The limit of stochasticity for a one-dimensional chain of interacting oscillators

G. P. Berman and A. R. Kolovskiĭ

L. V. Kirenskiĭ Physical Institute, Siberian Division of the Academy of Sciences of the USSR

(Submitted 28 March 1984)
Zh. Eksp. Teor. Fiz. **87**, 1938–1947 (December 1984)

We consider the problem of the stability of a nonlinear one-dimensional chain of coupled oscillators. We give an estimate of the limit of stochasticity as function of the parameters of the system. We show that below the limit of stochasticity the system can be described approximately by the nonlinear Schrödinger equation. We give a numerical comparison of the chain dynamics and of the nonlinear Schrödinger equation.

1. INTRODUCTION

At the present there is considerable progress in the study of the conditions for the occurrence of stochasticity of motion in Hamiltonian systems with a finite number of degrees of freedom. As there are, however, no rigorous analytical methods for the determination of a criterion for stochasticity for such systems (an exception are some systems of the Sinaĭ billiard type) normally the limit of the transition from regular to random motion is found by the use of various kinds of estimating considerations with the subsequent inclusion of a numerical experiment. Basically all existing criteria for the transition to chaos in Hamiltonian systems are based upon the estimates of parameters for which bifurcations occur which are connected with the destruction of various kinds of stable motions.[1–6] One of the most widely used criteria for the transition to chaos is the criterion for the overlap of nonlinear resonances.[7] This criterion corresponds to the destruction of the quasi-periodic motion of the system (i.e., the destruction of nontrivial integrals of motion) due to the "tangency" of the separatrix of the different nonlinear resonances. This leads to the locally unstable nature of the motion. Numerical calculations show that such a kind of motion is stochastic.[7] In Ref. 5 the criterion for the stochasticity for a Hamiltonian system with two degrees of freedom was determined from the conditions for the loss of global stability of the regular motion. In Ref. 6 an estimate of the parameters for which the transition to chaos occurs in a chain of nonlinearly coupled oscillator was determined from the condition for the loss of stability of a particular solution in the form of antiphased nonlinear oscillations. However, all these criteria of stochasticity of this kind possess, of course, a well known defect—the loss of stability of some regular motion may lead not to the occurrence of chaotic motion but to the transition to another (possibly, more complicated) kind of regular motion. As an example of systems to which such kind of criterion is inapplicable (although formally they can be used) we have the instances of completely integrable systems. Nonetheless for physical systems with a finite number of degrees of freedom the criteria used combined with numerical experiments turn out in many cases to be useful for the study of the features of the transition from regular to chaotic motion. When the number of degrees of freedom is increased the difficulties connected with the study of the conditions for the transition to chaos in nonlinear dynamical systems appreciably increases. In that case there occur in the system new collective kinds of motion which may lead to a change in the nature itself of such a transition.

The present paper is devoted to a study of the stability properties of the motion of a one-dimensional chain of coupled nonlinear oscillators with the Hamiltonian

$$H=\sum_{n=0}^{N-1}\left[\frac{p_n^2}{2}+\frac{1}{2}(u_{n+1}-u_n)^2+\frac{\beta}{4}(u_{n+1}-u_n)^4\right], \tag{1.1}$$

where p_n and u_n are the momentum and displacement of the nth oscillator, N the number of oscillators, and β the nonlinearity parameter. The system (1.1) is apparently the simplest physical model which is convenient for a study of the features of the transition from a stable to a chaotic motion for different values of the wavelengths which are excited, the nonlinearity, and the number N of degrees of freedom. Fermi, Pasta, and Ulam[8] started the numerical study of the stability of the system (1.1) in connection with an investigation of the conditions for the applicability of the statistical approach for the description of such kinds of systems. The numerical analysis of Ref. 8 led to a rather unexpected result: instead of the expected chaotic exchange of energies between the modes, the motion of the system (1.1) turned out to be almost periodic. Further analytical and numerical studies of the system (1.1) led to the establishment of the existence of a stochasticity limit separating the regions of stable and random motions.[1–4] The stochasticity limit was determined in Refs. 1 to 3 using the criterion for the overlap of nonlinear resonances. The negative result obtained in Ref. 8 was thus explained by the choice of the parameters of the system (1.1) corresponding to the stable regime of motions.

In our opinion, one can use as one of the conditions for estimates of the stochasticity limit the condition for the destruction of regular collective motions which are realized in such systems in well-defined limiting cases. In particular, such integrable limiting cases for the system (1.1) are the following: the long-wavelength approximation which under well defined conditions leads to a KdV type of equation,[3] and the "narrow packet" approximation $\delta k/k_0 \ll 1$, where k_0 is a characteristic wave number in the packet and δk the characteristic size of the packet in k.

In what follows we present an investigation of the sta-

 0038-5646/84/121116-06$04.00

bility properties of the system (1.1) with periodic boundary conditions in the case where high modes are initially excited. We show that under well defined conditions the narrow packet approximation is realized in the system (1.1) which leads to the nonlinear Schrödinger equation (NLS). It is well known that the latter is a completely integrable system.[9] We give a numerical comparative study of the dynamics of the NLS and of the nonlinear chain (1.1). We show that if the conditions for the narrow packet approximation are satisfied we have good agreement for the nonlinear chain of the well defined integrals of motion of the NLS equation the number of which can be enhanced due to the refinement of the conditions for the applicability of the narrow-packet approximation (e.g., through the increase in the number of oscillators N while leaving the other parameters of the system unchanged). The existence of such additional integrals means for the chain the presence of a well defined fraction of the regular component of the motion.

We give an estimate for the conditions for the applicability of the narrow-packet approximation and discuss the possibility of using it for an estimate of the stochasticity limit for the system (1.1). Such an estimate corresponds to the criterion for the destruction of regular motions described by the NLS equation. We show that the violation of the conditions for the applicability of the narrow-packet approximation in fact corresponds to the excitation of all modes of the chain. We give a comparison of the criterion for the violation of the narrow-packet approximation with the criterion for stochasticity obtained from the condition of overlap of nonlinear resonances and with the criterion for the loss of stability of antiphase motions.[6]

2. THE NARROW-PACKET APPROXIMATION

We consider the case with periodic boundary conditions: $u_0 = u_N, p_0 = p_N$ and change in (1.1) to the canonical variables a_k, a_k^*:

$$a_k=(2\omega_k)^{-1/2}(P_k-i\omega_k U_k^*),$$
$$U_k=\frac{1}{N^{1/2}}\sum_{n=0}^{N-1} u_n \exp\left(-i\frac{2\pi k}{N}n\right)=\dot{U}_{N-k},$$
$$P_k=\frac{1}{N^{1/2}}\sum_{n=0}^{N-1} p_n \exp\left(i\frac{2\pi k}{N}n\right)=\dot{P}_{N-k},$$
$$\omega_k=2\sin(\pi k/N),\quad 0\leq k\leq N-1. \tag{2.1}$$

In (2.1) $p_0 = 0$. AS we shall in what follows consider the narrow packet case:

$$\xi=\delta k/k_0\ll 1, \tag{2.2}$$

we write the Hamiltonian (1.1) in the variables a_k, a_k^* in the form

$$H=\sum_{k=0}^{N-1}\omega_k a_k a_k^* +\frac{1}{2}\sum_{k_{1-4}=0}^{N-1} V_{k_1k_2k_3k_4} a_{k_1}^* a_{k_2}^* a_{k_3} a_{k_4}\delta(k_1+k_2-k_3-k_4)$$
$$+O(a_1^* a_2 a_3 a_4+\text{c. c.};\ a_1a_2a_3a_4+\text{c.c.}), \tag{2.3}$$

where

$$V_{k_1k_2k_3k_4}=\frac{3\beta}{N}\left(\sin\frac{\pi k_1}{N}\sin\frac{\pi k_2}{N}\sin\frac{\pi k_3}{N}\sin\frac{\pi k_4}{N}\right)^{1/2}. \tag{2.4}$$

The terms $a_1^* a_2^* a_3 a_4$ in (2.3) describe resonance four-wave decay processes and under the condition (2.2) they are the decisive ones, while by O we indicate terms which in this approximation can be omitted (we discuss below the conditions for the applicability of the approximation (2.2)). In the same approximation we can considerably simplify the exact equations of motion

$$i\dot{a}_k=\delta H/\delta a_k^*,\quad i\dot{a}_k^*=-\delta H/\delta a_k. \tag{2.5}$$

We expand ω_k and $V_{k_1k_2k_3k_4}$ in the point k_0:

$$\omega_k=\omega_{k_0}+\lambda q-\Omega q^2,\quad q=k-k_0\ (|q|\ll k_0),$$
$$V_{k_1k_2k_3k_4}=V_0+W(q_1+q_2+q_3+q_4),$$
$$\lambda=2\frac{\pi}{N}\cos\frac{\pi k_0}{N},\quad \Omega=\frac{\pi^2}{N^2}\sin\frac{\pi k_0}{N},$$
$$V_0=\frac{3\beta}{N}\sin^2\frac{\pi k_0}{N},\quad W=\frac{3\pi\beta}{4N^2}\sin\frac{2\pi k_0}{N}. \tag{2.6}$$

Substituting (2.6) into (2.3) we get from (2.5)

$$i\dot{A}_q=-\Omega q^2 A_q+V_0\sum_{q_1q_2q_3}\dot{A}_{q_1}A_{q_2}A_{q_3}\delta(q+q_1-q_2-q_3),$$
$$A_q\equiv A_{k-k_0}=\exp[i(\omega_{k_0}+\lambda q)t]a_k. \tag{2.7}$$

Equations (2.7) represent the nonlinear chain of δq interacting oscillators. We get for the function

$$\Phi(\theta,t)=\sum_q A_q(t)e^{iq\theta}=\Phi(\theta+2\pi,t) \tag{2.8}$$

from (2.7) the NLS equation:

$$i\partial\Phi/\partial t=\Omega\partial^2\Phi/\partial\theta^2+V_0|\Phi|^2\Phi. \tag{2.9}$$

To facilitate the further analysis we introduce a dimensionless time $\tau = \Omega t$ and normalize the function $\Phi(\theta,t)$ to the integral of motion of Eq. (2.9):

$$I=\frac{1}{2\pi}\int_0^{2\pi}|\Phi(\theta,t)|^2 d\theta. \tag{2.10}$$

For the function

$$\psi(\theta,\tau)=\Phi(\theta,t)/I^{1/2} \tag{a}$$

we get from (2.9), (2.10)

$$i\partial\psi/\partial\tau=\partial^2\psi/\partial\theta^2+\varepsilon|\psi|^2\psi,\quad \psi(\theta+2\pi,\tau)=\psi(\theta,\tau),$$
$$\frac{1}{2\pi}\int_0^{2\pi}|\psi|^2 d\theta=1,\quad \varepsilon=\frac{V_0 I}{\Omega}. \tag{2.11}$$

We note that since the function ψ is normalized to unity, the parameter ε cannot be eliminated from Eq. (2.11). From (2.7) we get an equation for the Fourier amplitude $c_q = A_q/I^{1/2}$ of the function ψ:

$$i\frac{dc_q}{d\tau}=-q^2c_q+\varepsilon\sum_{q_1q_2q_3}c_{q_1}c_{q_2}c_{q_3}\delta(q+q_1-q_2-q_3). \qquad (2.12)$$

It is well known that the NLS Eq. (2.11), (2.12) is a completely integrable system which has an infinite number of integrals of motion.[9] We give a few of the first integrals:

$$I_1=\frac{1}{2\pi}\int_0^{2\pi}d\theta|\psi|^2=1,\qquad I_2=\frac{1}{2\pi}\int_0^{2\pi}d\theta\psi^*\frac{\partial\psi}{\partial\theta},$$

$$I_3=\frac{1}{2\pi}\int_0^{2\pi}d\theta\left(-\left|\frac{\partial\psi}{\partial\theta}\right|^2+\frac{\varepsilon}{2}|\psi|^4\right),$$

$$I_4=\frac{1}{2\pi}\int_0^{2\pi}d\theta\left(\psi^*\frac{\partial^3\psi}{\partial\theta^3}+2\varepsilon|\psi|^2\psi\frac{\partial\psi^*}{\partial\theta}+\frac{\varepsilon}{2}|\psi|^2\psi^*\frac{\partial\psi}{\partial\theta}\right). \qquad (2.13)$$

At the same time it is well known that Eq. (2.11) describes four-wave decay processes.[10] Indeed, Eq. (2.12) has exact solutions in the form of a finite amplitude wave:

$$c_q(\tau)=\exp[i(q^2-\varepsilon)\tau]\delta(q-q_0). \qquad (2.14)$$

We study the stability of the solution (2.14) with respect to the decay $2q_0\to(q_0+p)+(q_0-p)$. Let initially the following conditions be satisfied:

$$|c_{q_0+p}(0)|^2\sim|c_{q_0-p}(0)|^2\ll|c_{q_0}(0)|^2\approx1.$$

Linearizing Eq. (2.12) with respect to $c_{q_0\pm p}$, we get the set of equations

$$idc_{q_0+p}/d\tau=[\varepsilon-(q_0+p)^2]c_{q_0+p}+\varepsilon c_{q_0}^2\dot{c}_{q_0-p},$$

$$idc_{q_0-p}/d\tau=[\varepsilon-(q_0-p)^2]c_{q_0-p}+\varepsilon c_{q_0}^2\dot{c}_{q_0+p}, \qquad (2.15)$$

where according to (2.14) $c_{q_0}(\tau)\approx\exp[i(q_0^2-\varepsilon)\tau]$. It follows from (2.15) that $c_{q_0\pm p}\propto\exp(\nu_p\tau)$ where the instability growth rate has the form

$$\nu_p=|p|(2\varepsilon-p^2)^{1/2}. \qquad (2.16)$$

We note that the decay instability in the narrow packet approximation for small $|q|$ is analogous to the "negative pressure" kind of instability.[10] From (2.14) there follows a condition for the decay realization

$$2\varepsilon>p^2\quad(p=1,2,\ldots). \qquad (2.17)$$

3. CONDITION FOR APPLICABILITY OF THE NARROW PACKET APPROXIMATION. STOCHASTICITY LIMIT

Equation (2.11) was obtained in the approximation $\delta q_{\max}/k_0\ll1$, where $\delta q_{\max}$ is the maximum effective number of excited modes ($\delta q(\tau)\leqslant\delta q_{\max}$).

We obtain from the energy balance an estimate of how $\delta q_{\max}$ depends on ε. To do this we write the Hamiltonian I_3, (2.13), of the system (2.11), (2.12) in the form

$$I_3=-\sum_q q^2|c_q|^2+\frac{\varepsilon}{2}\sum_l\left|\sum_q\dot{c}_{q-l}c_q\right|^2. \qquad (3.1)$$

For the sake of simplicity we consider the case when for $\tau=0$ in practice a single mode with $q=0$ is excited. We then have from (3.1) $I_3\approx\varepsilon/2$ at any time. Using the condition that the function $\psi(\theta,\tau)$ be normalized to unity, we get an estimate for the sums in (3.1)

$$\sum_q q^2|c_q|^2\leqslant(\delta q_{max})^2,\qquad\sum_l\leqslant\delta q_{max}.$$

Hence we get the estimate

$$\delta q_{max}\sim\varepsilon, \qquad (3.2)$$

which satisfactorily agrees with the results of a numerical test (see Sec. 4). The condition for the applicability of the narrow packet approximation for the chain (1.1) can thus be written in the form

$$\xi=\delta q_{max}/k_0\sim\varepsilon/k_0=3\beta E_{k_0}N/2\pi^2k_0\ll1, \qquad (3.3)$$

where we have used Eqs. (2.6), (2.11) and instead of I in the definition of the parameter ε have introduced the quantity $E_{k_0}=\omega_{k_0}I$ which characterizes some characteristic energy in the packet.

In sense indicated in the Introduction, the violation of the condition for the applicability of the narrow-packet approximation (3.3) can be considered as the condition for the stochastization of the system (1.1). This gives an estimate of the stochasticity limit ($\xi\sim1$)

$$\beta=\beta_0=2\pi^2k_0/3E_{k_0}N. \qquad (3.4)$$

The numerical analysis shows (see also Refs. 2 and 4) that in the parameter range (3.4) there occurs a stochastic excitation of practically all modes. We shall give in what follows an additional discussion of the estimate (3.4) for the stochasticity limit to compare it with the numerical analysis of the system (1.1).

4. PECULIARITIES OF THE DYNAMICS IN THE NARROW PACKET APPROXIMATION

In this section we discuss some characteristic features of the behavior of the system (1.1) in the case when the conditions for the narrowness of the packet are satisfied: $\xi\ll1$ ($\beta\ll\beta_0$) and the dynamics of the chain can with a good degree of accuracy be described by Eqs. (2.11), (2.12). We want here to draw attention to the following two facts.

The system (2.11), (2.12) is an example of a nonlinear system which, being completely integrable, turns out to be unstable under the condition $\varepsilon>1/2$. Applied to the chain (1.1) condition (2.17) means

$$\beta>\beta_R=\pi^2/3E_{k_0}N. \qquad (4.1)$$

This instability is observed in a numerical experiment with (1.1) in the parameter range

$$\beta_R<\beta\ll\beta_0 \qquad (4.2)$$

and can erroneously be interpreted as a local instability of the stochastic motion (see, e.g., Ref. 6), although in actual fact the motion takes place in the region where the narrow packet is applicable.

It is also well known that for completely integrable sys-

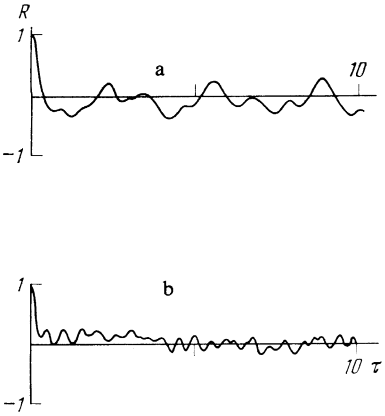

FIG. 1. Form of the autocorrelation function of mode occupations $|c_q(\tau)|^2$ for the NLS equation, $q=0$: a—$\varepsilon=5$, b—$\varepsilon=10$.

tems an appreciable decrease of the temporal correlation functions may be connected with the increase in the number of degrees of freedom involved in the dynamics. This leads to the fact that in the numerical analysis of such systems it turns out to be rather complicated to distinguish the dynamic motion from the stochastic motion using the shape of the temporal correlation functions.[11] In the case of the system (2.12) the damping of the temporal correlations with time must be amplified even more with an increase in δq due to the instability of the motion in the parameter range (4.2). We give in Fig. 1 the form of the temporal auto-correlations of the mode populations $|c_q(\tau)|^2$ for the system (2.12),

$$R_q(\tau)=\frac{\langle|c_q(\tau+\tau')|^2|c_q(\tau')|^2\rangle-\langle|c_q(\tau')|^2\rangle^2}{\langle|c_q(\tau')|^4\rangle-\langle|c_q(\tau')|^2\rangle^2}, \qquad (4.3)$$

for different values of the parameter ε; $\langle\ldots\rangle$ indicates averaging over the time τ'. It is clear from Fig. 1 that with increasing ε (increasing the width of the packet) the damping of the correlation functions is enhanced.

With the aim of checking the ε-dependence (3.2) of the width of the packet δq we integrated the system (2.12) numerically for different values of the parameter ε. A control on the accuracy of the calculations was given by the condition that the normalization I_1 and the integral I_3 retained their values. We chose as initial conditions $c_q^{(0)}$ in all experiments we did the following: $|c_0(0)|^2=0.998$, $|c_1(0)|^2=|c_{-1}(0)|^2=0.001$ with random phase distributions. The numerical calculations show that the packet of excited modes is localized in q and the population drops exponentially with increase of the mode q. The average width of the packet in that case increases with increases ε according to a law which is close to the linear law (3.2).

5. NUMERICAL ANALYSIS OF THE CHAIN

With the aim of ascertaining the possibility of using the narrow packet approximation for a description of the dynamics of the chain, we performed direct numerical calculations of the equations of motion of the chain. To compare with the solution of the system (2.12), we expressed the dynamics of the chain in the slow variables (see Sec. 2):

$$c_q(\tau)=a_{k_0+q}(t)I^{-1/2}\exp[i(\omega_{k_0}+\lambda q)t], \quad \tau=\Omega t. \qquad (5.1)$$

We chose our initial conditions as before: $|c_1(0)|^2=|c_{-1}(0)|^2=0.001$, $|c_0(0)|^2=0.998$. We chose the values of k_0 and I in (5.1) as follows: $k_0=N/2$, $I=N$, which corresponds to the initial excitation of the chain approximately in the form $u_0\sim 1$, $u_1\sim -1$, $u_2\sim 1$, $u_3\sim -1$, and so on. Our calculations show that when conditions (3.3) for the narrow-packet approximation are satisfied the characteristic dynamic properties (oscillation time and packet width of the NLS equation and of the chain agree qualitatively.

Moreover, it follows from the results of Sec. 2 that in the range of parameters where the narrow packet approximation is valid the chain must possess the approximate additional integrals of motion (2.13). We calculated numerically several first integrals of motion of the NLS equation for the system (1.1). As an example we give in Fig. 2a the time dependence of I_3 for a chain of $N=64$ particles and the value $\varepsilon=10.0$. The anomalous large oscillations of I_3 in the region of $\tau=0.6$ correspond to the moment of largest spreading of the packet of excited modes. We note that notwithstanding the fact that in the given case the condition for the narrow packet approximation was satisfied poorly, we did not observe an irreversible change in I_3 at large times. For instance, for the case depicted in Fig. 2a we observed a periodic destruction and re-establishment of I_3 at times $\tau\approx 20$, approximately 40 times longer than the characteristic time of the packet oscillation due to the exchange of energy between the modes. Figure 2b corresponds to the value $\varepsilon=20$ and the integral I_3 is completely destroyed. Figure 2c corresponds to a chain of $N=128$ particles and $\varepsilon=10$. As one should expect in comparison with the case depicted in Fig. 2a, the integral I_3 is much better conserved. Fixing the value of ε and increasing the number of oscillators N for the chain (1.1), can thus achieve good conservation of the integrals of motion of the NLS equation.

The calculations show that for fixed N for the chain (1.1) the integrals (2.13) with low numbers are better conserved. This property may be explained on the basis of the spectra

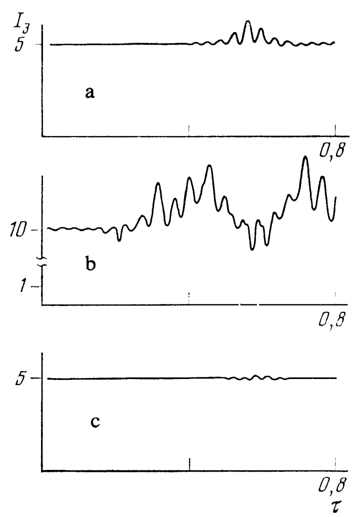

FIG. 2. Time dependence of I_3 for the chain: a—$\varepsilon=10$, $N=64$; b—$\varepsilon=20$, $N=64$; c—$\varepsilon=10$, $N=128$.

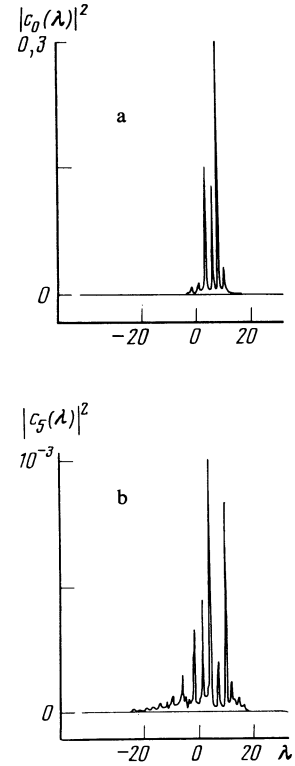

FIG. 3. Spectrum of $c_q^{(\lambda)}$ for the NLS equation: a—$q = 0$; b—$q = 5$.

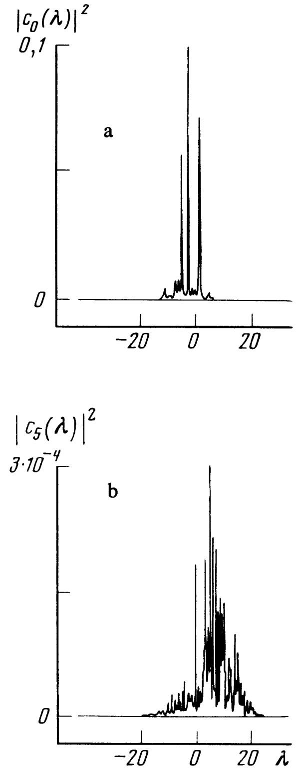

FIG. 4. Spectrum of $c_q^{(\lambda)}$ for the chain, $N = 64$: a—$q = 0$; b—$q = 5$.

$c_q(\lambda)$ of the NLS and of the chain:

$$c_q(\lambda) = \langle c_q(\tau) e^{-i\lambda\tau} \rangle \tag{5.2}$$

($\langle \ldots \rangle$ indicates as before averaging over the time). It is clear from Fig. 3 ($\varepsilon = 5$) that the spectrum $c_q(\lambda)$ for the NLS equation is well resolved irrespective of the value of the quantity q. The evaluation of the spectrum of the chain $c_q(\lambda)$ for $N = 64$ is given in Fig. 4. The value $\varepsilon = 5$ which we chose is at the limit of the applicability of the narrow packet approximation. It is clear from Fig. 4a that the spectrum of the chain is as before well resolved for $q = 0$ but when q is increased (Fig. 4b) the spectrum of the chain becomes appreciably different from that of the NLS equation. Since, as follows from the explicit form of the integrals (2.13), the role of $c_q(\tau)$ increases with large $|q|$ when the number of the integral increases, the fact of the destruction of just the higher integrals becomes understandable.

We also note here that the integral I_1 (normalization) is satisfactorily conserved up to very high values of ε which allows us practically always to eliminate one of the parameters of the chain. In particular, for $N = 64$ the integral I_1 is conserved with a 5% accuracy up to $\varepsilon = 100$.

6. CONCLUSION

The results show that the stochastization of the chain (1.1) in the case of a large number of degrees of freedom N and an initial excitation in the region of the high modes proceeds as follows. The first necessary condition for stochastization is that the nonlinearity parameter β exceeds the value $\beta_R = \pi^2/3E_{k_0}N$ of (4.1). When $\beta > \beta_R$ there occurs in the system a decay of modes and the characteristic size of the packet of the excited modes will be determined by the quantity $\varepsilon = 3E_{k_0}N/2\pi^2$. We note that when $\beta > \beta_R$ there will occur decays in the system (1.1) in contrast to the system (2.12) also in the case when a single mode is populated, as small perturbations due to nonresonant terms are able to "push" the system out of the unstable solution (2.14).

The existence of the instability limit (4.1) was noted in all numerical experiments on the dynamics of the chain (1.1) (see Refs. 2,4). We note also the recent Ref. 6 where from a completely different approach a similar estimate (up to replacing $\pi^2/3 \leftrightarrow 3.226$) was obtained for the instability limit of antiphase oscillations.

The further dynamics of the system (1.1) will depend on the ratio of the parameters ε and k_0. In the case $\varepsilon/k_0 \ll 1$ (region of validity of the narrow packet approximation) the dynamics of the chain will remain stable and can approximately be described by the NLS equation. As β increases, i.e., as the narrow packet approximation worsens, there occurs a gradual destruction of the additional integrals (2.13), and at $\beta \sim \beta_0$ [Eq. (3.4)] practically all nontrivial integrals turn out to be destroyed. Condition (3.4) can thus be considered to be the upper limit of the stochastization of (1.1). One shows easily that the limit of stochasticity in the form (3.4) means in fact that the interaction energy in (1.1) must be of the same order as the energy of the unperturbed system. Complete stochastization of the motion of the chain must thus be expected at sufficiently high excitation energies.

We compare the results obtained here for the estimate of the stochasticity limit for the chain (1.1) with an estimate for the stochasticity limit using the criterion of the overlap of resonances which in the region of an initial excitation of high modes has the form[1,2]

$$\beta>\beta_c=\frac{10m(N-1)}{3EN^2}\left(\frac{k_0}{N}\right)^2, \quad N-k_0\ll N, \tag{6.1}$$

where E is the total energy of the system, m an empirical parameter which has the meaning of the number of excited modes. One should note that the criterion (6.1) is obtained for zero boundary conditions ($u_0=u_N=0$, $\omega_k=2\sin(\pi k/2N)$) so that the comparison can only be made to order of magnitude. Putting in (6.1) $k_0\sim N\gg 1$, we have

$$\beta_c\sim 3.3m/EN. \tag{6.2}$$

It follows from (6.2) that when $m\sim 1$ the parameter β_0 agrees with the estimate of the stochasticity limit obtained from the condition for destruction of the narrow packet approximation: $\beta_c\approx\beta_0$ (with $k_0=N/2$ in (3.4)).

In conclusion we consider the possibility of using the scheme given above for the estimate of the stochasticity limit for other systems. It is well known that in discrete systems of the kind considered here and also in systems of fields various approximations are possible which lead to an important class of completely integrable systems.[9] The study of the conditions for applicability of such approximations (of the kind of the narrow packet approximation considered here) can turn out to be useful to determine the conditions for the transition from dynamic to stochastic motion. Studies of this kind are now underway (see, e.g., Ref. 12).

We express our gratitude to G. M. Zaslavskiĭ for his interest in this work and to B. V. Chirikov, F. M. Izraĭlev, and D. L. Shepelyanskiĭ for useful discussions.

[1]F. M. Izraĭlev and B. V. Chirikov, Dokl. Akad. Nauk SSSR **166**, 57 (1966) [Sov. Phys. Doklayd **11**, 30 (1966)].
[2]F. M. Izraĭlev, A. I. Khisamutdinov, and B. V. Chirikov, Preprint No 252, Inst. Nucl. Phys., Novosibirsk, 1968.
[3]B. V. Chirikov, F. M. Izraĭlev, and V. A. Tayursky, Computer Phys. Commun. **5**, 11 (1973).
[4]N. Ooyama, H. Hirooka, and N. Saito, J. Phys. Soc. Japan **27**, 815 (1969).
[5]N. N. Kozhukhovskiĭ and A. A. Martynyuk, Yad. Fiz. **37**, 508 (1983) [Sov. J. Nucl. Phys. **37**, 301 (1983)].
[6]N. Budinsky and T. Bountis, Physica **D8**, 445 (1983).
[7]B. V. Chirikov, Phys. Repts. **52**, 263 (1979).
[8]E. Fermi, J. Pasta, and S. Ulam, Los Alamos Report LA-1940 (1955): Collected Papers of Enrico Fermi, Vol. II (Univ. Chicago Press, 1965) p. 978.
[9]V. E. Zakharov, S. V. Manakov, S. P. Novikov, and L. P. Pitaevskiĭ, Teoriya solitonov, Metod obratonoĭ zadachi (Soliton theory; inverse scattering method) Nauka, Moscow, 1980 Ch. 1, §10 [English translation published by Consultants Bureau].
[10]V. E. Zakharov, Zh. Eksp. Teor. Fiz. **51**, 1107 (1966) [Sov. Phys. JETP **24**, 740 (1967)].
[11]G. Casati, IXth Internat. Conf. on Nonlinear Oscillations, Kiev, 1981.
[12]V. M. Eleonskiĭ and N. E. Kulagin, Zh. Eksp. Teor. Fiz. **85**, 1437 (1983) [Sov. Phys. JETP **58**, 833 (1983)].

Translated by D.ter Haar