K[G,\boldsymbol{\tau}]},$$
$$K[G,\boldsymbol{\tau}]=-G(\mathbf{r})-G(\mathbf{r}+\boldsymbol{\tau})+\sum_i\frac{2}{\tau_i}\int_{x_i}^{x_i+\tau_i}G(\mathbf{r}')\,dx_i'.$$

Commuting T_∇ and $U(\mathbf{r})$, we rewrite (2.7) in the form

$$\left\{-\frac{\hbar^2}{2m_e}\Delta+U(\mathbf{r})\exp[\mathcal{F}(\mathbf{r},\boldsymbol{\alpha})]-E-n\hbar\omega\right\}\Psi_n(\mathbf{r})=0,$$
$$\mathcal{F}(\mathbf{r},\boldsymbol{\alpha})=K[\ln U(\mathbf{r}),-\sin\varphi\boldsymbol{\alpha}]. \qquad (2.8)$$

Assuming the potential $U(\mathbf{r})$ to be smooth, we obtain

$$\mathcal{F}(\mathbf{r},\boldsymbol{\alpha})=-\sin\varphi\boldsymbol{\alpha}\frac{\partial U}{\partial\mathbf{r}}U^{-1}. \qquad (2.9)$$

Equation (2.8) is a Schrödinger equation with a noncentral potential (the field-induced noncentrality is significant), and can be solved by the eikonal method. The conditions for the applicability of the method, in analogy with the scattering in the absence of a field, are[8]

$$(e^{\mathcal{F}}U)_{max}\ll E+n\hbar\omega=\hbar^2q_n^2/2m_e,\qquad q_nR\gg 1$$

(R is the characteristic effective radius of the potential). In this approximation, the solution can be written in the form

$$\Psi_n^-(\mathbf{r})=(2\pi)^{-3/2}\exp\left\{i\mathbf{q}_n\mathbf{r}+\frac{i}{2q_n}\int_{-\infty}^{z}V(x,y,z')\times\exp[\mathcal{F}(x,y,z')]dz'\right\} \qquad (2.10)$$

under the condition that the momentum of the incident electron is directed along the z axis. We have introduced in (2.10) the notation $2m_e\hbar^{-2}U=V$. Using (2.4), we obtain the wave function of the final state

$$\Psi^-(\mathbf{r},\varphi)=\sum_{n-\infty}^{\infty}e^{-in\varphi}\Psi_n^-(\mathbf{r}-\boldsymbol{\alpha}\sin\varphi). \qquad (2.11)$$

The wave function of the initial state with momentum $\hbar\mathbf{q}_i$ is

$$\varphi(\mathbf{q}_i,\mathbf{r},\varphi)=(2\pi)^{-3/2}\exp(i\mathbf{q}_i\mathbf{r}-i\boldsymbol{\alpha}\mathbf{q}_i\sin\varphi). \qquad (2.12)$$

The scattering amplitude is given by

$$f(\mathbf{q}_i,\mathbf{q})=-\frac{(2\pi)^3}{4\pi}(\Psi_{\mathbf{q}}^-,V(\mathbf{r})\varphi_{\mathbf{q}_i}), \qquad (2.13)$$

and the scalar product is defined in (2.2). Using (2.12), (2.11), and (2.10) we can represent the scattering amplitude as a sum of partial amplitudes corresponding to scattering with absorption or emission of s-photons,

$$f=\sum_{s=s_0}^{\infty}f_s(\mathbf{q}_i,\mathbf{q}_s),$$

$$f_s(\mathbf{q}_i,\mathbf{q}_s)=-\frac{1}{4\pi}\int_0^{2\pi}\frac{d\varphi}{2\pi}\int d^2\mathbf{b}\int_{-\infty}^{\infty}dz\,V(\mathbf{b},z)\exp\{i\Theta(z,\mathbf{b},\varphi)\},$$
$$\Theta=s\varphi+\tilde{\mathbf{q}}_s\boldsymbol{\alpha}\sin\varphi+\chi-\tilde{\mathbf{q}}_s\mathbf{b}; \qquad (2.14)$$

$$\chi(q_s,\mathbf{b},\varphi,z)=-\frac{1}{2q_s}\int\limits_{-\infty}^{z-\alpha_z\sin\varphi} dz' V(\mathbf{b}-\boldsymbol{\alpha}_\perp\sin\varphi,z')$$
$$\times\exp\{\mathcal{F}(\mathbf{b}-\boldsymbol{\alpha}_\perp\sin\varphi,z')\},$$
$$\tilde{\mathbf{q}}_s=\mathbf{q}_s-\mathbf{q}_i,\quad \hbar^2q_s^2=\hbar^2q_i^2+2m_es\hbar\omega.$$

Using the properties of the shift operator, we can integrate in (2.14) with respect to z in analogy with the field-free case. As a result we have

$$f_s(\mathbf{q}_i,\mathbf{q}_s)=-\frac{q_si}{2\pi}\int\limits_0^{2\pi}\frac{d\varphi}{2\pi}\int d^2\mathbf{b}[\exp\{i\chi(q_s,\mathbf{b},\varphi)\}-1]$$
$$\times\exp(is\varphi+i\tilde{\mathbf{q}}_s\boldsymbol{\alpha}\sin\varphi+\eta),$$
$$\chi(\mathbf{q}_s,\mathbf{b},\varphi)=-\frac{1}{2q_s}\int\limits_{-\infty}^{+\infty}dzV(\mathbf{b},z)\exp\{\mathcal{F}(\mathbf{b},z)\},\qquad(2.15)$$
$$\eta(\mathbf{b},\varphi)=K[e^{i\chi}-1,-\boldsymbol{\alpha}_\perp\sin\varphi].$$

The expression for χ from (2.15) can be called the eikonal for the scattering problem in the presence of an electromagnetic wave.

We write the differential cross section for a process with emission or absorption of s-photons in the form

$$d\sigma/d\Omega=(q_s/q_i)|f_s|^2.$$

Since an essential role is played in the eikonal approximation by a small scattering angle, we have for the solid-angle element $d\Omega=d\gamma q_s^{-2}\tilde{q}_sd\tilde{q}_s$, where γ is the azimuthal angle. The cross section of the process is determined by the expression

$$\sigma_s=\int\limits_0^{2\pi}d\gamma\int\frac{1}{2\pi}\tilde{q}_s\,d\tilde{q}_s|B(\tilde{q}_s)|^2,$$
$$B(\tilde{q}_s)=\int\limits_0^{2\pi}\frac{d\varphi}{2\pi}\exp(is\varphi+i\tilde{\mathbf{q}}_s\boldsymbol{\alpha}\sin\varphi)$$
$$\times\int d^2\mathbf{b}\exp(-i\tilde{\mathbf{q}}_s\mathbf{b}+\eta)[e^{i\chi}-1].$$

We consider now the limiting cases for the scattering amplitude.

3. FIRST EIKONAL APPROXIMATION

Assume that the eikonal is a small quantity, i.e.,

$$(e\mathcal{F}U)_{max}R\ll\hbar^2q_s^2/4m_e.$$

The scattering amplitude then takes the form

$$f_s^{(1)}(\mathbf{q}_i,\mathbf{q}_s)=-\frac{iq_s}{2\pi}\int\limits_0^{2\pi}\frac{d\varphi}{2\pi}\int d^2\mathbf{b}\chi(q_s,\mathbf{b},\varphi)\exp(is\varphi-i\tilde{\mathbf{q}}_s\mathbf{b})$$
$$\times\exp(i\tilde{\mathbf{q}}_s\boldsymbol{\alpha}\sin\varphi)$$
$$=-\frac{1}{4\pi}\int d^2\mathbf{b}\int\limits_{-\infty}^{+\infty}dz\exp(-i\tilde{\mathbf{q}}_s\mathbf{b})V(\mathbf{r})J_s\left(\tilde{\mathbf{q}}_s\boldsymbol{\alpha}+i\frac{\boldsymbol{\alpha}}{V}\frac{\partial V}{\partial\mathbf{r}}\right),\qquad(3.1)$$

where (2.9) was used; the potential is assumed to be smooth.

In the first eikonal approximation the SB is determined by a Bessel function whose argument depends on the ratio of the oscillations in the wave to the characteristic dimension of the potential and to the electron wavelength as determined by the momentum transfer.

It can be seen from (3.1) that the SB is determined by two parameters

$$X=\frac{e\mathbf{F}\tilde{\mathbf{q}}_s}{m_e\omega^2},\quad X_1=\frac{e\mathbf{F}}{m_e\omega^2}U^{-1}\frac{\partial U}{\partial\mathbf{r}},$$

whereas in the Born approximation the SB is determined by the one parameter X.

Let $X_1\ll1$, $X>X_1(R\tilde{q}_s\gg1)$; the dependence of the scattering amplitude on the field characteristics can then be factorized. If the "rotation" $\mathbf{q}_i\to\mathbf{q}_s$ takes place at $X<1$, the field-oscillator excitation probability has a power-law smallness, and at $X>1$ the most probable is excitation of an oscillator with $s\lesssim X$. At $X\ll1$ and $X_1>X$, for example, at small momentum transfers the SB is determined by the parameter X_1. For scattering in a CO_2-laser field by potentials with atomic effective radii, the parameter X_1 becomes of the order of unity at intensities $F\gtrsim10^{-4}$ a.u. At lower intensities, the passage through the region of the potential will be adiabatic, i.e., there will be no photon absorption or emission in the course of the collision. It is therefore important to take the parameter X_1 into account in the case of scattering in strong fields and at small momentum transfers.

4. PERTURBATION THEORY IN THE PARAMETER X_1

Under the conditions $X_1\ll1$ or $|\mathcal{F}|\ll1$, when perturbation theory can be used, the eikonal can be represented as a sum of two terms:

$$\chi(q_s,\mathbf{b},\varphi)=\chi_0(q_s,\mathbf{b})+\frac{1}{2q_s}\boldsymbol{\alpha}\sin\varphi\int\limits_{-\infty}^{+\infty}\frac{\partial V}{\partial\mathbf{r}}dz',$$

where

$$\chi_0(q_s,\mathbf{b})=-\frac{1}{2q_s}\int\limits_{-\infty}^{+\infty}V(x,y,z')dz'$$

is the eikonal for scattering in the absence of a field. The scattering amplitude is reduced to the form

$$f_s(\mathbf{q}_i,\mathbf{q}_s)=-\frac{iq_s}{2\pi}J_s(\tilde{\mathbf{q}}_s\boldsymbol{\alpha})\int d^2\mathbf{b}\exp\{-i\mathbf{q}_s\mathbf{b}\}$$
$$\times[\beta_s\exp\{i\chi_0(q_s,\mathbf{b})\}-1],\qquad(4.1)$$
$$\beta_s=J_s\left(\tilde{\mathbf{q}}_s\boldsymbol{\alpha}+\frac{\boldsymbol{\alpha}}{2q_s}\int\limits_{-\infty}^{+\infty}\frac{dV}{d\mathbf{r}}dz'\right)\Big/J_s(\tilde{\mathbf{q}}_s\boldsymbol{\alpha}).$$

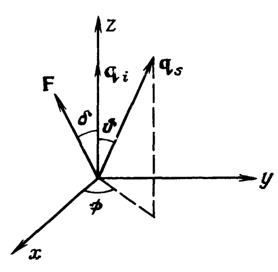

FIG. 1. Scattering $\mathbf{q}\to\mathbf{q}_s$ in a linearly polarized field of intensity $\mathbf{F}=(F\sin\delta,F\cos\delta)$.

TABLE I. Differential cross section (in a.u.) for scattering at $X_1 = 0.1$; $s = 1$.

ϑ	Without field	in the LF approximation	In calculation with the amplitude from (4.2)
7°	275,26	4·0,01	0,69
10°	66,09	0,66·0,01	0,67
15°	13,22	0,53·0,01	0,219
20°	4,2	0,38·0,01	0,107
25°	1,742	0,28·0,01	0,061
30°	0,852	0,21·0,01	0,0395

To use (4.1) in the calculation one must know the eikonal of the potential and of its derivatives. In the case of a smooth potential the conditions for the validity of perturbation theory are written, using (2.9) in the form:

$$V^{-1}\frac{\partial V}{\partial \mathbf{r}}\alpha \ll 1, \quad \frac{eF}{m_e\omega^2 R} \ll 1.$$

We consider now scattering by a truncated Coulomb potential

$$U(r)=\begin{cases} 0 & r>R_0,\\ -Ze^2/r & r<R_0.\end{cases}$$

The geometry of the scattering process is shown in Fig. 1. The scattering amplitude amplitude in first order in the parameter $X_1 = eF/m_e\omega^2 a_0$, where a_0 is the Born radius, takes the form

$$|f_s| = \left| -\frac{2m_e Ze^2}{\hbar^2 \tilde{q}_s^2} \tilde{J}_s(\mathbf{q}_s\alpha) + \frac{8Ze^2}{q_s} X_1 (J_{s-1}(X) - J_{s+1}(X)) \cos\phi \right|. \quad (4.2)$$

The calculation was carried out at the following values of the quantities contained in the formula:

$$Z=1, \quad \hbar^2 q_i^2/2m_e = 110 \text{ eV} \approx 4 \text{ a.u.}, \quad \delta=\pi/4, \quad \phi=0.$$

Table I lists the cross sections for the case when the parameter X_1 is equal to 0.1. The results illustrate the statement that the low-frequency (LF) approximation calls for refinement for scattering in strong fields and at small momentum transfers.

5. OPTICAL THEOREM, TWO SCATTERING CENTERS, BICHROMATIC FIELD

The optical theorem for scattering in the presence of an electromagnetic wave takes the form

$$\operatorname{Im} f_0(0) = \frac{q_i}{4\pi}\sigma_t, \quad \sigma_t = \sum_s \sigma_s,$$

i.e., the total cross section with absorption and emission of an arbitrary number of photons is determined by the imaginary part of the amplitude of elastic scattering through zero angle. We shall show that the scattering amplitude (2.15) agrees with the optical theorem. We introduced below the notation $R(\varphi) = \exp(i\chi(q,\mathbf{b},\varphi)) - 1$.

By definition, we have

$$\sigma_t = \sum_s \int d\Omega_f |f_s(\mathbf{q}_i, \mathbf{q}_s|^2$$

$$= \sum_s \int d\Omega_f \int_0^{2\pi} \frac{d\varphi\, d\varphi'}{(2\pi)^2} \int d^2\mathbf{b}\, d^2\mathbf{b}' \left(\frac{2q_s}{4\pi}\right)^2$$

$$\times \exp\{is(\varphi-\varphi') + i\mathbf{q}_s\alpha(\sin\varphi - \sin\varphi') - i\tilde{\mathbf{q}}_s(\mathbf{b}-\mathbf{b}')\} R(\varphi) R^*(\varphi')$$

$$= \int_0^{2\pi} \frac{d\varphi}{2\pi} \int d^2\mathbf{b} |R(\varphi)|^2.$$

We have used here for the solid angle the eikonal-approximation expression $d\Omega_f = q_s^{-2} d^2\mathbf{q}_s$, the definition of the two-dimensional δ-function, and the equality

$$\sum_s \exp\{is(\varphi-\varphi')\} = 2\pi\delta(\varphi-\varphi').$$

On the other hand,

$$\operatorname{Im} f_0(0) = \int_0^{2\pi} \frac{d\varphi}{2\pi} \frac{q_i}{2\pi} \int d^2\mathbf{b}\, \operatorname{Re}(1-e^{i\chi}).$$

Comparison of these expressions shows that the amplitude obtained satisfies the optical theorem.

The scattering amplitude (2.15) allows us to consider, in the eikonal approximation, scattering by two scattering centers, $U(\mathbf{r})$ and $U(|\mathbf{r}-\mathbf{c}|)$. We then have

$$\exp\{i\chi(U) + i\chi(U_c)\}$$

$$= 1 - \frac{2\pi}{i} \sum_s \int d^2\tilde{\mathbf{q}}_s \exp\{-is\varphi - i\tilde{\mathbf{q}}_s\alpha\sin\varphi + i\tilde{\mathbf{q}}_s\mathbf{b}\} \frac{\tilde{f}_s(\tilde{\mathbf{q}}_s, q_s)}{q_s}.$$

We express the amplitude $\tilde{f}_s$ for scattering by two centers in terms of the amplitude for scattering by one center:

$$\tilde{f}_s(\mathbf{q}_s, q_s) = f_s(\tilde{\mathbf{q}}_s, q_s) + \exp\{-i\tilde{\mathbf{q}}_s\mathbf{c}_\perp\} f_s(\mathbf{q}_s, q_s)$$

$$+ 2\pi i \exp\{-i\tilde{\mathbf{q}}_s\mathbf{c}_\perp/2\} \sum_{m=-\infty}^{\infty} \frac{q_s}{q_m q_{s-m}} \int d^2\mathbf{b} \frac{1}{2} e^{i\mathbf{b}\mathbf{c}_\perp}$$

$$\times \left\{ f_m\left(\mathbf{b} + \frac{\tilde{\mathbf{q}}_s}{2}\right) f_{s-m}\left(\frac{\tilde{\mathbf{q}}_s}{2} - \mathbf{b}\right) + f_{s-m}\left(\mathbf{b} + \frac{\tilde{\mathbf{q}}_s}{2}\right) f_m\left(\frac{\tilde{\mathbf{q}}_s}{2} - \mathbf{b}\right) \right\}.$$

The momentum-transfer vector is perpendicular to the direction of the incident electron. The difference from the case of field-free scattering lies in the appearance of the sum over m, which corresponds to reradiation of the photons on scattering from two centers.

In the eikonal approximation we can obtain the amplitude of scattering in a bichromatic field. The Schrödinger equation for an electron in the field of two monochromatic waves of frequencies ω_1 and ω_2 is

$$\left\{ -\frac{\hbar^2}{2m_e}\Delta + U(\mathbf{r}) - E_0 + \sum_{k=1,2} \left[\frac{\hbar\omega_k}{i} \frac{\partial}{\partial\varphi_k} - \frac{ie\hbar(\mathbf{F}_k\nabla)}{m_e\omega_k} \cos\varphi_k + N_k\hbar\omega_k \right] \right\} \Psi = 0.$$

We seek the solution in the form

$$\Psi=\sum_{n,m}\exp(-in\varphi_1-im\varphi_2)\prod_{j=1,2}\hat{T}_j\Psi_{n,m}(\mathbf{r}),$$

$$\hat{T}_j=\exp(-\sin\varphi_j\boldsymbol{\alpha}_j\nabla),\qquad \boldsymbol{\alpha}_j=e\mathbf{F}_j/m_e\omega_j^2.$$

We can then obtain for the function $\Psi_{n,m}(\mathbf{r})$ the equation

$$\left[-\frac{\hbar^2}{2m_e}\Delta+U(\mathbf{r})\exp(\mathscr{F}_1+\mathscr{F}_2)-E\right]\Psi_{n,m}(\mathbf{r})=0,$$

$$\mathscr{F}_j=K[\ln U(\mathbf{r}),-\sin\varphi_j\alpha_j],\qquad E=E_0+\sum_k N_k\hbar\omega_k.$$

In analogy with the case of a monochromatic field, the scattering amplitude can be represented as a sum of partial amplitudes with absorption of s quanta of frequency ω_1 and s' quanta of frequency ω_2:

$$f_{ss'}(\mathbf{q}_i,\mathbf{q}_{ss'})=\frac{-iq_{ss'}}{2\pi}\int_0^{2\pi}\frac{d\varphi_1\,d\varphi_2}{(2\pi)^2}\int d^2\mathbf{b}[e^{i\chi}-1]$$

$$\times\exp\{-is\varphi_1-is'\varphi_2+i\tilde{\mathbf{q}}_{ss'}(\boldsymbol{\alpha}_1\sin\varphi_1+\boldsymbol{\alpha}_2\sin\alpha_2)-i\tilde{\mathbf{q}}_{ss'}\mathbf{b}\},$$

$$\chi(q_{ss'},\mathbf{b},\varphi_1,\varphi_2)=-\frac{1}{2q_{ss'}}\int_{-\infty}^{+\infty}dz\,V(\mathbf{b},z)\exp\{\mathscr{F}_1+\mathscr{F}_2\},\qquad (5.1)$$

$$\tilde{\mathbf{q}}_{ss'}=\mathbf{q}_{ss'}-\mathbf{q}_i,\qquad \hbar^2q_{ss'}^2=\hbar^2q_i^2+2m_e\hbar(s\omega_1+s'\omega_2).$$

Under conditions when the first eikonal approximation is valid, Eq. (5.1) reduces to

$$f_{ss'}^{(1)}(\mathbf{q}_i,\mathbf{q}_{ss'})=-\frac{1}{4\pi}\int d^2\mathbf{b}\int_{-\infty}^{+\infty}dz\exp(-i\tilde{\mathbf{q}}_{ss'}\mathbf{b})\,V(\mathbf{r})$$

$$\times J_s\left(\boldsymbol{\alpha}_1\left(\tilde{\mathbf{q}}_{ss'}+iV^{-1}\frac{\partial V}{\partial\mathbf{r}}\right)\right)J_{s'}\left(\boldsymbol{\alpha}_2\left(\tilde{\mathbf{q}}_{ss'}+iV^{-1}\frac{\partial V}{\partial\mathbf{r}}\right)\right),$$

from which it follows that absorption of quanta of different frequencies is determined by the Bessel functions that correspond to these frequencies. The analysis of the result is similar to that in Sec. 3.

The authors thank M. V. Fedorov for a discussion of the main results.

APPENDIX

We shall use below the relations

$$\exp(-i\omega ta^+a)|m\rangle=\exp(-i\omega tm)|m\rangle,$$

$$(x+i\varepsilon)^{-1}=-i\int_0^\infty\exp(ixt-\varepsilon t)\,dt,\qquad \varepsilon\to0.\qquad (A.1)$$

Substituting (1.10) in (1.9) we have

$$f_s^{(2)}(\mathbf{q}_i,\mathbf{q}_f)=-\frac{m_e}{2\pi\hbar^2}\sum_{m=0}^{\infty}\int\frac{d\mathbf{q}}{(2\pi)^3}\left(-\frac{i}{\hbar}\right)\int_0^\infty dt$$

$$\times\exp\left\{\frac{i}{\hbar}(E-\varepsilon_q-m\hbar\omega+i\varepsilon)t\right\}U(\mathbf{q}_f-\mathbf{q})U(\mathbf{q}-\mathbf{q}_i)$$

$$\times\langle n+s|D(-\rho_f)$$

$$\times D(\rho_q)|m\rangle\langle m|D(-\rho_q)D(\rho_i)|n\rangle\exp(-|\rho_q|^2).$$

Using (A.1) we can convolute the sum over m. We use next the fact that

$$e^{-i\omega ta^+a}D(\rho)=D(\rho e^{-i\omega t})e^{-i\omega ta^+a},$$

the property of the shift operator

$$D(\beta)D(\alpha)=\exp\{(\beta\alpha^*-\beta^*\alpha)/2\}D(\beta+\alpha),$$

the expression for the matrix element

$$\langle n+s|D(\rho)|n\rangle=[(n+s)!/n!]^{1/2}e^{-\nu/2}e^{i\gamma s}(-\nu^{1/2})^{-s}L_{n+s}^{-s}(\nu),$$

$$\rho=\nu^{1/2}e^{i\gamma}$$

and a transition to the classical-field limit by using (1.7); we obtain ultimately

$$f_s^{(2)}=-\frac{m_e}{2\pi\hbar^2}\int\frac{d\mathbf{q}}{(2\pi)^3}U(\mathbf{q}_f-\mathbf{q})U(\mathbf{q}-\mathbf{q}_i)\left(-\frac{i}{\hbar}\right)\int_0^\infty dt$$

$$\times\exp\left\{\frac{i}{\hbar}(\varepsilon_i-\varepsilon_q+i\varepsilon)t\right\}\exp(i\gamma(t)s)J_s[2(n\nu(t))^{1/2}],\qquad (A.2)$$

$$\nu^{1/2}e^{i\gamma}=\rho_q-\rho_f+e^{-i\omega t}(\rho_i-\rho_q).$$

After using the summation formula[9]

$$J_\nu([x^2+y^2-2xy\cos\phi]^{1/2})$$

$$=\left(\frac{x-ye^{i\phi}}{x-ye^{-i\phi}}\right)^{\nu/2}\sum_{k=-\infty}^{\infty}J_{\nu+k}(x)J_k(y)e^{ik\phi}$$

we obtain Eq. (1.11) of the text, where

$$X_{fq}=(\mathbf{q}_f-\mathbf{q})e\mathbf{F}/m_e\omega^2,\qquad X_{iq}=(\mathbf{q}_i-\mathbf{q})e\mathbf{F}/m_e\omega^2.$$

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