Theory of magnetic noise with the $1/\omega$ spectrum and its generalizations

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We make a detailed theoretical investigation of a mechanism for the formation of the magnetic flicker noise. The mechanism is based on the remembering and forgetting of the priming white noise. We show that this mechanism leads to slow fluctuations with a nearly $1/\omega$ spectrum at $\omega \to 0$, with the shape of the spectrum depending on the dispersion relation of the priming noise only at finite frequencies. We obtain a relation between the spectral intensity of the flicker noise and the spectral intensity of the priming noise and use it to evaluate the frequency region in which the $1/\omega$ spectrum dominates over the uniform background. We show that systems of other types have a $1/\omega$ noise spectrum if the noise is generated by randomly occurring pulses whose duration ϑ has an asymptotic (at $\omega \to \infty$) probability distribution that is like the typical distribution for the magnetic noise considered here. Such pulses characteristically arise for highly nonlinear nonequilibrium systems which exhibit hysteresis.

I. INTRODUCTION

The theory of noise with the $1/\omega$ spectrum has been the subject of many papers. Some of these have attributed this noise to the gradual aging of the system, in which case it must be treated as nonstationary.[1] Other papers treat this noise as a stationary process. Equilibrium fluctuations of the resistance, which are always present, give a stationary noise whose spectrum grows as $\omega \to 0$, but this noise, as a rule, is masked by noises of other origin (see, e.g., Ref. 2).

In the present paper we consider a mechanism for the $1/\omega$ noise in systems with a memory in the presence of a priming white noise. An example of a physical system having a memory is a ferromagnet in a static or periodically changing external field.

Back in 1957 Néel[3] showed theoretically that if a demagnetized ferromagnet is suddenly subjected to a static external field having a small admixture of a priming random field with a normal dispersion relation, then the magnetization will change (over a certain time interval) as the square of the logarithm of the time. This result, which has been confirmed experimentally,[4] suggests that the magnetization fluctuations have a $1/\omega$ spectrum.

Subsequently magnetic noise (flucutations of the magnetic susceptibility or magnetization) with the $1/\omega$ spectrum (or $1/\Delta\omega$, see below) has been observed experimentally[5,6] in ferromagnets subjected to a harmonically varying external magnetic filed. Visual observations of magnetization reversal in samples having a small number of domains (a ferromagnetic film with three domains) have revealed that this noise is due to random shifts of the domain walls, with certain parts of the walls remaining immobile for long periods of time and then being set in motion as if they had been hit by a random jolt. Therefore, their motion can be followed by the naked eye at external field frequencies of the order of a hundred hertz. In many-domain samples the magnetization reversal will occur in the same way, since regions containing a small number of domains interact only weakly with one another and the magnetic characteristics of the whole sample are obtained by averaging over these regions.

On the basis of these observations the following mechanism has been proposed for the magnetic noise.[5,6] The magnetic state of the sample (as a whole) is characterized by a random sequence $\xi(t)$ at the times $t = \dots -2\theta, -\theta, 0, \theta, 2\theta, \dots$ when the external magnetic field $H = H_0 \cos \omega_0 t$ is maximum in absolute value ($2\theta = 2\pi/\omega_0$ is its period). To the external field we add a random magnetic field (an additive priming noise $x(t)$, which, at a given θ, can be considered white). This random field determines a value of $\xi(t)$ which is maintained until at one of the subsequent times t the random magnetic field exceeds the value that had up till then determined $\xi(t)$ and "overthrows" this value of $\xi(t)$. A new value of $\xi(t)$ is established at the next time in the sequence in accordance with the value of $x(t)$ at that time (see Sec. 2). The priming noise might be broad-band Barkhausen noise. The correctness of this mechanism is confirmed, in particular, by the fact that the most intense noise $\xi(t)$ occurs in a comparatively narrow interval of values of H_0 and falls off at smaller and larger values of H_0.[7]

Elementary considerations[5,6] lead to the conclusion that this mechanism can give a nearly $1/\omega$ spectrum (at $\omega \to 0$). The calculation presented below confirms this conclusion; it turns out that the random sequence $\xi(t)$ can be either stationary or nonstationary, depending on whether or not one allows for a spontaneous [independent of $x(t)$] overthrow of $\xi(t)$ with an exceedingly small probability p.

Since $\xi(t)$ determines the magnetic susceptibility of the sample, for a linear coupling of the magnetization and field the mechanism in question gives a noise spectrum of the type $1/\Delta\omega$ about the frequency ω_0 of the external field; for a nonlinear coupling the same will be true of the noise spectrum around the harmonics of ω_0. If $\xi(t)$ determined the magnetization, then the latter would have a $1/\omega$ spectrum concentrated at low frequencies.

It seems to us that mechanism described above has a more general significance and can in many cases give rise to

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low-frequency noise with a nearly $1/\omega$ spectrum. Therefore, in elaboratng the theory we shall not specify the nature of $\xi(t)$ and the nature of the priming noise $x(t)$.

1. SPECTRUM OF THE RANDOM SEQUENCE FORMED BY NONOVERLAPPING PULSES

Let us consider a random sequence $\xi(t)$, where t is an integral time variable ($t = 0, \pm 1, \pm 2, \ldots$), consisting of a superposition of randomly occurring pulses of a specified shape and having a random amplitude a_ν and a random duration ϑ_ν, where a_ν and ϑ_ν are statistically related to each other but statistically independent of a_μ and ϑ_μ for $\mu \neq \nu$. Such a sequence can be represented

$$\xi(t) = \sum_\nu a_\nu f\left(\frac{t-t_\nu}{\vartheta_\nu}\right), \tag{1}$$

where t_ν is the random time at which the pulse arises (different times t_ν are statistically independent); $f(y)$ is a rectangular function, equal to zero for $y < 0$ and $y \geqslant 1$ and equal to one for $0 \leqslant y < 1$; $\vartheta_\nu = t_{\nu+1} - t_\nu$ is the duration of the νth pulse, which ends when the next pulse begins. The autocorrelation function

$$R_\xi(\tau) = \overline{\xi(t)\xi(t-\tau)} = \sum_{\nu,\mu} \overline{a_\nu a_\mu f\left(\frac{t-t_\nu}{\vartheta_\nu}\right) f\left(\frac{t-t_\mu-\tau}{\vartheta_\mu}\right)} \tag{2}$$

is related to the spectral intensity $S_\xi(\omega) = S_\xi(\omega \pm 2\pi)$ by the well-known equations

$$S_\xi(\omega) = \sum_{\tau=-\infty}^{\infty} R_\xi(\tau) e^{i\omega\tau}, \quad R_\xi(\tau) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_\xi(\omega) e^{-i\omega\tau} d\omega. \tag{3}$$

Introducing the notation

$$F(\omega,\vartheta) = \sum_{t=-\infty}^{\infty} f\left(\frac{t}{\vartheta}\right) e^{i\omega t} = \sum_{t=0}^{\vartheta-1} e^{i\omega t} = \frac{1-e^{i\omega\vartheta}}{1-e^{i\omega}}, \tag{4}$$

we can write $S_\xi(\omega)$ in the form

$$S_\xi(\omega) = \frac{1}{2\pi}\sum_{\nu,\mu}\int_{-\pi}^{\pi} \overline{a_\nu a_\mu F(\tilde\omega,\vartheta_\nu) F^*(\omega,\vartheta_\mu) \exp[i\omega(t_\nu - t_\mu)] \exp[i(\tilde\omega-\omega)(t_\nu - t)]}\, d\tilde\omega. \tag{5}$$

Here the first exponential depends on the differences $t_\nu - t_\mu$, which are determined by the duration of the pulses, while the second exponential depends on the times t_ν at which the pulses appear and so can be taken out and averaged separately. As a result we obtain the series

$$\sum_{\nu=-\infty}^{\infty} \overline{e^{-i\Omega t_\nu}} = \frac{1}{\bar\vartheta}\sum_{t=-\infty}^{\infty} e^{-i\Omega t} = \frac{2\pi}{\bar\vartheta}\sum_{r=-\infty}^{\infty} \delta(\Omega - 2\pi r), \tag{6}$$

which gives a periodic delta function; $\bar\vartheta$ is the average duration of a pulse, so $1/\bar\vartheta$ is the probability for the appearance of a new pulse at some time t.

Substituting relation (6) into the integral in (5) and introducing the function

$$\varphi(\omega) = \overline{e^{-i\omega\vartheta}} = \sum_{\vartheta=1}^{\infty} P(\vartheta) e^{-i\omega\vartheta}, \tag{7}$$

where $P(\vartheta)$ is the probability that a pulse has duration ϑ, we obtain the spectral intensity of the random sequence $\xi(t)$ as

$$S_\xi(\omega) = \frac{1}{\bar\vartheta}[\mathrm{K}(\omega) + 2\,\mathrm{Re}\,\Lambda(\omega)],$$

$$\mathrm{K}(\omega) = \overline{a^2|F(\omega,\vartheta)|^2}, \quad \Lambda(\omega) = \frac{\overline{aF(\omega,\vartheta)e^{-i\omega\vartheta}}\;\overline{aF^*(\omega,\vartheta)}}{1-\varphi(\omega)}, \tag{8}$$

where the function $K(\omega)$ takes into account the intrapulse correlation ($\nu = \mu$) and the funtion $\lambda(\omega)$ takes into account the interpulse correlation ($\nu \neq m$). The summation over ν and μ is done in the same way as in the analogous problem for a random process (see, e.g., Ref. 1). The dispersion relation of the randon quantities a and ϑ can be found by specifying the mechanism which excites and quenches the pulses. This is done in the next section.

2. EXCITATION AND QUENCHING OF WHITE-NOISE PULSES

Let us suppose that the random sequence $\xi(t)$ considered above is generated by white noise $x(t)$—a stationary random sequence whose values at different (integral) times t are statistically independent. The probability that an any integral time t the value of $x(t)$ will not exceed a is given in the usual form

$$W(a) = \int_{-\infty}^{a} w(x)\,dx, \tag{9}$$

where $w(x)$ is the corresponding probability density; $w(x)dx$ is the probability that $x < x(t) < x + dx$.

We further assume that the value $\xi(t) = a_\nu$ arising at time t_ν is maintained up till the time $t_{\nu+1} - 1$ (inclusive), and at the time $t_{\nu+1}$ is replaced by the new value $a_{\nu+1}$. The time $t_{\nu+1}$ is determined by the condition $x(t_{\nu+1} - 1) > a_\nu$: the white noise, exceeding a_ν at $t = t_{\nu+1} - 1$, dumps the value $\xi(t) = a_\nu$ at $t = t_{\nu+1}$. The new value $a_{\nu+1}$ is determined by $x(t_{\nu+1})$, specifically, $\xi(t) = a_{\nu+1} = x(t_{\nu+1})$ for $t \geqslant t_{\nu+1}$. This value is again maintained up till a time $t_{\nu+2}$ such that $x(t_{\nu+2} - 1) > a_{\nu+1}$ and is replaced for $t \geqslant t_{\nu+2}$ by the value $a_{\nu+2} = x(t_{\nu+2})$ (Fig. 1).

Thus the random sequence $\xi(t)$ retains a memory of the value $x(t_\nu)$ until a higher value $x(t_{\nu+1} - 1)$ comes along and erases this memory, and afterwards $\xi(t)$ will retain a memory of the next value $x(t_{\nu+1})$. The random sequence $\xi(t)$, unlike $x(t)$, turns out to be correlated; as we shall show, its spectral intensity increases without bond as $\omega \to 0$.

In such a formulation of the problem the average value of $\xi(t)$ is nonzero; we shall call $\xi(t)$ a random sequence of the first kind (RS-1 for short). A random sequence of the second kind (RS-2) is obtained by assuming that a_ν can with equal probability take on both positive and negative values (i.e.,

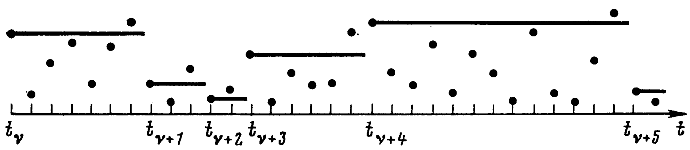

FIG. 1. Random sequence of the first kind (RS-1).

the function $w(x)$ is even), and the pulses are quenched under the condition $|x(t_{\nu+1}-1)|>|a_\nu|$, when $x(t_{\nu+1}-1)$ and a_ν have different signs (Fig. 2). Here we obviously have $\xi(t)=0$ and, as we shall see $\Lambda(\omega)=0$.

For RS-1 the probability that a pulse of amplitude a will have duration ϑ is clearly

$$P_a(\vartheta)=W^{\vartheta-1}(a)[1-W(a)], \tag{10}$$

since for this it is necessary that $x(t)\leqslant a_\nu$ for $\vartheta-1$ times and that $x(t)>a$ on the ϑ th time; the probability $P(\vartheta)$ in (7) is given by

$$P(\vartheta)=\int_{-\infty}^{\infty} w(a)P_a(\vartheta)\,da=\frac{1}{\vartheta}-\frac{1}{\vartheta+1}=\frac{1}{\vartheta(\vartheta+1)}. \tag{11}$$

This probability, which does not depend on the form of $w(x)$, yields the value $\bar{\vartheta}=\ln\infty$, and so Eqs. (6) and (8) cannot be used. This situation arises because $\xi(t)$ is not a stationary random sequence for infinite $\bar{\vartheta}$. In order to arrive at a finite $\bar{\vartheta}$ and a stationary sequence SR-1, we must have, in addition to the "induced" quenching of pulses with the aid of the priming noise $x(t)$, a "spontaneous" quenching with a small probability p which does not depend on $x(t)$. Then formulas (10) and (11) become

$$P_a(\vartheta)=[qW(a)]^{\vartheta-1}[1-qW(a)],\quad P(\vartheta)=\frac{q^{\vartheta-1}}{\vartheta}-\frac{q^{\vartheta}}{\vartheta+1}, \tag{12}$$

where $q=1-p\approx 1$ is the probability that the spontaneous quenching does not occur. We then get

$$\bar{\vartheta}=\frac{1}{q}\ln\frac{1}{p},\quad \overline{\vartheta^2}=\frac{2}{p}-\frac{1}{q}\ln\frac{1}{p},$$
$$\varphi(\omega)=1-\frac{1}{q}(1-e^{i\omega})\ln(1-qe^{-i\omega}). \tag{13}$$

For SR-2 the probability $P_a(\vartheta)$ is obtained from the first formula in (12) by replacing $W(a)$ with $W(|a|)$, and, since $w(a)$ is even and $W(0)=1/2$, the expression for $P(\vartheta)$ is

$$P(\vartheta)=2\left[\frac{q^{\vartheta-1}}{\vartheta}\left(1-\frac{1}{2^{\vartheta}}\right)-\frac{q^{\vartheta}}{\vartheta+1}\left(1-\frac{1}{2^{\vartheta+1}}\right)\right], \tag{14}$$

so that

$$\bar{\vartheta}=\frac{2}{q}\ln\frac{1+p}{2p},\quad \overline{\vartheta^2}=\frac{4}{p(1+p)}-\frac{2}{q}\ln\frac{1+p}{2p},$$
$$\varphi(\omega)=1-\frac{2}{q}(1-e^{i\omega})\left[\ln(e^{i\omega}-q)-\ln\left(e^{i\omega}-\frac{q}{2}\right)\right]. \tag{15}$$

The average values which figure in (8) must be evaluated using the relation

$$\overline{\Phi_1(a)\Phi_2(\vartheta)}=\int_{-\infty}^{\infty} w(a)\Phi_1(a)\,da\sum_{\vartheta=1}^{\infty}P_a(\vartheta)\Phi_2(\vartheta),$$

and on summing over ϑ we get

$$\sum_{\vartheta}P_a(\vartheta)|F(\omega,\vartheta)|^2=\frac{1}{1-\cos\omega}\operatorname{Re}\frac{e^{i\omega}-1}{e^{i\omega}-qW},$$
$$\sum_{\vartheta}P_a(\vartheta)F(\omega,\vartheta)e^{-i\omega\vartheta}=\frac{1}{e^{i\omega}-qW},$$
$$\sum_{\vartheta}P_a(\vartheta)F^*(\omega,\vartheta)=\frac{e^{i\omega}}{e^{i\omega}-qW},$$

where $W=W(a)$ for SR-1 and $W=W(|a|)$ for SR-2. Introducing the notation

$$J_k(\omega)=\int_{-\infty}^{\infty}\frac{w(a)a^k\,da}{e^{i\omega}-qW(a)}, \tag{16}$$

where therefore get for SR-1

$$K(\omega)=\frac{1}{1-\cos\omega}\operatorname{Re}[(e^{i\omega}-1)J_2(\omega)],$$
$$\Lambda(\omega)=\frac{e^{i\omega}}{1-\varphi(\omega)}J_1^2(\omega), \tag{17}$$

while for SR-2 we must replace $W(a)$ by $W(|a|)$ in integrals (16). Thus the problem of evaluating the spectrum of $\xi(t)$ with formula (8) has reduced to the problem of evaluating integrals of the form (16) for $k=1$ and 2; these integrals depend both on the frequency and on the form of the function $w(a)$.

3. SPECTRAL INTENSITY AT $\omega\to 0$

In evaluating the integrals $J_1(\omega)$ and $J_2(\omega)$ for SR-1 one must recognize that $W(a)$ is a monotonic function of a which goes from $W(-\infty)=0$ to $W(\infty)=1$; therefore, the inverse

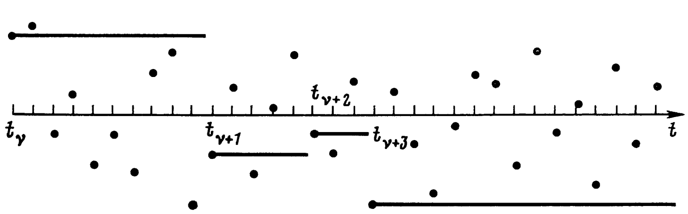

FIG. 2. Random sequence of the second kind (RS-2).

function $a = a(W)$ is single-valued, and the integral $J_k(\omega)$ can be written

$$J_k(\omega)=\int_0^1 \frac{a^k(W)\,dW}{e^{i\omega}-qW}, \tag{18}$$

and if a varies over finite limits, i.e., if $b_0 \leqslant a \leqslant b_1$, where $b_0 = a(0)$ and $b_1 = a(1)$, then an integration by parts easily yields the expression

$$J_k(\omega)=-\frac{b_1^k}{q}\ln(e^{i\omega}-q)+\frac{b_0^k}{q}i\omega + \frac{1}{q}\int_0^1 \ln(e^{i\omega}-qW)\frac{da^k(W)}{dW}dW,$$

which is convenient for evaluating $J_k(\omega)$ at $\omega \ll 1$ and $p \ll 1$. Setting

$$p+i\omega=(p^2+\omega^2)^{1/2}e^{i\psi}, \quad \psi=\mathrm{arctg}(\omega/p), \quad 0<\psi<\pi/2 \tag{19}$$

and introducing the large logarithm

$$L=\ln\frac{1}{(p^2+\omega^2)^{1/2}}, \tag{20}$$

we get in the low-frequency limit

$$J_k(\omega)=b_1^k(L-i\psi)-I_k, \quad I_k=-\int_0^1 \ln(1-W)\frac{da^k(W)}{dW}dW,$$

where I_k is a real number which depends on the dispersion relation $W(a)$. The spectral intensity of SR-1 is of the following form for $\omega \ll 1$ and $p \ll 1$:

$$S_\xi(\omega)=\frac{2I_1^2}{\bar{\vartheta}}\frac{\psi}{\omega(L^2+\psi^2)}\approx\frac{\pi I_1^2}{\bar{\vartheta}}\frac{1}{\omega(\ln^2\omega+\pi^2/4)}, \quad \text{if} \quad \omega\gg p. \tag{21}$$

For the SR-2 we easily see that $J_1(\omega) = 0$ and $\Lambda(\omega) = 0$, while the integral $J_2(\omega)$ can be written

$$J_2(\omega)=2\int_{1/2}^1 \frac{a^2(W)\,dW}{e^{i\omega}-qW} =-\frac{2}{q}b_1^2\ln(e^{i\omega}-q)+\frac{2}{q}\int_{1/2}^1 \ln(e^{i\omega}-qW)\frac{da^k(W)}{dW}dW,$$

from which, for $\omega \ll 1$ and $p \ll 1$, we get

$$J_2(\omega)=-2b_1^2\ln(p+i\omega)-I_2,$$

$$I_2=-2\int_{1/2}^1 \ln(1-W)\frac{da^2(W)}{dW}dW$$

and

$$S_\xi(\omega)=\frac{2b_1^2}{\bar{\vartheta}}\frac{\psi}{\omega}\approx\frac{\pi b_1^2}{\bar{\vartheta}}\frac{1}{\omega}, \quad \text{if} \quad \omega\gg p. \tag{22}$$

In these relations it is assumed that the white noise $x(t)$ lies withing finite limits, so that $b_0 \leqslant a \leqslant b_1$ for SR-1) and $-b_1 \leqslant a \leqslant b_1$ for SR-2.

The results, specifically (21) and (22), can be summed up by the following remarks. We have introduced a small probability p which makes the random sequence $\xi(t)$ stationary and the average duration ϑ finite; for example, for SR-1 we have $\bar{\vartheta} \approx 23$ at $p = 10^{-10}$. The nature and size of p remain open questions; evidently p can be due to different causes in different systems. Nevertheless, it is important that for $p \ll \omega \ll 1$ the spectrum of SR-1 is nearly proportional to $1/\omega$ and the spectrum of SR2 is exactly proportional to $1/\omega$.

Even though the distinction between the spectra of SR-1 and SR-2 is barely perceptible at low frequencies on account of the slow change in $\ln \omega$, the integrals

$$\int_0^\pi S_\xi(\omega)\,d\omega$$

behave differently at $p \to 0$: for SR-1 the integral goes to zero [since the factor $1/(\ln^2\omega + \pi^2/4)$ ensures the convergence of the integral, and $\bar{\vartheta} \to \infty$], while for SR-2 the integral approaches a finite limit determined by the value of $\overline{\xi^2(t)}$. The difference is due to the fact that SR-1 is a fixed-sign sequence (see Fig. 1), and so for it $\overline{\xi(t)} > 0$ and the spectrum contains another term proportional to $\delta(\omega)$, whereas for SR-2 we have $\overline{\xi(t)} = 0$ and there is no such term. In both cases $S_\xi(\omega)$ goes to finite limits for $\omega \ll p$:

$$S_\xi(0)=2I_1^2/(\bar{\vartheta})^3p \quad \text{(SR-1)},$$
$$S_\xi(0)=2b_1^2/\bar{\vartheta}p \quad \text{(SR-2)}. \tag{23}$$

We have obtained all these results by assuming that the duration ϑ of a pulse depends on its amplitude a. If, on the other hand, we assume that a and ϑ are independent random quantitiies and that the probability $P(\vartheta)$ is, as before, taken in the form (12) or (14), then for SR-1 we obtain the simple expression

$$S_\xi(\omega)=\frac{\overline{a^2}-(\bar{a})^2}{\bar{\vartheta}}\frac{1-\mathrm{Re}\,\varphi(\omega)}{1-\cos\omega}, \tag{24}$$

which can also be used when a varies over an infinite interval. At small p and ω expression (24) becomes

$$S_\xi(\omega)=\frac{\overline{a^2}-(\bar{a})^2}{\bar{\vartheta}}\frac{2\psi}{\omega}\approx\frac{\overline{a^2}-(\bar{a})^2}{\bar{\vartheta}}\frac{\pi}{\omega}, \quad \text{if} \quad \omega\gg p, \tag{25}$$

and here

$$S_\xi(0)=\frac{\overline{a^2}-(\bar{a})^2}{\bar{\vartheta}}\frac{2}{p}. \tag{26}$$

Setting $\bar{a} = 0$ in (24) and taking $\varphi(\omega)$ according to (15), we obtain $S_\xi(\omega)$ for SR-2; expressions (25) and (26) must be doubled for SR-2. We see that a $1/\omega$ law is obtained in this case also, but then $P(\vartheta)$ must be introduced in a purely formal way.

According to (21), the function

$$s(\omega)=\frac{1}{\omega(\ln^2\omega+\pi^2/4)} \tag{27}$$

determines the spectrum of SR-1; this function differs little from $1/\omega$. The spectrum $S_\xi(\omega)$ is often represented (in separate regions) in the form B/ω^λ, where the exponent.

$$\lambda=-d\ln S_\xi(\omega)/d\ln\omega$$

is a slowly varying function of ω. For the function in (27) we have $\lambda < 1$, with $\lambda \to 1$ at $\omega \to 0$. The correlation function corresponding to spectrum (27) falls off at $\tau \to \infty$ as $1/\ln(\gamma\tau)$, $\gamma = 1.781...$(see Ref. 7).

4. SPECTRAL INTENSITY AT FINITE ω

The behavior of the spectral intensity over the entire interval $0 < \omega \leqslant \pi$ depends on the probability density $w(a)$. To discover this dependence, let us take the functions $w(a)$ and $W(a)$ for $0 \leqslant a \leqslant 1$ in the form

$$w(a)=\frac{(1+\rho)\rho}{(\rho+a)^2},\quad W(a)=\frac{(1+\rho)a}{\rho+a},\quad a(W)=\frac{\rho W}{1+\rho-W}, \tag{28}$$

where ρ is a real parameter ($\rho > 0$ or $\rho < -1$). For $\rho \ll 1$ small values of a are the most probable, for $\rho \approx -1$ the most probable values are close to unity, and for $\rho = \infty$ the probability density $w(a)$ becomes uniform in the interval $0 \leqslant a \leqslant 1$ (Fig. 3). For the functions in (28) the integrals in (18) are easily evaluated, and we get

$$\begin{aligned} J_1(\omega) &= -\frac{\rho}{q(1+\rho)-e^{i\omega}}\left[I_1+\frac{e^{i\omega}}{q}\ln(1-qe^{-i\omega})\right],\\ I_1 &= (1+\rho)\ln\frac{1+\rho}{\rho},\\ J_2(\omega) &= \frac{\rho^2}{[q(1+\rho)-e^{i\omega}]^2}\Big\{[q(1+\rho)-2e^{i\omega}]I_1\\ &\quad -\frac{1+\rho}{\rho}[q(1+\rho)-e^{i\omega}]-\frac{e^{2i\omega}}{q}\ln(1-qe^{-i\omega})\Big\}. \end{aligned} \tag{29}$$

The expression for I_1 shows that the spectral intensity (21) for $\rho \ll 1$ is proportional to $\ln^2(1/\rho)$, i.e., gets larger as the probabilities of comparatively large overshoots gets smaller, while for $\rho \approx -1$ the value of I_1^2 is small. These expressions enable one to evaluate $S_\xi(\omega)$ for SR-1. To eliminate the parameter p, we introduce the normalized function $s(\omega) = \bar{\vartheta} S_\xi(\omega)/\pi I_1^2$ and, assuming $\omega \gg p$ and $p \ll 1$, we obtain $p = 0$ and $q = 1$.

Figure 4 shows graphs of the normalized functions $s(\omega)$ evaluated by the exact formulas (8), (17), and (29); at small ω they coincide with the low-frequency asymptote (27). Figure 4 shows that at finite value of ω the exponent λ can be larger or smaller than one (depending on the parameter ρ) and changes rather slowly with frequency. Figure 4 also shows

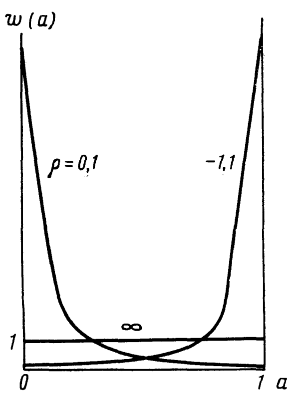

FIG. 3. The function $w(a)$ according to Eq. (28).

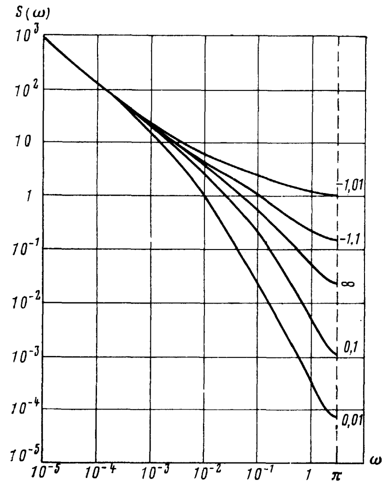

FIG. 4. Normalized spectral intensities for SR-1 at various values of ρ.

that a function $w(a)$ for which small values of a are more probable leads to a relatively fast decay of $S_\xi(\omega)$ as $\omega \to \pi$. This result, like the large values of $S_\xi(\omega)$ for $\omega \ll 1$, is explained by the fact that such a function $w(a)$ leads to prolonged overshoots of $\xi(t)$.

For SR-2 we give the probability density as

$$w(a)=(1+\rho)\rho/2(\rho+|a|)^2,\quad -1 \leqslant a \leqslant 1, \tag{30}$$

and then the integral $J_2(\omega)$ can also be expressed in terms of elementary functions. Figure 5 shows the normalized functions $s(\omega = 4\bar{\vartheta} S_\xi(\omega)/\pi b_1^2$ evaluated at $p = 0$. For $\omega \to 0$ they approach the $1/\omega$ asymptote from below, unlike the case of Fig. 4, where the curves approach the asymptotic curve (27) from both sides.

The functions we have obtained not only give a $1/\omega$ spectrum but also establish a quantitative relation between this spectrum and that of the priming noise. Formulas (8), (13), (15), (16), and (17) imply that for $\omega = \pi$ the spectral

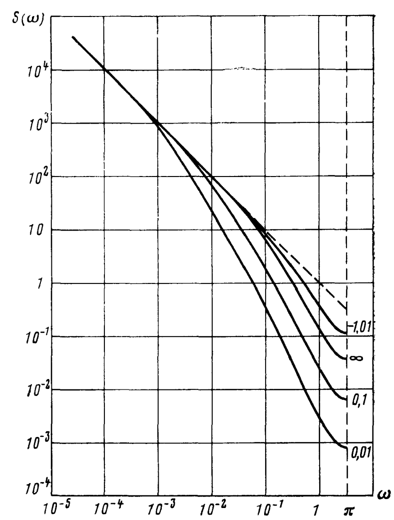

FIG. 5. Normalized spectral intensities for SR-2 at various values of ρ.

density of $1/\omega$ noise satisfies the equations (for $q\sim 1$)

$$\bar{\vartheta}S_\xi(\pi)=\int_{-\infty}^{\infty}\frac{a^2w(a)\,da}{1+W(a)}-\frac{1}{\ln 2}\Big[\int_{-\infty}^{\infty}\frac{aw(a)\,da}{1+W(a)}\Big]^2\quad\text{for RS-1},$$

$$\bar{\vartheta}S_\xi(\pi)=\int_{-\infty}^{\infty}\frac{a^2w(a)\,da}{1+W(|a|)}\quad\text{for RS-2},$$

the right-hand side of which to within a factor of order one coincide with the constant spectral density $S_x = a^2 - (a)^2$ of the priming noise. For example, in the case of a uniform distribution [Eqs. (28) and (30) for $\rho = \infty$] the right-hand side is equal to $0.69S_x$ for SR-1 and $0.54S_x$ for SR-2. Thus the frequency interval in which the $1/\omega$ noise is noticeable is determined primarily by $\bar{\vartheta}$, which depends on the spontaneous-overthrow probability p [Eq. (13)]; for any reasonable values of p the quantity $\bar{\vartheta}$ is of the order of tens. Consequently, the $1/\omega$ noise will exceed the priming background at frequencies which differ from the harmonics of the magnetization-reversal frequency ω_0 by a quantity of the order of 0.1 ω_0 or less. It is just such values that have been observed experimentally.[7]

CONCLUSION

To the best of our knowledge, noise with a $1/\omega$ spectrum or nearly $1/\omega$ spectrum has never been considered in terms of the mathematical theory of random processes and sequences. In the physics papers such noise was first interpreted as a pulse process with pulses decaying as $t^{-1/2}$ at $t\to\infty$.[9] Subsequently, since pulses of this shape had not been observed in physical systems, the $1/\omega$ noise began to be regarded as either the superposition of relaxation processes with different relaxation times[10] or as a pulse process with pulses of fixed shape but of different durations and amplitudes.[11] The main difficulty with these interpretations is to find a physical justification for the requisite relaxation-time or pulse-length distributions that would give a $1/\omega$ spectrum.

The above analysis of magnetic noise shows that in systems with a memory a suitable distribution [see Eq. (11) and following] arises in a natural way. The mechanism we have considered that leads to the $1/\omega$ spectrum is "coarse": a spectrum proportional to $1/\omega$ at $\omega\to 0$ is obtained regardless of the distribution of the priming noise and the particular method of establishing and overthrowing pulses.

Another extremely interesting question is, what other mechanisms besides a memory can give the required pulse-length distribution for the generation of $1/\omega$ noise? One such mechansim, found in a recent paper,[12] is a random sequence $\xi(t)$ consisting of zero and one and approximating (for low frequencies) an iteration process in which a regular change [slow, $\xi(t)=0$] alternates with a stochastic change [fast, $\xi(t)=1$]. If $P(l)$ is the probability of having ϑ zeros in a row (which for $\vartheta\to\infty$ is prorportional to $1/\vartheta^2$), one obtains a spectrum for $\omega \ll 1$ that is proportional to the function (27). The random sequences $\xi(t)$ and $1-\xi(t)$ have identical spectra for $\omega\neq 0$, and for the second random sequence $P(\vartheta)$ refers to the length of rectangular pulses separated by empty spaces of random length. Be that as it may, the function $P(\vartheta)$ which leads to the $1/\omega$ spectrum has the same asymptotic expression at $\vartheta\to\infty$ as in the above theory of magnetic noise.

The theory elaborated in the present article can be undoubtedly be generalized to random processes. In this case the $1/\omega$ spectrum would follow from a probability density $P(\vartheta) = 2\vartheta_0/\pi(\vartheta_0^2+\vartheta^2)$ for $\omega\vartheta_0 \ll 1$. It should be noted that this expression is meaningful at all values of ϑ and is integrable, whereas the probability density adopted in the traditional spectral approach[10] is proportional to $1/\vartheta$ and cannot be used at $\vartheta\to 0$ and $\vartheta\to\infty$. In our approach $S_\xi(\omega)$ is rendered integrable at $\omega = 0$ by introducing a factor $\exp(-p\vartheta/\vartheta_0)$, in $P(\vartheta)$, with $p \ll 1$. Moreover, if the pulse duration is related to the activation energy E by the relation $\vartheta = \vartheta_0\text{-}\exp(E/kT)$, then for $P(\vartheta)\sim 1/\vartheta^2$ the probability density for the energy E is proportional to $\exp(-E/kT)$, which is more sensible physically than a uniform energy distribution.[10]

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