Angular momentum of a Heisenberg ferromagnet with a magnetic dipole interaction

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(Submitted 17 December 1983)
Zh. Eksp. Teor. Fiz. **87**, 1330–1335 (October 1984)

Because of the isotropic character of the exchange interaction in a Heisenberg ferromagnet, the total spin (magnetic moment) vector is conserved. If allowance is made for the relativistic dipole-dipole interaction, which is anisotropic with respect to rotations in spin space, this conservation law breaks down. In this case, however, the Hamiltonian of the ferromagnet is invariant with respect to transformations which include rotations in both spin and coordinate space. In the present paper we construct the corresponding integral of motion.

THE LAGRANGIAN

The classical description of a ferromagnet in the long-wavelength approximation is based on the Landau-Lifshitz equation[1]

$$\dot{\mathbf{M}}=g[\mathbf{M},\times\mathbf{H}_{\text{eff}}]. \tag{1}$$

Here $\mathbf{M}=\mathbf{M}(\mathbf{r},t)$ is the magnetization vector ($\mathbf{M}^2=\text{const}$), g is the gyromagnetic ratio, and $\mathbf{H}_{\text{eff}}$ is the effective magnetic field, given by the variational derivative

$$\mathbf{H}_{\text{eff}}=-\delta E/\delta\mathbf{M}, \tag{2}$$

where E is the energy of the system. In the case under consideration, the expression for E is of the form[1]

$$E=\int\left\{\frac{\alpha}{2}\left(\frac{\partial\mathbf{M}}{\partial x_i}\right)^2-\mathbf{Mh}-\frac{\mathbf{h}^2}{8\pi}\right\}dV. \tag{3}$$

The first term in braces is the density of the inhomogeneous exchange energy. The other two terms describe the magnetic dipole interaction; here the field $\mathbf{h}$ satisfies the magnetostatic equations

$$\operatorname{curl}\mathbf{h}=0,\qquad \operatorname{div}\mathbf{h}=-4\pi\operatorname{div}\mathbf{M}. \tag{4}$$

Expression (3) does not contain terms corresponding to other interactions (such as the anisotropy energy or the Zeeman interaction with a constant external field). We shall later elucidate the effect of such terms on the conservation of the angular momentum of the system.

It is easy to see that the projection m_z of the angular-momentum density vector $\mathbf{m}=\mathbf{M}/g$ onto an arbitrary axis z and the corresponding aziumuthal angle $\varphi=\arctan(m_y/m_x)$ are the canonical conjugates of the momentum and coordinate:

$$\{m_z(\mathbf{r}),\ \varphi(\mathbf{r}')\}=-\delta(\mathbf{r}-\mathbf{r}') \tag{5}$$

(on the left is the classical Poisson bracket). Relation (5) follows from

$$\{m_z(\mathbf{r}),\ m_+(\mathbf{r}')\}=-im_+(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}'),\qquad m_+=m_x+im_y,$$

which is the classical limit obtained from the Poisson bracket for the spin components.

The Hamilton equations obtained from (3) with such a choice of canonical variables coincides with the equations for m_z and φ obtained from (1). This was pointed out in Ref. 2. The Lagrangian of the system is defined in the usual way[1]:

$$\mathcal{L}=m_z\dot{\varphi}-\mathcal{H}, \tag{6}$$

where the energy density (3) is

$$\mathcal{H}=\frac{\alpha}{2}g^2\left\{\frac{m^2}{m^2-m_z^2}\left(\frac{\partial m_z}{\partial x_i}\right)^2+(m^2-m_z^2)\left(\frac{\partial\varphi}{\partial x_i}\right)^2\right\}$$
$$-g\{(m^2-m_z^2)^{1/2}(h_x\cos\varphi+h_y\sin\varphi)+m_zh_z\}-\frac{\mathbf{h}^2}{8\pi}. \tag{7}$$

Here we have used the relation

$$m_x=(m^2-m_z^2)^{1/2}\cos\varphi,\qquad m_y=(m^2-m_z^2)^{1/2}\sin\varphi.$$

As the "true" Lagrangian variables in (6) one should take the generalized coordinate φ and generalized velocity $\dot{\varphi}$. The momentum m_z is eliminated with the aid of the equation

$$\dot{\varphi}=\delta E/\delta m_z. \tag{8}$$

Here the system is described, as it should be, by a single Lagrange's equation of second order in time and by Eqs. (4). If one chooses as the generalized coordinate for the field $\mathbf{h}$ the scalar potential

$$\mathbf{h}=\nabla\psi, \tag{9}$$

then the first of Eqs. (4) is satisfied identically, while the second is the Lagrange's equation with respect to the variables ψ, $\dot{\psi}$, $\nabla\psi$.

Actually, it may be difficult to eliminate one of the canonical variables, m_z or φ, if the coupling between them is nonlocal. In the case under consideration the nonlocality will be independent of which variable is eliminated. Substantially simpler in this respect is the continuum Ising model, in which the expression for the energy contains spatial derivatives only of m_z:

$$E_{\text{Ising}}=\int\left\{\frac{\alpha g^2}{2}\left(\frac{\partial m_z}{\partial x_i}\right)^2+f(m_z,\varphi)\right\}dV,$$

0038-5646/84/100764-03$04.00

where the second term in braces includes any interactions which contain the field $\mathbf{m}(\mathbf{r}, t)$ but not its derivatives (this would include the dipole interaction). Using the equation

$$\dot{m}_z = -\frac{\delta E}{\delta\varphi} = -\frac{\partial f(m_z, \varphi)}{\partial\varphi},$$

one can express φ in terms of m_z and $\dot{m}_z$, after which the equation $\dot{\varphi} = \delta E/\delta m_z$ becomes a Lagrange's equation with m_z and $\dot{m}_z$ as the generalized coordinate and velocity, respectively.

Returning to Lagrangian (6), let us treat the variables m_z and φ as independent generalized coordinates. Then the Lagrange's equations are the same as the Hamilton's equations for a system whose energy is given by (3). Thus, the description of the system by means of a Lagrange's equation of second order in time for one generalized coordinate is equivalent to the description by means of two first-order equations for m_z and φ treated as independent functions. In this respect a magnet described by a magnetization field $\mathbf{M}$ is no way different from any classical mechanical system. In fact, let us take the example of a one-dimensional system with Hamiltonian $\mathscr{H}(p, q)$. If the corresponding Lagrangian $\mathscr{L} = p\dot{q} - \mathscr{H}(p, q)$ is treated formally as a function of the generalized coordinates p and q and velocities $\dot{p}$ and $\dot{q}$, then the Lagrange's equations are the same as the Hamilton's equations for the initial one-dimensional system. An analgous possibility is used to obtain the Schrödinger equation from a Lagrangian[3] in which the wavefunctions ψ and ψ^* are treated as independent even though they are canonical conjugates.

ANGULAR MOMENTUM OF THE MAGNETIC SYSTEM OF A FERROMAGNET

Starting from Lagrangian (6), let us construct the energy-momentum tensor by the standard rules[4]:

$$T_{ik} = q_{,i}\frac{\partial\mathscr{L}}{\partial q_{,k}} - \mathscr{L}\delta_{ik}$$

($i, k = 1,2,3,4$; $x_{1,2,3} = x, y, z$; $x_4 = t$). This tensor yields conserved energy and momentum owing to the invariance of the action with respect to space-time translations. The stress tensor

$$T_{\mu\nu} = -\alpha g^2\left\{\frac{m^2}{m^2-m_z^2}\frac{\partial m_z}{\partial x_\mu}\frac{\partial m_z}{\partial x_\nu} + (m^2-m_z^2)\frac{\partial\varphi}{\partial x_\mu}\frac{\partial\varphi}{\partial x_\nu}\right\} + \frac{1}{4\pi}\frac{\partial\psi}{\partial x_\mu}\frac{\partial\psi}{\partial x_\nu} + gm_\nu\frac{\partial\psi}{\partial x_\mu} - \mathscr{L}\delta_{\mu\nu} \qquad (10)$$

in the presence of a magnetic dipole interaction is nonsymmetric, with the result that the orbital angular momentum

$$L_\lambda = \int e_{\lambda\mu\nu}x_\mu T_{\nu\lambda}dV, \qquad \lambda = 1, 2, 3 \qquad (11)$$

is not conserved:

$$\frac{dL_\lambda}{dt} = -g\int e_{\lambda\mu\nu}m_\mu\frac{\partial\psi}{\partial x_\nu}dV. \qquad (12)$$

However, since the magnetic system under study, including the field $\mathbf{h}$, is isotropic, an integral of motion should exist. To construct this integral of motion one usually uses Noether's theorem,[5] according to which an invariance of the action with respect to some transformation implies that the corresponding physical quantity is conserved. In the present case the transformation in question is a spatial rotation of the system. For an infinitesimal transformation we have

$$\delta x_\alpha = {}^1/_2 e_{\alpha\beta\gamma}e_{\beta\mu\nu}x_\gamma\delta\omega_{\mu\nu},$$
$$\delta m_\alpha = {}^1/_2 e_{\alpha\beta\gamma}e_{\beta\mu\nu}m_\gamma\delta\omega_{\mu\nu}, \quad \delta\psi = 0, \qquad (13)$$

where $\delta\omega_{\mu\nu} = -\delta\omega_{\nu\mu}$ are the parameters of the rotation.

Energy (3) is invariant with respect to arbitrary transformations (13), whereas the part of the integral of motion that is due to the "kinetic" term $m_z\,\dot{\varphi}$ in (6) is invariant only under rotations about the z axis, which has no physical distinction whatsoever.

Let us first consider a rotation about the z axis. In this case $\delta m_z = 0$, $\delta\varphi = \delta\omega \equiv \delta\omega_{xy}$, and the variation of Lagrangian (6) under such a rotation is equal to zero. On the other hand, using the Lagrange's equations

$$\frac{\partial}{\partial x_i}\left(\frac{\partial\mathscr{L}}{\partial q_{,i}}\right) - \frac{\partial\mathscr{L}}{\partial q} = 0 \quad (q = m_z, \varphi, \psi;\ i = 1, 2, 3, 4), \qquad (14)$$

we obtain

$$\delta\mathscr{L} = \frac{\partial}{\partial x_i}\left(\frac{\partial\mathscr{L}}{\partial\varphi_{,i}}\bar{\delta}\varphi + \frac{\partial\mathscr{L}}{\partial m_{z,i}}\bar{\delta}m_z + \frac{\partial\mathscr{L}}{\partial\psi_{,i}}\bar{\delta}\psi\right) + \frac{\partial\mathscr{L}}{\partial x_\mu}\delta x_\mu. \qquad (15)$$

Here $\bar{\delta}q = \delta q - q_{,\mu}\,\delta x_\mu$ is a variation of the form of the generalized coordinates, a commutation with differentiation with respect to the coordinates and time.[5] It follows from (13) that under rotation we have

$$\frac{\partial\mathscr{L}}{\partial x_\mu}\delta x_\mu = \frac{\partial}{\partial x_\mu}(\mathscr{L}\delta x_\mu),$$

which gives a conservation law for the projection of the angular momentum onto the z axis:

$$\frac{\partial J_{z4}}{\partial t} + \frac{\partial J_{z\mu}}{\partial x_\mu} = 0, \qquad (16)$$

where

$$J_{z4} = m_z - m_z[\mathbf{r}, \nabla\varphi]_z \qquad (17)$$

is the density of the z component of the total angular momentum, and

$$J_{z\mu} = -\alpha g^2(m^2-m_z^2)\frac{\partial\varphi}{\partial x_\mu} + \alpha g^2(m^2-m_z^2)\frac{\partial\varphi}{\partial x_\mu}[\mathbf{r}, \times\nabla\varphi]_z + \alpha g^2\frac{m^2}{m^2-m_z^2}\frac{\partial m_z}{\partial x_\mu}[\mathbf{r}, \times\nabla m_z]_z - gm_\mu[\mathbf{r}, \times\nabla\psi]_z - \frac{1}{4\pi}\frac{\partial\psi}{\partial x_\mu}[\mathbf{r}, \times\nabla\psi]_z - \mathscr{L}e_{\mu z\gamma}x_\gamma \qquad (18)$$

is the corresponding flux density. The second term in (17) is the density of the z projection of the orbital angular momentum (11), and the first term can be regarded as the density of the spin angular momentum. In our case the latter density coincides with the density of the magnetic moment of the ferromagnet (to within a factor of g). Equation (16) is thus the generalization of the conservation law for the angular momentum of the magnetic system of a ferromagnet having a magnetic dipole-dipole interaction in addition to isotropic exchange. If we neglect the dipole-dipole interaction, the continuity equation (16) is valid for each term of (17) separately. In this case the flux density of the spin angular momentum is equal to the first term in (18), while the remaining terms (for $\psi\equiv 0$) give the density of the orbital angular momentum. Equation (16) for the spin density in this case, as is easily seen, coincides with the Landau-Lifshitz equation for m_z. Conservation of the spin and orbital angular momenta separately when only the exchange interaction is taken into account is a consequence of the invariance of the system energy with respect to independent rotations in spin and coordinate space.[2] As we have already mentioned, Lagrangian (6) is not invariant with respect to rotations about the X and Y axes, so that one cannot use Noether's theorem directly. One can, however, use a modification of this theorem to construct the corresponding integrals of motion.

Under a rotation about the X axis the variation δm_z is given by formula (13), and

$$\delta\varphi = -\frac{m_z \cos\varphi}{m^2 - m_z^2}\,\delta\omega,$$

$$\delta\dot{\varphi} = -\left[\frac{m^2\cos\varphi}{(m^2-m_z^2)^{3/2}}\dot{m}_z + \frac{m_z \sin\varphi}{(m^2-m_z^2)^{1/2}}\dot{\varphi}\right]\delta\omega.$$

Direct evaluation of the variation of the Lagrangian then gives

$$\delta\mathscr{L} = -\frac{\partial}{\partial t}\left[g\frac{m^2\cos\varphi}{(m^2-m_z^2)^{1/2}}\right]\delta\omega$$

($\delta\omega$ is the rotation angle), i.e., although $\delta\mathscr{L}\neq 0$, it does reduce to a total time derivative. Therefore, subtracting the resulting expression from (15), we obtain the conservation law

$$\frac{\partial J_{x4}}{\partial t} + \frac{\partial J_{x\mu}}{\partial x_\mu} = 0,$$

where the angular momentum density is

$$J_{x4} = (m^2-m_z^2)^{1/2}\cos\varphi - m_z[\mathbf{r}, \nabla\varphi]_x,$$

and the flux density is

$$J_{x\mu} = \alpha g^2 (m^2-m_z^2)^{1/2} m_z \cos\varphi \frac{\partial\varphi}{\partial x_\mu} - \alpha g^2 \frac{m^2}{(m^2-m_z^2)^{1/2}}\sin\varphi\frac{\partial m_z}{\partial x_\mu}$$
$$-g m_\mu [\mathbf{r}, \nabla\psi]_x - \frac{1}{4\pi}\frac{\partial\psi}{\partial x_\mu}[\mathbf{r}, \nabla\psi]_x + \alpha g^2 (m^2-m_z^2)\frac{\partial\varphi}{\partial x_\mu}[\mathbf{r}, \nabla\varphi]_x$$
$$+\alpha g^2 \frac{m^2}{m^2-m_z^2}\frac{\partial m_z}{\partial x_\mu}[\mathbf{r}, \nabla m_z]_x + \mathscr{L} e_{\mu x \gamma} x_\gamma.$$

Expressions for J_{y4} and $J_{y\mu}$ are found in an analogous manner.

As a result, we obtain a conservation law for the total angular momentum vector[3]

$$\mathbf{J} = \int (\mathbf{m} - m_z[\mathbf{r}, \times\nabla\varphi])\,dV. \qquad (19)$$

Angular momentum (19) is not conserved as a vector if the Zeeman interaction with an external magnetic field and the anisotropy energy are incorporated in addition to the exchange and dipole interactions. If the anisotropy is uniaxial and the field is directed along the preferred axis, then the projection of $\mathbf{J}$ onto this axis is conserved. If the field is directed at an angle to the anisotropy axis or if the anisotropy is biaxial, then none of the angular momentum projections is conserved.

On the other hand, if one neglects the dipole interaction then the orbital angular momentum vector, but not the spin, will be conserved for any configuration of the field and anisotropy axis. This is a consequence of the symmetry of the system with respect to rotations in coordinate space and the lack of symmetry in spin space. In this case the stress tensor, as it should be, is symmetric.

Finally, for a magnet with an exchange interaction which is anisotropic in configuration space, i.e., when

$$E = \frac{\alpha_{ik}}{2}\int \frac{\partial \mathbf{M}}{\partial x_i}\frac{\partial \mathbf{M}}{\partial x_k}dV,$$

the orbital angular momentum is not conserved (the stress tensor in nonsymmetric) but the total spin is conserved (we have isotropy in spin space).

Expression (19) for angular momentum $\mathbf{J}$ is not symmetric in the projections of angular momentum $\mathbf{m}$. One can easily see, however, that going over from m_z and $\varphi = \arctan(m_y/m_x)$ to other corresponding pairs will change the integrand in (19) by derivatives of some function with respect to the coordinates. Therefore $\mathbf{J}$ is, as it should be, independent of the choice of axes.

[1]Lagrangian (6) corresponds to choosing φ as the generalized coordinate. If m_z is taken for the generalized coordinate, then $\mathscr{L} = \dot{m}_z\varphi - \mathscr{H}$.
[2]The term $g\mathbf{m}\nabla\psi$ in the energy density "intermingles" these spaces.
[3]The analogous integral of motion in the linear spin-wave approximation is found in Ref. 6. An expression for the second term in the integral in (19) is given in the monograph of Ref. 7.

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Translated by Steve Torstveit