Electromagnetic waves in a randomly inhomogeneous metal

V. A. Ignatchenko, Yu. I. Man'kov, and F. V. Rakhmanov

L. V. Kirenskiĭ Institute of Physics, Siberian Branch, Academy of Sciences of the USSR

(Submitted 10 October 1983; resubmitted 29 February 1984)
Zh. Eksp. Teor. Fiz. **87**, 228–233 (July 1984)

The modifications to the dispersion relation and damping of electromagnetic waves in a metal made necessary by large-scale inhomogeneities are analyzed. A random potential with a finite correlation length $1/k_0$ is adopted as a model for the inhomogeneities. An unusual "microscope effect" occurs: Inhomogeneities with a scale dimension $1/k_0$ are manifested in the spectrum and in the damping of the electromagnetic waves in a greatly magnified form—with a scale dimension $c/v_F k_0$. The physical mechanism for this effect is discussed.

The modifications to the dispersion relation and damping of plasma waves made necessary by the scattering of these waves by large-scale inhomogeneities were studied in Refs. 1 and 2 for metals and semiconductors respectively. A random potential with a finite correlation length was adopted as a model for the inhomogeneities. It was shown that experiments on the modified dispersion relation could in principle yield information on the basic stochastic characteristics of the random potential: its standard deviation and correlation length.

Our purpose in the present paper is to see whether the same characteristics can be found from the modifications to the dispersion relation for electromagnetic waves in a metal. There has been no previous study of this topic, despite the existence of an extensive literature which deals with a variety of extremely subtle and complicated aspects of the interaction of electromagnetic waves with inhomogeneities (see the reviews in Refs. 3–5, for example).

To analyze electromagnetic waves in an inhomogeneous metal we adopt the approximation of a small and rather smooth random potential, in which case we can switch from the quantum-mechanical approach to the classical approach.[6] As in Ref. 1, we assume a single-band metal with a quadratic and isotropic dispersion relation for conduction electrons. We can then apply Eq. (2.1) of Ref. 1 to $f(\omega,\mathbf{k},\mathbf{p})$, which is the Fourier transform of the nonequilibrium increment in the electron distribution function caused by the propagation of an electromagnetic wave through the metal. This equation was derived without any assumption regarding the wave polarization. When combined with Maxwell's equations and an equation which relates the current density of f, it forms a complete system of equations for our problem.

We consider waves of arbitrary polarization (instead of restricting the analysis to longitudinal excitations, as in Refs. 1 and 2). From Maxwell's equations we have a relation between the electric field $\mathbf{E}$ and the current density $\mathbf{j}$:

$$\mathbf{E}(\mathbf{k},\omega)=-\frac{4\pi i}{\omega}[\mathbf{k}(\mathbf{kj})-k_c^2\mathbf{j}][k^2-k_c^2]^{-1}, \tag{1}$$

where ω and $\mathbf{k}$ are respectively the frequency and wave vector of the wave, and $k_c=\omega/c$.

We can derive a closed expression for the current density $\mathbf{j}(\mathbf{k},\omega)$:

$$\mathbf{j}=\frac{\omega_p^2}{\omega^2}\chi(\mathbf{j})+w^2\hat{I}\mathbf{v}\left\{D_{\parallel}^{-1}\frac{\mathbf{kv}}{k^2}\hat{I}\mathbf{kv}\{\mathbf{Kj},\rho\},\rho\right\}$$

$$+w^2k_c^2\hat{I}\mathbf{v}\left\{D_{\perp}^{-1}\mathbf{v}\hat{I}\,\frac{[\mathbf{k}\times[\mathbf{k}\times\mathbf{v}]]}{k^2(k^2-k_c^2)}\{\mathbf{Kj},\rho\},\rho\right\}. \tag{2}$$

Here

$$\chi=\frac{k_c^2}{k^2-k_c^2}\left\{\left(\frac{\mathbf{k}(\mathbf{kj})}{k_c^2}-\mathbf{j}\right)\left(1+\frac{3}{5}q^2\right)-\frac{2}{5k^2}q^2[\mathbf{k}[\mathbf{kj}]]\right\},$$

$$D_{\parallel}=1-\frac{\omega_p^2}{\omega^2}\left(1+\frac{3}{5}q^2\right),\qquad D_{\perp}=1-\frac{\omega_p^2/c^2}{k_c^2-k^2}\left(1+\frac{1}{5}q^2\right), \tag{3}$$

$$\mathbf{K}=[\mathbf{k}(\mathbf{kv})-k_c^2\mathbf{v}](k^2-k_c^2)^{-1},\qquad q=kv_F/\omega.$$

The effect of the operator $\breve{I}$ is defined by

$$\hat{I}g=\hat{I}(\mathbf{k})g(\mathbf{p})=-\frac{8\pi e^2}{h^3\omega}\int d\mathbf{p}\frac{f_0''g(\mathbf{p})}{\omega-\mathbf{kv}},$$

where f_0'' is the second derivative of a Fermi distribution function with respect to the energy, and the rest of the notation is that of Ref. 1.

Equation (2) has been written approximately. Its derivation used relation (1) and an equation relating the current density to the distribution function $f(\mathbf{k},\mathbf{p},\omega)$. In addition, a formal "solution" of the equation of f is substituted (twice) into integral convolutions of the same equation. Only terms with w raised to a power no higher than two are retained.

Averaging this equation over an ensemble of random realizations of the function $\rho(\mathbf{k})$, we find a general dispersion relation for waves of arbitrary polarization (the vector $\mathbf{E}$) in a metal (the z axis is directed along $\mathbf{k}$):

$$\|D_i\delta_{ij}-w^2\hat{I}v_i\hat{Q}K_j(\mathbf{k})\|=0. \tag{4}$$

Here $i,j=x,y,z; D_z=D_{\parallel}, D_{x,y}=2D_{\perp}$; and we have introduced the operator

$$\hat{Q}g=\int d\mathbf{k}'S(\mathbf{k}-\mathbf{k}')$$

$$\times\left\{\frac{\mathbf{k}'\mathbf{v}}{k'D_{\parallel}(k')}\hat{I}'\frac{\mathbf{k}'\mathbf{v}}{k'}+\frac{k_c^2\mathbf{v}}{k'^2D_{\perp}(k')}\hat{I}'\frac{[\mathbf{k}\times[\mathbf{k}\times\mathbf{j}]]}{k'^2-k_c^2}\right\}g(\mathbf{p}), \tag{5}$$

where $S(\mathbf{k})$ is the spectral density of the correlation function of the random function $\rho(\mathbf{r})$, and $\rho(\mathbf{r}),\widehat{I}'\equiv\widehat{I}(\mathbf{k}')$.

For a homogeneous metal ($w=0$) we find from (4) dispersion relations for plasma and electromagnetic waves:

$$\omega^2=\omega_p^2(1+{}^3/_5q^2),\qquad \omega^2=\omega_p^2(1+{}^1/_5q^2)+k^2c^2. \tag{6}$$

It follows from the form of determinant (4) that electromagnetic and plasma waves interact with each other in an inhomogeneous metal. This interaction may take the form of the decay of a coherent electromagnetic wave into fluctuat-

 0038-5646/84/070133-03$04.00

ing plasma waves and, correspondingly, the decay of a coherent plasma wave into fluctuating electromagnetic waves. In certain cases there may also be conversions of coherent plasma and electromagnetic waves into each other. This effect is seen in an inhomogeneous Maxwellian plasma (see Ref. 7, for example), where the inhomogeneity stems from spatial-temporal thermodynamic fluctuations.

Working from the general form of dispersion relation (4), we can identify a class of spectral functions (of the random-process type) for which the conversion of coherent plasma and electromagnetic waves into each other does not occur. It can be shown that if $S(\mathbf{k}-\mathbf{k}')$ does not depend on the azimuthal angle of $\mathbf{k}'$ then all the off-diagonal elements in (4) vanish upon integration (this assertion is made only for the approximation used here; the correlation functions corresponding to powers higher than w^2 have not been studied). This condition is satisfied by $S(\mathbf{k})$, which depends only on $|\mathbf{k}|$, and thus by an isotropic correlation function. Restricting the discussion to processes of this type, we find from (4) the dispersion relation

$$(D_\perp-{}^1/{}_2w^2\hat{I}v_+\hat{Q}K_-)(D_\perp-{}^1/{}_2w^2\hat{I}v_-\hat{Q}K_+)(D_\parallel-w^2\hat{I}v_z\hat{Q}K_z)=0, \tag{7}$$

where $v_\pm=v_x\pm iv_y$ and $K_\pm=K_x\pm iK_y$. Equating the last factor to zero, we find an expression for the spectrum and damping of plasma waves in an inhomogeneous metal. The first two factors correspond to the modified dispersion relation for left- and right-hand-polarized electromagnetic waves.

To find some estimates we adopt a simple exponential correlation function for the inhomogeneities, as in Refs. 1 and 2. We restrict the discussion to the case $q\ll 1$, and we find a modified dispersion relation for electromagnetic waves (the expressions for the right- and left-hand-polarized waves are identical):

$$\left(\frac{\omega'}{\omega_p}\right)^2=1+s-15\frac{\gamma^2}{q_0^2}\times\left\{\frac{1}{1+s}\frac{u-\beta^2}{4u^{5/2}}G_1+\beta^2\left(\frac{1}{1+4u}+\frac{1}{4u^{5/2}}G_2\right)\right\},$$

$$G_1=\operatorname{arctg}\sqrt{u}-\frac{\beta^2+u}{\beta^2-u}\operatorname{arctg}\left(\frac{\beta^2-u}{\beta^2+u}u^{1/2}\right), \tag{8}$$

$$G_2=\operatorname{arctg}\sqrt{u}-(1+2u)\operatorname{arctg}\frac{u^{1/2}}{1+2u}.$$

Here $s=(kc/\omega_p)^2$, $u=(k/k_0)^2$, $\beta^2=3v_F^2/5c^2$, k_0 is the correlation wave number of the inhomogeneities ($1/k_0$ is the correlation length), and $\gamma=w/2\epsilon_F$.

The damping of the electromagnetic waves is determined by

$$\frac{\omega''}{\omega_p}=15\frac{\gamma^2}{q_0^2}\frac{1}{(1+s)^{1/2}}\left\{\frac{1}{1+s}\frac{\beta}{4u^{3/2}}F_1+\beta^2\left(\frac{u^{1/2}}{1+4u}-\frac{1}{4u^{3/2}}F_2\right)\right\},$$

$$F_1=\frac{\beta^2+u}{4\beta u}\ln\frac{\beta^2+u(1+2\beta)}{\beta^2+u(1-2\beta)}-1, \tag{9}$$

$$F_2=\frac{1+2u}{4}\ln(1+4u)-1.$$

Analysis of these expressions reveals an unexpected result. As in the case of plasma waves, we find a characteristic modification of the dispersion relation and maximum in the damping, but not near the correlation wave number k_0: All these effects occur at a much smaller wave number, $k_u=\beta k_0$. We find an unusual "microscope effect": The electromagnetic waves sense not the actual size of the inhomogeneities in the sample, $r_0\sim 1/k_0$, but a size which is some two or three orders of magnitude greater ($r_u=1/k_u$) and which is absent from the actual structure of the inhomogeneities. Near the characteristic wave number k_u, the modification to the dispersion relation and the damping can be described highly accurately by

$$\frac{\omega'}{\omega_p}=\left(1+\frac{k^2}{k_p^2}\right)^{1/2}-\frac{5}{2}\frac{\gamma^2}{q_0^2}\frac{1-k^2/k_u^2}{(1+k^2/k_p^2)^{3/2}(1+k^2/k_u^2)^2}$$

$$\frac{\omega''}{\omega_p}=\frac{5\gamma^2}{q_0^2}\frac{k/k_u}{(1+k^2/k_p^2)^{3/2}(1+k^2/k_u^2)^2}, \tag{10}$$

where

$$k_p=\omega_p/c,\quad q_0=k_0v_F/\omega_p=({}^5/{}_3)^{1/2}(k_u/k_p).$$

Figure 1 shows the modified dispersion relation $\omega'(k)$ and the damping $\omega''(k)$ for electromagnetic waves in an inhomogeneous metal. The modification is basically of the same form as that of plasma waves: Near the characteristic wave number k_u, the $\omega'(k)$ curve deviates from the corresponding curve in a homogeneous sample. The deviation is toward lower frequencies. At $k=0$, this deviation reaches a maximum, which is equal in magnitude to the deviation for plasma waves. The damping reaches a maximum in the same region, at $k=k_u/\sqrt{3}$.

There are also some differences between the modifications of the electromagnetic and plasma waves. Under the inequality

$$12q_0^4/\gamma^2<1 \tag{11}$$

a slight maximum can be seen on the $\omega'(k)$ curve at $k\approx k_u\sqrt{3}$; correspondingly, we find a region of anomalous dispersion to the right of this maximum.

The primary distinction between the modification of electromagnetic waves and that of plasma waves, however, is the renormalization of the characteristic correlation wave number of the inhomogeneities, k_0. Let us examine the physical mechanism for this effect.

In an inhomogeneous metal with an isotropic correla-

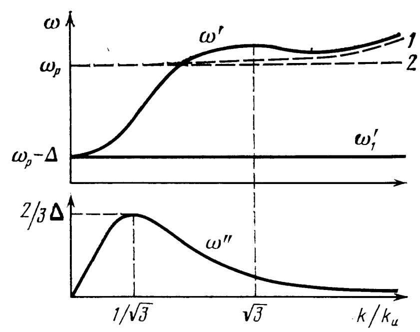

FIG. 1. Modified dispersion relation $\omega'(k)$ and damping $\omega''(k)$ of electromagnetic waves in an inhomogeneous metal ($\Delta=5\gamma^2\omega_p/2q_0^2$). Dashed lines—Dispersion curves for electromagnetic (1) and plasma (2) waves in a homogeneous metal; $\omega_l'(k)$—modified dispersion curve for plasma waves in an inhomogeneous metal (at $k\lesssim k_u$ this curve is seen as the line $\omega=\omega_p-\Delta$).

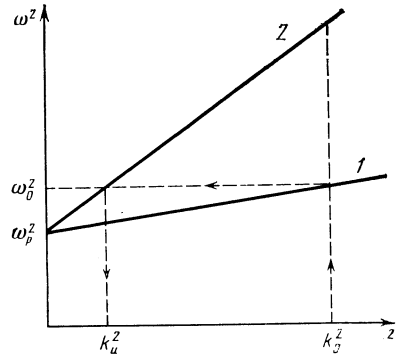

FIG. 2. Renormalization of the correlation number of the inhomogeneities in the modified dispersion relation for electromagnetic waves (the "microscope effect"). 1—Plasma waves; 2—electromagnetic waves.

tion function, the equations for the expected values of the projections of the current density $\langle j_i \rangle$ decay into three independent equations, as can be seen from (7). One of these new equations describes plasma waves with $\langle \mathbf{E} \rangle \| \mathbf{k}$, while the other two describe electromagnetic waves with $\langle \mathbf{E} \rangle \perp \mathbf{k}$. However, the independence of the equations for the expected values means only that there is no conversion of the coherent components of the plasma and electromagnetic waves into each other; it does not imply that there are no interactions between these waves. Each of these independent equations and, correspondingly, each of the three independent dispersion relations (7) reflects both the fluctuating electromagnetic waves and the fluctuating plasma wave. In other words, there are both direct and cross mechanisms for the decay of a coherent wave into fluctuating waves. The second terms in the curly brackets in (8) and (9) correspond to the direct mechanism, and the first terms to the cross mechanism, for the decay of a coherent electromagnetic wave into fluctuating electromagnetic and plasma waves. The cross mechanism is predominant (the direct mechanism corresponds to small terms on the order of β^2). It is this circumstance which gives rise to the "microscope effect," which can be conveniently explained by means of Fig. 2, which shows the unperturbed dispersion curves for plasma waves (1) and electromagnetic waves (2) (this figure is not drawn to scale).

The waves which interact most effectively with the inhomogeneities with a correlation wave number k_0 are the plasma waves with $k = K_0$ (i.e., with $\omega = \omega_0$). As a result, near $k \approx k_0$ we see a modification of the dispersion relation and a maximum in the damping of the plasma waves.[1]

We find a different picture for the electromagnetic waves. The decay of an electromagnetic wave into fluctuating electromagnetic waves near k_0 occurs again [see the last terms, proportional to β^2, in (8) and (9)]: at $k \approx k_0$ we see a modification of the dispersion relation and an increase in the damping of the electromagnetic waves. These effects, however, are negligibly small in comparison with those caused by the transfer to the electromagnetic waves of those processes which occur in the plasma waves. This transfer is implemented by the cross decays of electromagnetic and plasma waves which occur at the frequency ω_0. From expressions (6) we easily find that the wave numbers of the plasma and electromagnetic waves corresponding to a given frequency convert into each other with a scaling factor β. This circumstance again gives rise to a microscope effect: The events which actually occur with the plasma waves at $k \sim k_0$ are manifested in the electromagnetic waves at $k \sim \beta k_0$. We might note that the dispersion relation and damping of the plasma waves which follow from general relation (7) also reflect both direct and cross scattering mechanisms. For the plasma waves, in contrast with the electromagnetic waves, it is the direct mechanism which is predominant; it was studied in detail in Refs. 1 and 2.

CONCLUSION

We have studied the effect of a random inhomogeneous potential with a finite correlation length on long-wavelength electromagnetic and plasma excitations in a metal. In the first nonvanishing order of perturbation theory we have derived a dispersion relation [relation (4)] for the expected values of these excitations.

The case of an isotropic correlation function has been studied in detail. In this case the dispersion relations for the expected values of the plasma waves and electromagnetic waves become independent. On the other hand, the modification of the dispersion relation and the damping of the plasma waves are determined by the decay of these waves into both fluctuating electromagnetic waves and fluctuating plasma waves; the same is true of electromagnetic waves. The modifications of the plasma waves and of the electromagnetic waves are both caused primarily by the decay of these waves into fluctuating plasma waves (i.e., by the direct process in the case of the plasma waves and by the cross process in the case of the electromagnetic waves). This result is, in particular, a justification for the approximation used in Refs. 1 and 2, where the decay of plasma waves into fluctuating electromagnetic waves was ignored.

The results of this paper show that experiments on the dispersion relation for electromagnetic waves (like experiments on the dispersion relation for plasma waves) can in principle reveal two basic characteristics of the correlation function of the random potential: the relative mean square fluctuation γ and the correlation length $r_0 = k_0^{-1}$. When plasma waves are used to determine the correlation length, the range of wave numbers $k \sim k_0$ must be studied. For electromagnetic waves, on the other hand, there is a microscope effect: In order to determine the same correlation length of the inhomogeneities one must study considerably longer waves, with $k \sim (v_F/c)k_0$.

[1]V. A. Ignatchenko, Yu. I. Man'kov, and F. V. Rakhmanov, Zh. Eksp. Teor. Fiz. **81**, 1771 (1981) [Sov. Phys. JETP **54**, 939 (1981)].
[2]V. A. Ignatchenko, Yu. I. Man'kov, and F. V. Rakhmanov, Fiz. Tverd. Tela (Leningrad) **24**, 2292 (1982) [Sov. Phys. Solid State **24**, 1301 (1982)].
[3]Yu. I. Barabanenkov, Yu. A. Kravtsov, S. M. Rytov, and V. I. Tatarskiĭ, Usp. Fiz. Nauk **102**, 3 (1970) [Sov. Phys. Usp. **13**, 551 (1970)].
[4]Yu. A. Ryzhov and V. V. Tamoĭkin, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **13**, 356 (1970).
[5]A. Ishimaru, Wave Propagation and Scattering in Random Media. Vol. 2, Academic Press, New York, 1978, Ch. II (Russ. transl. Mir, Moscow, 1981).
[6]I. M. Lifshitz, M. Ya. Azbel', and M. I. Kaganov, Élektronnaya teoriya metallov (Electron Theory of Metals), Nauka, Moscow, 1971, p. 15.
[7]A. I. Akhiezer (editor), Élektrodinamika plazmy (Plasma Electrodynamics), Nauka, Moscow, 1974, p. 587.

Translated by Dave Parsons