Generalized Rayleigh waves and the geometry of isofrequency surfaces of sound oscillation waves in crystals

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(Submitted 28 September 1984)
Zh. Eksp. Teor. Fiz. **88**, 1089–1097 (March 1985)

The transition with respect to the anisotropy coefficient $\eta = (C_{11} - C_{12})/2C_{44}$ from ordinary to generalized Rayleigh waves is investigated for cubic symmetry crystals. The necessary and sufficient conditions for such transitions are found. Attention is drawn to the fact that a change in the geometry of the cross section of the isofrequency surface for long-wave sound oscillations affects the physical characteristics of the surface waves. The generalized Rayleigh waves in strongly anisotropic crystals are also analyzed.

Rayleigh waves are the basic types of surface sound waves in solids. The amplitude of the vibrations in such waves falls off exponentially into the bulk of the crystal. The decrease in the amplitude takes place monotonically (ordinary Rayleigh waves) or nonmonotonically, with oscillations (generalized Rayleigh waves).[1] Although these and other types of Rayleigh waves in anisotropic matter have been studied for a long time, and in great detail, the actual transition from ordinary to generalized waves still remains unclear. This is connected with the fact that, owing to the complexity of the dispersion equations, numerical calculations have been used in most studies.[1–4] The present work is devoted to the analysis of the conditions for the transition from ordinary to generalized Rayleigh waves, and to the study of the connection of the basic characteristics of surface waves with the geometry of isofrequency long-wavelength surface acoustic waves. It is found that the shape of the isofrequency phonon surface (its local geometry) affects the physical characteristics of the surface waves in a manner similar to the way in which the singularities of the Fermi surface appear in the electron properties of metals,[5] including the surface properties.[6] In contrast to the situation involving the electron properties, many of which are essentially determined by the shape of the Fermi surface, few examples are known of acoustical properties being determined by the shape of the isofrequency surfaces of crystals.

TRANSITION FROM ORDINARY TO GENERALIZED RAYLEIGH WAVES

It is known that generalized Rayleigh waves can propagate only in anisotropic crystals, since the damping of the amplitude of a surface wave in an isotropic solid always takes place monotonically.[1] As will be shown below, the basic features of the phenomena we are considering can be illustrated by the example of a cubic crystal, which is described by three independent elastic moduli, instead of two for the isotropic solid. We shall analyze the symmetric directions of wave propagation and only such slices of the surface for which the sagittal plane is a plane of mirror symmetry of the crystal. The Rayleigh wave in this case is a double mode, and the characteristic equation of bulk oscillations is a biquadratic.

We consider the double-mode Rayleigh wave, propagating in the [100] direction on the (001) boundary of a cubic crystal. The displacements in such a wave have the form

$$u_x = (A_1 e^{\gamma_1 kz} + A_2 e^{\gamma_2 kz}) e^{ik(x-Vt)},\qquad u_z = i(\Gamma_1 A_1 e^{\gamma_1 kz} + \Gamma_2 A_2 e^{\gamma_2 kz}) e^{ik(x-Vt)}, \tag{1}$$

where V is the velocity and k is the wave number of the wave. In the expression (1), γ_1, γ_2 are the roots of the biquadratic characteristic equation (2), which is obtained from the bulk equations of motion:

$$C_{44}C_{11}\gamma^4 - \gamma^2[C_{44}(C_{44}-\rho V^2) + C_{11}(C_{11}-\rho V^2) - (C_{12}+C_{44})^2] + (C_{44}-\rho V^2)(C_{11}-\rho V^2) = 0 \tag{2}$$

for displacements of the form $u_i = u_i^0 e^{ik(x-Vt)+\gamma kz}$ (the x, y, z axes coincide with the crystallographic axes; $i = x, z$, and the z axis is directed along the external normal; C_{11}, C_{12}, C_{44} are the elastic moduli). The quantities Γ_1 and Γ_2 are the eigenvectors of the bulk equations of motion ($\Gamma = u_x/u_z$), corresponding to the eigenvectors γ_1, γ_2, and have the following form ($i = 1, 2$):

$$\Gamma_i = -\frac{\gamma_i(C_{12}+C_{44})}{\rho V^2 - C_{11} + C_{44}\gamma_i^2} = \frac{\rho V^2 + C_{11}\gamma_i^2 - C_{44}}{\gamma_i(C_{12}+C_{44})}. \tag{3}$$

The ratios between A_1 and A_2 are found from the boundary conditions $\sigma_{zz} = \sigma_{zx} = 0$ on the surface $z = 0$:

$$\frac{A_2}{A_1} = -\frac{1+\Gamma_1\gamma_1}{1+\Gamma_2\gamma_2} = -\frac{C_{11}\gamma_1 - C_{12}\Gamma_1}{C_{11}\gamma_2 - C_{12}\Gamma_2}. \tag{4}$$

Using (3) and (4), we can write the dispersion relation for the surface wave in the following form:

$$[(C_{12}+\rho V^2)(C_{44}-\rho V^2)C_{12} + C_{11}^2 C_{44}\gamma_1^2\gamma_2^2 + C_{11}C_{12}(C_{44}-\rho V^2) \times(\gamma_1^2+\gamma_2^2+\gamma_1\gamma_2) - C_{11}C_{44}\gamma_1\gamma_2(C_{12}+\rho V^2)](\gamma_1-\gamma_2) = 0. \tag{5}$$

This equation is equivalent to two equations. At $\gamma_1 \neq \gamma_2$ Eq. (5) coincides with the Stoneley equation[2] for the velocity of a surface wave in a cubic crystal of the specified geometry. The solution of the Stoneley equation is shown in Fig. 1 as a function of the anisotropy parameter (factor)

$$\eta = \frac{C_{11}-C_{12}}{2C_{44}} \tag{6}$$

in the case $C_{12} + C_{44} > 0$ and at fixed values of $\xi = C_{11}/C_{44} > 1$ (curve *I*, $V_{T_1} = (C_{44}/\rho)^{1/2}$). Examples are known in which η varies continuously (for example, as a function of the temperature[8]). Curve *I*, as a function of η, describes both

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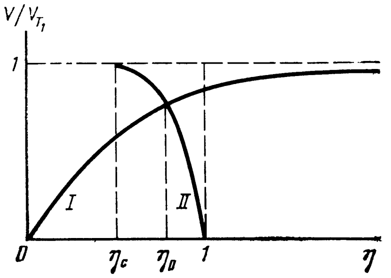

FIG. 1. Dependence of the velocity of Rayleigh waves on the anisotropy parameter (curve I); solution of the equation $\gamma_1 = \gamma_2$ corresponding to the surface wave (curve II); plane (001), direction of the wave [100].

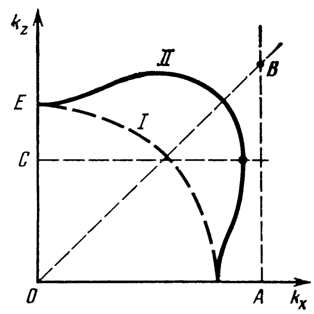

FIG. 2. Convex (I) and nonconvex (II) cross sections relative to the direction k_x.

the ordinary Rayleigh wave (γ_1 and γ_2 are real) and the generalized wave—when γ_1 and γ_2 are complex and have the form $\gamma_{1,2} = \gamma' \pm i\gamma''$. The transition from one wave to the other takes place at $\gamma_1 = \gamma_2$. This condition means that the point of transition (in the anisotropy parameter η) is determined by the simultaneous solution of the Stoneley equation[2] and the equation obtained from the condition $\gamma_1 = \gamma_2$. The condition that the roots of the characteristic equation (2) be equal leads to the following expressions:

$$(\rho V^2)^2\left(\frac{1}{C_{11}}-\frac{1}{C_{44}}\right)^2-2\rho V^2\left(\frac{1}{C_{11}}+\frac{1}{C_{44}}\right)\left[\frac{C_{44}}{C_{11}}+\frac{C_{11}}{C_{44}}-\frac{(C_{12}+C_{44})^2}{C_{11}C_{44}}-2\right]+\left[\frac{C_{44}}{C_{11}}+\frac{C_{11}}{C_{44}}-\frac{(C_{12}+C_{44})^2}{C_{11}C_{44}}-2\right]\times\left[\frac{C_{44}}{C_{11}}+\frac{C_{11}}{C_{44}}-\frac{(C_{12}+C_{44})^2}{C_{11}C_{44}}+2\right]=0. \quad (7)$$

The only root of Eq. (7) corresponding to the surface wave possesses the following properties. In the case $C_{11} - C_{12} = 2C_{44}$, i.e., on an isotropic solid ($\eta = 1$), the equation has a zero root ($V_0 = 0$); in the case $(C_{12} + C_{44})^2 = C_{11}(C_{11} - C_{44})$, which corresponds to $\eta = \eta_c < 1$, the root V_0 is identical with the bulk transverse velocity V_{T_1}.

Thus, the desired solution exists in the inerval $\eta_c < \eta \leqslant 1$. Curve II in Fig. 1 represents this solution. In such a "wave," as is seen from (1), (3), and (4), the total displacements u_z and u_x are equal to zero. We note that the analogous root, corresponding to the "wave" with $V_0 = 0$, is also in the Rayleigh equation for an isotropic solid[9] ($\eta = 1$ on curve). In the case $\eta_c < \eta < 1$, such a wave corresponds to the nonzero velocity V_0. The existence of a general root of the two equations (5) (the presence of the point intersection η_0 of the curves I and II) is the necessary and sufficient condition for transition from ordinary to generalized Rayleigh waves. In Fig. 1, the region $\eta < \eta_0$ corresponds to the generalized Rayleigh wave, $\eta > \eta_0$, to the ordinary. At the point of intersection, $\eta = \eta_0$, as also in all branches of II, the total displacement **u** in the double-mode Rayleigh wave vanishes.

The resulting transition criterion, according to the anisotropy parameter (6), from ordinary to generalized Rayleigh waves can be transferred to the case of surface wave propagation in an arbitrary direction in the plane of mirror symmetry of the crystal. Here the surface wave is a triple mode; however, the characteristic equation remains bicubic, as also for the high-symmetry directions. This circumstance allows us to speak of a transition from ordinary to generalized (in displacements in the sagittal plane) surface waves in terms of the other continuous variable, the angle of rotation of the propagation of the wave in such a plane. Such a transition was obtained numerically for Ni in the (110) plane at an angle of 63° relative to [001], and also in the (001) plane in W.[1] For cuts intermediate between (100) and (110), the characteristic equation of the bulk oscillations is not bicubic (or biquadratic), and therefore the ordinary Rayleigh waves do not generally exist at $\eta \neq 1$.

The point η_0 possesses still another interesting property. Analysis carried out in the limit where the transverse velocities V_{T_1} and V_{T_2} are small in comparison with the longitudinal, shows that at this point, the velocity of the Rayleigh wave for these cases is identical with the velocity of the bulk transverse wave polarized perpendicular to the sagittal plane.

ANALYSIS OF THE FEATURES OF THE ISOFREQUENCY SURFACE

The appearance in the crystal of the generalized Rayleigh wave can be connected with the shape of the isofrequency phonon surface. It is well known that for various relations between the elastic moduli allowed by the conditions of elastic stability,[1)] the corresponding cross sections of the isofrequency surface can be either convex of nonconvex.[10] For analysis of the Rayleigh wave, the properties of the cross section in the k_x k_z plane (xz is the sagittal plane) of the isofrequency surface of the bulk transverse mode, polarized in the sagittal plane, are important. In this cross section, there is interest in its local geometry near the direction of the wave vector of the surface wave. In the case of a convex cross section relative to the given direction of **k**, the lines perpendicular to this direction have two (equivalent) points of intersection with the cross section, and in the case of concave cross sections, more than two.

Naturally, it makes sense to introduce the concepts of convexity and nonconvexity only for the high-symmetry directions (of the type [100], [110]). We now consider the properties of the cross section of the isofrequency surface of mode V_{T_1}, near the k_x direction. An example of part of the cross section[2)] of the isofrequency surface of a cubic crystal is shown in Fig. 2. The curve I corresponds to a convex cross section, relative to the [100] direction, while curve II corresponds to the nonconvex. The entire analysis of the properties of the cross sections of isofrequency surfaces in the k_x k_z plane can be carried out on the basis of Eq. (2). The condition for the k_x k_z cross section to be concave relative to the [100] direction has the form

$$\frac{(C_{12}+C_{44})^2-C_{11}(C_{11}-C_{44})}{C_{11}-C_{44}}>0, \qquad (8)$$

while the condition for nonconvexity relative to the [110] direction is

$$\frac{(C_{11}-C_{12})(2C_{11}+C_{12})-C_{44}(5C_{11}+3C_{12})}{C_{12}+C_{44}}>0. \qquad (9)$$

We can obtain the connection of the critical value of the anisotropy parameter η_c, corresponding to the transition from convex to nonconvex cross section of the isofrequency surface of mode V with the parameter $\xi = C_{11}/C_{44}$ from (6), (8) and (9). For the [100] direction,

$$\eta_c<\frac{(1+\xi)-(\xi(\xi-1))^{1/2}}{2}, \quad \xi>1, \qquad (10)$$

while for the [110] direction,

$$\eta_c>\frac{3(\xi+1)-(9\xi^2-14\xi+9)^{1/2}}{4}, \quad 2\eta_c-\xi<1. \qquad (11)$$

We note that crystals for which $\xi<1$ (i.e., $C_{44}<C_{11}$) and $2\eta_c-\xi>1$ (i.e., $C_{12}+C_{44}<0$), do not occur in nature. Regions of convexity and nonconvexity of the cross section of the isofrequency surface of mode V_{T_1} in the k_x k_z plane relative to the [001] and [110] directions of a cubic crystal are shown in Fig. 3 in the variables ξ, η. The region 1 corresponds to nonconvexity relative to the [001] direction, region 2—relative to the [110] direction, region 3—to a cross section that is convex relative to both directions, and region 4—to nonphysical relations between the elastic parameters of the crystal. For most cubic crystals, the anisotropy parameter η is smaller than 0.75 in region 1) (Na, Pb, Ni, Cu, Ag, Fe, Ge, Si, Au, GaAs, MgO, LiF, ZnS, ZnSb, . . .). At $\eta=1$ (isotropic case), all the isofrequency surfaces have the shape of concentric spheres, and an arbitrary cross section is circular.

The value of η_c determined by the expression (10) is identical with the value of η_c in curve *II* of Fig. 1. It then follows (this can established directly) that the transition from ordinary to generalized Rayleigh waves takes place when the isofrequency surface is still convex. Therefore, the nonconvexity of the isofrequency surface of a bulk transverse mode, polarized in the sagittal plane, is a sufficient condition for the existence of a generalized Rayleigh wave (without its being the necessary condition). The boundaries between the regions in which generalized and ordinary Rayleigh waves exist are indicated in Fig. 3 by the dashed lines. Region 1′, which includes region 1, corresponds to generalized Rayleigh waves along [100] on the (001) surface, and region 2′, which includes the region 2, to generalized Rayleigh waves along [1$\bar{1}$0] on the (110) plane. Thus, for example, for crystals of Al, diamond and Ni, the Rayleigh wave on the (001) plane along [100] is a generalized one.[1,2] At the same time, the corresponding cross section of the isofrequency surface of mode V_{T_1} (see Fig. 3) for Al and diamond is convex, while for Ni (and also for most cubic crystals) it is

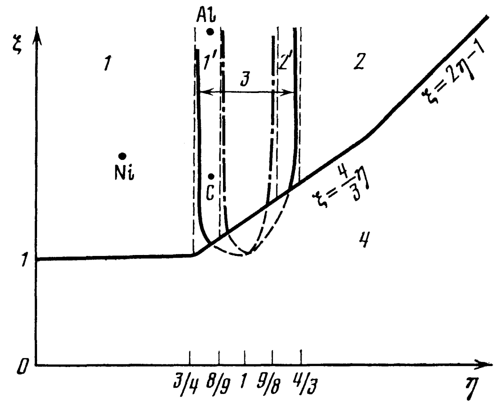

FIG. 3. Values of the parameters η, ξ at which the corresponding cross sections are convex or nonconvex—solid lines; boundaries of transition from ordinary to generalized Rayleigh waves—dashed lines (see text). The values of the elastic moduli for nickel, diamond and aluminum were taken from Ref. 1.

nonconvex. We note that in crystals with an anisotropy parameter close to unity (for example, W, $\eta \approx 1.005$), the isofrequency surfaces of both transverse modes are convex in all symmetric directions. Nevertheless, even in such crystals, directions exist (in the plane of mirror symmetry), that are different from symmetric, in which the Rayleigh wave is a generalized one (but displaced in the sagittal plane)[1]. Only when the anisotropy parameter is strictly equal to unity is the Rayleigh wave an ordinary one in an arbitrary geometry of the cubic crystal.

The nonconvexity of the corresponding cross section of the isofrequency surface as a sufficient condition for the existence of a generalized Rayleigh wave has a graphic geometric interpretation. In Fig. 2, the line AB corresponds to a surface wave of a given frequency, traveling along [100] with wave number $0A$ on the (001) boundary. The curve OC for a nonconvex isofrequency surface (curve *II*) is equal to $\gamma'' k$ in the wave, i.e., it determines the period of oscillation of the amplitude of the generalized Rayleigh wave as a function of depth.

GENERALIZED RAYLEIGH WAVES IN STRONGLY ANISOTROPIC CRYSTALS

In the preceding section, we analyzed the condition of transition from ordinary to generalized Rayleigh waves and the connection of this transition with the singularities of the isofrequency surfaces of the bulk transverse wave. As was pointed out, this transition and the change in the local geometry also take place at values of η close to unity. We now analyze the situation of a strongly anisotropic cubic crystal, with an anisotropy parameter $\eta=(C_{11}-C_{12})/2C_{44}$, greatly different from unity. Many alkali metals belong to such a group (Li, K, Na, . . .) and also compounds that are close to a structural phase transition, connected with the soft transverse velocity $V_{T_2}=(C_{11}-C_{12})/2\rho)^{1/2}$ (for example, A-15 compounds Nb_3, Sn, V_3, Ge), in which $\eta \ll 1$, and also piezoelectric crystals with a piezoeffect in the paraphase (KDP, Rochelle salt) near the piezoelectric transition, when the soft velocity $V_{T_1}=(C_{44}/\rho)^{1/2}$ ($\eta \gg 1$) occurs). In both cases ($\eta \ll 1$

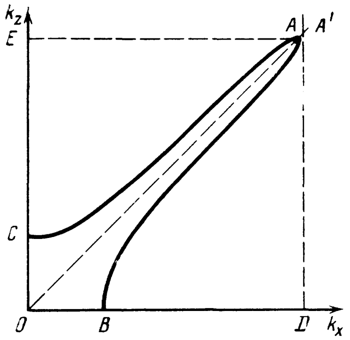

FIG. 4. Cross section in the k_x k_z plane of the isofrequency surface of the transverse bulk mode V_{T_1} in the case $\eta \ll 1$.

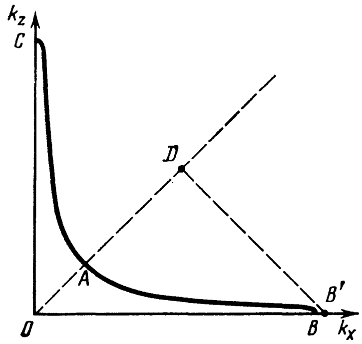

FIG. 5. Cross section in the k_x k_z plane of the isofrequency surface of the transverse bulk mode V_{T_1} in the case $\eta \gg 1$.

and $\eta \gg 1$) the nonconvexity of the isofrequency surface is most sharply expressed, while the period of the oscillations of the amplitude of the generalized Rayleigh wave is finite with depth and comparable with the wavelength. At the same time, close to the point of transition from ordinary to generalized Rayleigh waves, the period of the oscillations greatly exceeds the wavelength and the penetration depth ($\gamma'' < \gamma'$). We shall first consider the surface wave on the (001) boundary, traveling along the [100] direction in a crystal with $\eta \ll 1$. The presence of the small parameter η allows us to carry out the entire calculation analytically (the numerical calculation for the Nb_3Sn crystal was given in Ref. 9). To within linear terms in η this wave has the following characteristics:

$$\rho V^2 = 2\eta C_{44}[1 - {}^1/_2\eta(1 + C_{44}/C_{11})],$$
$$\gamma_{1,2} = \pm[1 - {}^1/_2\eta(1 + C_{44}/C_{11})]i + {}^1/_2\eta(1 + C_{44}/C_{11}). \qquad (12)$$

It is seen from (12) that in the case $\eta \to 0$ $\gamma_{1,2} \to \pm i$, $V \to V_{T_2}\sqrt{2}$. The limiting relations can be interpreted geometrically with the help of the topology of the isofrequency surface. The cross section in the k_x k_z plane of the isofrequency surface is perpendicular to the bulk wave, polarized in the sagittal plane, is shown in Fig. 4 in the case $\eta \ll 1$ (see note 2). The line $A'D$ in this plot corresponds to a surface wave with the wave number $0D$. As is seen from (12), as $\eta \to 0$ the value of the separation of the surface wave from the spectrum of bulk transverse waves decreases—the line $A'D$ actually touches the corresponding cross section at the point A. The parameters of this cross section are the following: $0B = 0C = \omega/V_{T_1}$, $0A = \omega/V_{T_2}$, i.e., in the limit $\eta \to 0$, this cross section is extended along the bisector of the coordinate angle, actually coinciding with it ($0A \gg 0B$). Therefore, in the limit we have $0D = 0A/\sqrt{2}$, which corresponds to $V = V_{T_1}\sqrt{2}$. Moreover, from the condition $0E = 0D$, we obtain $\gamma_{1,2} = \pm i$, i.e., the period of the oscillations $(\gamma'')^{-1}$ (with increasing depth) tends to the wavelength, while the penetration depth significantly exceeds the wavelength $\gamma' < \eta \ll 1$. For strongly anisotropic crystals, the increase in the penetration depth in comparison with the wavelength is a natural result of the wave.[11,12] Thus, as the parameter η from $\eta_0 < 1$ decreases to the limiting value determined by the condition of elastic stability ($\eta = 0$), the period of the oscillation $(\gamma'')^{-1}$ decreases from a maximum, equal to infinity at $\eta = \eta_0$, to a minimum, equal to the wavelength at $\eta = 0$.

A similar consideration of the generalized Rayleigh wave can be carried out for another type of strongly anisotropic crystal, with $\eta \gg 1$. In such a crystal, we consider the surface wave traveling along $[10\bar{1}]$ on a boundary of type (101). The corresponding cross section is shown in Fig. 5 (see note 2). The parameters of this cross section $0B = 0C$ and $0A$ are the same as in Fig. 4, with, however, the opposite relation between them: $0A \ll 0B$, i.e., the cross section is taken along the coordinate axes. The line $B'D$ corresponds to a surface wave with wave number $0D$. To the conditions $0D = 0B/\sqrt{2}$ and $0D = DB$ corresponds $V = V_{T_1}\sqrt{2}$ and $\gamma_{1,2} = \pm i$. The same characteristics of the generalized Rayleigh wave we are considering are obtained from analytical considerations to lowest order in $1/\eta$. As also in the previous case, ($\eta \ll 1$, boundary (001), direction of the wave [100'], in the present case ($\eta \gg 1$, boundary (101), direction of the wave $[10\bar{1}]$), the minimum period of the oscillations of the wave amplitude with depth is equal to the wavelength (as $\eta \to \infty$). At a structural phase transition, which is associated with the destruction of the elastic stability of the crystal, the penetration depth of the Rayleigh wave is a maximum,[13] while the period of oscillation of the amplitude of the generalized wave is a minimum.

CONCLUSION

We have analyzed the transition under the change in the anisotropy parameter η of a cubic crystal from ordinary to generalized Rayleigh (two-partial) waves, and we have calculated the necessary and sufficient conditions for the transition. At the transition point, the displacement in the wave vanishes. This same property is also possessed by oscillations on all branches of surface "waves," corresponding to the condition $\gamma_1 = \gamma_2$, and complement the branch of surface Rayleigh waves. This branch exists in the interval of values of η and η_c to 1, while η_c can be either greater than or less than unity. The existence of the point of intersection η_0 of the branch corresponding to the Rayleigh wave branch is the necessary and sufficient condition for the transition from ordinary to generlized Rayleigh waves. The value of η_0 lies in the interval between $\eta = \eta_c$ and $\eta = 1$. In the case $\eta_0 < 1$, the generalized Rayleigh waves exist at $\eta < \eta_0$ in the case in which $\eta_0 > 1$, at values $\eta > \eta_0$. The point $\eta = \eta_0$ corresponds to a change in the local geometry in one of the symmetric directions of the isofrequency surface of the branch of bulk transverse oscillations polarized in the sagittal plane. At this point, the cross section of the isofrequency surface in the k_x k_z plane (xz is the sagittal plane) become nonconvex relative to the direction of propagation of the wave. The nonconvexity of the cross section of the isofrequency surface relative to the direction of propagation of the wave is a sufficient condition for the existence of generalized Rayleigh waves.

The root $V_0 = 0$, which corresponds to the condition $\gamma_1 = \gamma_2$, is also in the Rayleigh equation for an isotropic solid. Since the total displacement in such a "wave" is equal to zero, this "wave" never appears at the boundary between a solid and a vacuum. For its appearance, a second medium is necessary—in the simplest case, such a medium can be a liquid (or gas). It is well known[14] that on the boundary between an isotropic solid and a rarefied medium, for example, a gas ($\rho_2/\rho \ll 1$, $V_2/V_T \ll 1$), in addition to the damped Rayleigh wave there also exists an undamped surface wave. It is characteristic for this wave that longitudinal and transverse components of the wave actually penetrate to the same depth, close to a wavelength into the interior of the solid ($\gamma_1 \approx \gamma_2 \approx 1$). In anisotropic crystals with $V_R > V_0 > 0$ (V_R is the velocity of the Rayleigh wave on the boundary with a vacuum), a similar choice of parameters of the liquid can also assure the existence of undamped surface waves, in which $\gamma_1 \approx \gamma_2$ and the velocity of the wave is finite.

Still another example of the appearance of such a "wave" in an isotropic solid is a system consisting of a layer with high density and high rigidity on the isotropic substrate. As shown in Ref. 15, in such a system there exists a surface wave with a quadratic dispersion law, similar to the dispersion law of the bending wave in a thin free film. This wave is the slowest of all the elastic waves, including acoustic surface waves. As in the case of a surface wave on the boundary of an isotropic solid with a liquid of low density and sound velocity, longitudinal and transverse oscillations penetrate close to a wavelength into the isotropic substrate. The displacements in such a wave are small, and are proportional to the small value of its velocity (frequency).

In strongly anisotropic crystals with an anisotropy parameter very different from unity, the nonconvexity of the corresponding cross sections of the isofrequency surfaces of the transverse modes is very clearly delineated. In such crystals, the analysis of the basic characteristics of the generalized Rayleigh waves in terms of the shape of the cross section of the isofrequency surface is found to be useful. For specific cases, such surface wave properties as the velocity and period of oscillations as a function depth, found analytically, can be interpreted physically with the help of geometric analysis of the corresponding cross sections.

APPENDIX

Equation (5) in the limit of small transverse velocities

$$V_{T_1} = (C_{44}/\rho)^{1/2}, \quad V_{T_2} = [(C_{11} - C_{12})/2\rho]^{1/2}$$

(the region $\xi \gg 1$ in Fig. 3) has the form

$$64\eta^3\Delta + 16\eta^2(3\Delta^2 - 5\Delta + 2\Delta^{1/2}) + 4\eta(3\Delta^3 - 13\Delta^2 + 9\Delta + 1 - 8\Delta^{3/2} + 8\Delta^{1/2}) + \Delta^4 - 8\Delta^3 + 10\Delta^2 - 3 - 8\Delta^{5/2} + 16\Delta^{3/2} - 8\Delta^{1/2} = 0, \quad (A.1)$$

and the corresponding Stoneley equation is

$$16\eta^2\Delta - 8\eta\Delta(1-\Delta) + (\Delta - 1)^3 = 0. \quad (A.2)$$

We have introduced the following notation in (A.1) and (A.2):

$$\eta = (C_{11} - C_{12})/2C_{44}, \quad \Delta = 1 - V^2/V_{T_1}^2$$

as well as the conditions $(C_{11} - C_{12})/C_{11} \ll 1$, $C_{44}/C_{11} \ll 1$.

The wave is propagated on the (001) plane in the [100] direction.

The root of the Stoneley equation corresponding to positive values of η is equal to

$$\eta_{St} = \frac{(1-\Delta)(\Delta + \Delta^{1/2})}{4\Delta}, \quad (A.3)$$

while Eq. (A.1) has the following roots:

$$\eta_1 = \eta_2 = \eta_{St}, \quad (A.4)$$

$$\eta_3 = \frac{3 + 2\Delta^{1/2} - \Delta}{4}. \quad (A.5)$$

Thus Eq. (A.1) contains a doubly degenerate root of the Stoneley equation and an additional root corresponding to the condition $\gamma_1 = \gamma_2$.

From (A.3) and (A.4), we can find the point of intersection of the branch of Rayleigh waves with the branch corresponding to $\gamma_1 = \gamma_2$: $\eta_0 = 8/9$, $\Delta_0 = 1/9$. The value η_c at which the transition from convex to nonconvex cross section the isofrequency surface of mode V_{T_1} takes place is found from the condition $\Delta = 0$: $\eta_c = 3/4$.

For propagation of the wave along the $[1\bar{1}0]$ direction on the (110) plane, in the limit of small transverse velocities V_{T_1} and V_{T_2}, we obtain the following values of the parameters of interest: $\eta_c = 4/3$, $\eta_0 = 9/8$.

The Stoneley equation (A.2) at $\eta = 0$ has a triply degenerate root $\Delta = 1$. At $\eta \ll 1$ this equation has a single solution ($\Delta = 1 - 2\eta$), which describes the surface wave, and two complex-conjugate roots.

At $\eta = 1$, we have $\Delta \approx 1/11$, while the other two roots are complex. At $\eta \gg 1$, there is a single real root ($\Delta = 1/16\eta^2$), and two complex roots.

Thus, for all possible values of the anisotropy parameter η (in every case, in the limit of small transverse velocities, $\xi \gg 1$) the Stoneley equation (A.2) has a single real solution, corresponding to a surface Rayleigh wave (ordinary or generalized).

[1)]For a cubic crystal, these conditions are the following[3]:

[2)]The remaining part of the cross section is obtained by reflection of the reduced coordinates relative to the axes.

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Translated by R. T. Beyer