Formation of current layers in nonequilibrium magnetic configurations

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Two-dimensional equilibrium configurations in magnetic hydrodynamics are investigated. Particular attention is devoted to bounded configurations, i.e., those in which the transverse magnetic field vanishes outside some closed surface. It is shown that the equilibrium configuration must be axially symmetric. It is also shown that there will be no equilibrium when the boundary conditions (magnetic tube pinched by the "walls") or the initial conditions (for example, for the configurations shown in Fig. 3) are such that the system departs from axial symmetry. The resulting plasma motion leads to the appearance of a current layer, and there is fast field dissipation that ultimately takes the magnetic configuration to an axially symmetric equilibrium state.

The formation of current layers in a highly conducting medium has played a central role in many astrophysical and geophysical applications. Processes involving the reconnection of magnetic lines of force can also develop under laboratory conditions.[1–3] When the kinetic energy is high (in comparison with magnetic energy), plasma motion determines the behavior of the magnetic field and the formation of current layers presents no difficulties. In particular, in dynamo theory, field generation is always connected with a reduction in its scale, down to the dimensions for which dissipation becomes appreciable. The current layer is thus formed, and this is followed either by field generation or rapid field dissipation (see, for example, Ref. 4). The current-layer problem is quite different in the case of slow motions (or, generally, zero initial plasma velocity). Whereas the dynamo theory deals with weak magnetic fields, the opposite limiting case of slow motion is commonly referred to as the strong-field approximation.[5] In the latter case, it is precisely the magnetic (and not the kinetic) energy that produces heating of the plasma, accelerated particles, emission of radiation, and intensive plasma motion, which leads to flare processes in the solar corona and to phenomena involving the reconnection of lines of force in the Earth's magnetosphere.

In a recent paper, Parker[6] has put forward the hypothesis that the appearance of a current layer is connected not with the instability of the magnetic configuration, but with a loss of equilibrium under definite boundary conditions. It will be shown below that the equilibrium conditions impose a restriction on the magnetic field configuration, so that only certain definite classes of equilibrium states are possible. When a nonequilibrium configuration is established at the initial time, and it does not have a topologically equivalent class of equilibria, equilibrium can be achieved only by turning on dissipation mechanisms that destroy the freezing-in of the magnetic field. The substantial departure from equilibrium that arises in the course of the transition leads to a fast and considerable dissipation of magnetic field energy. This process is related to the reconnection of magnetic lines of force, and is similar to the development of disruptive instability[2,1] and the coalescence of current filaments in the tokamak.[3] Because of the difference between the field boundary conditions in the tokamak and in cosmic electrodynamics, the formation of current layers in space requires separate analysis. The formulation of the problem given below is designed exclusively for astrophysical and geophysical applications.

§1. EQUILIBRIUM OF TWO-DIMENSIONAL CONFIGURATIONS

The dynamics of a magnetic field is a complicated nonlinear problem, so that it is natural to begin not with the general three-dimensional situation, but with a somewhat idealized two-dimensional problem. It is precisely this approach that was adopted earlier in the analysis of solutions near singular lines.[7,8] Parker[6] has considered a layer of plasma with an initially uniform field perpendicular to the surface of the layer (and parallel to the z axis). Slow motion of the plasma that disturbs the field occurs on the lower boundary of the layer with the result that a new magnetic field configuration is established. When the characteristic scale l of the motion is small in comparison with the layer thickness L, the field established in this way may be expected to be a slowly-varying function of z. In other words, the lines of force are uniformly twisted in the z direction by the motion on the lower boundary. The problem thus becomes two-dimensional and the equilibrium condition can be written in the form

$$4\pi\nabla p=[\mathrm{rot}\,\mathbf{H},\times\,\mathbf{H}],\quad\text{or}\quad \Delta A+4\pi dP(A)/dA=0,\qquad(1)$$

where $P=p+H_z^2/8\pi$ (see Ref. 9), A is the z component of the vector potential of the transverse field $\mathbf{H}_\perp=\{H_x,H_y,0\}$ (the other components of the vector potential are all zero), and P is a function of A only (and so are both p and H_z separately).

In the formulation of the Parker problem,[6] it is assumed that the motion in the lower part of the layer can be described by isolated cells. The field between the cells is then undisturbed, and the solution (1) takes the form of isolated "islands." In other words, it can be assumed that $A\neq 0$ only in the disturbed regions. The next assumption is that the

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curve drawn around a cell is always convex because the tension in the lines of force must compensate the pressure gradient which is a maximum in the interior of the cell. If we now use some method to press two such cells against each other, and the cells contain magnetic fields with opposite directions on the contact boundary, then either the edge of one of the cells becomes concave or both edges become straight. In either case, the edge of the cell will no longer be described by a convex curve, and it is impossible to have an equilibrium state.[1]

There are several aspects of this picture that are not clear. Nevertheless, we shall show below that conclusions relating to the coalescence of islands are entirely correct and, moreover, the results can be considerably strengthened. To begin with, we shall show that the two-dimensional equilibrium state of a bounded cell (i.e., a cell with zero transverse field on the cell boundary) is axially symmetric. In other words, the cell boundary is not simply convex but specifically circular.

Let us confine our attention to an incompressible medium, which means that

$$P \gg H_\perp^2/8\pi. \tag{2}$$

and consider an axially symmetric initial equilibrium configuration with $\mathbf{H}_\perp = 0$ on the boundary. There is no doubt that this configuration exists (see, for example, Refs. 9 and 10). Deformation in the xy plane will, of course, modify the configuration, and our problem is to determine whether incompressible deformation will lead the initial equilibrium configuration to a final configuration which is also a state of equilibrium but, naturally is not identical with the initial state. The answer to this question is in the negative. In fact, let us introduce a coordinate frame (s_1, s_2, z), in which the coordinate line $s_1 = \text{const}$, $s_2 = \text{const}$ runs along the line of force of the field $\mathbf{H}_\perp$, i.e., along the $A = \text{const}$ curve. It is obvious that, for given s_1, the quantity A is conserved in incompressible motion (this is so by definition and corresponds to a frozen-in field), and so is P. The transformation Jacobian $\sqrt{g}$ is also conserved. For the initial configuration, $x = s_1 \cos s_2$, $y = s_1 \sin s_2$, so that $\sqrt{g} = s_1$. Let us suppose that deformation has resulted in a new equilibrium state. It satisfies (1), which can be written in the following form in terms of our coordinates:

$$\frac{d^2A}{ds_1^2}\left[\left(\frac{\partial s_1}{\partial x}\right)^2+\left(\frac{\partial s_1}{\partial y}\right)^2\right]+\frac{dA}{ds_1}\left(\frac{\partial^2 s_1}{\partial x^2}+\frac{\partial^2 s_1}{\partial y^2}\right)=-4\pi\frac{dP}{dA},$$
$$(\partial s_1/\partial x)^2+(\partial s_1/\partial y)^2=g_{22}/g, \quad g=g_{11}g_{22}-g_{12}^2, \tag{3}$$

where g_{ij} is a component of the metric tensor. We then have $\mathbf{x} = \mathbf{x}(\mathbf{x}_0)$, and the transformation Jacobian is $(x, y)/(x_0, y_0) = 1$. The quantity s_1 varies from zero to the boundary value $s_1 = s_1^{(0)}$, and s_2 varies form 0 to 2π. When $s_1 = s_1^{(0)}$, we have $dA/ds_1 = 0$ and, as we have said, the values of A, P and, hence, d^2A/ds_1^2 and dP/dA, are the same as on the boundary of the initial configuration. Hence, for $s_1 = s_1^{(0)}$, it follows from (3) that g_{22} retains its initial value.

This means that the length of the bounding curve is

$$\int_0^{2\pi} \sqrt{g_{22}}\, ds_2 = 2\pi s_1^{(0)},$$

i.e., it is the same as the length of the initial circle $s_1 = s_1^{(0)}$. Hence, it follows that the deformation has not modified the outer curve and, possibly, has affected only the internal curves. The point is that any incompressible deformation of the outer curve unavoidably leads to an increase in its length because the length of a circle is the minimum length of a curve bounding a given area. Thus, the curve $s_1 = s_1^{(0)}$ is a circle. To elucidate the field configuration for $s_1 < s_1^{(0)}$, consider (3) in terms of the orthogonal coordinates (s_1, s_2', z):

$$\frac{d^2A}{ds_1^2}\frac{1}{g_{11}}+\frac{dA}{ds_1}\frac{1}{\sqrt{g}}\frac{\partial}{\partial s_1}\frac{\sqrt{g}}{g_{11}}=f(s_1). \tag{4}$$

A coordinate line can be determined from the equation

$$\partial x/\partial s_1=-\sqrt{g_{11}}\sin\Phi, \quad \partial y/\partial s_1=\sqrt{g_{11}}\cos\Phi, \tag{5}$$

$$\partial x/\partial s_2'=\sqrt{g_{22}}\cos\Phi, \quad \partial y/\partial s_2'=\sqrt{g_{22}}\sin\Phi;$$
$$\frac{1}{\sqrt{g_{11}}}\frac{\partial\sqrt{g_{22}}}{\partial s_1}=\frac{\partial\Phi}{\partial s_2'}, \quad \frac{1}{\sqrt{g_{22}}}\frac{\partial\sqrt{g_{11}}}{\partial s_2'}=-\frac{\partial\Phi}{\partial s_1}. \tag{6}$$

Thc curvature of a line of force k and of an orthogonal line k' can then be expressed in terms of the components of the metric tensor, as follows:

$$k=\frac{1}{\sqrt{g}}\frac{\partial}{\partial s_1}\sqrt{g_{22}}, \quad k'=-\frac{1}{\sqrt{g}}\frac{\partial}{\partial s_2'}\sqrt{g_{11}}.$$

Let us expand all the quantities in (4) and (6) into series in powers of $s_1 - s_1^{(0)}$. From the zero-order approximation to (4) and (3), it follows that $\sqrt{g_{11}}$, $\sqrt{g_{22}}$, are independent of s_2' for $s_1 = s_1^{(0)}$. Since, in addition, the outer curve is a circle (i.e., k and $\partial\sqrt{g_{22}}/\partial s_1$ are independent of s_2') and $\partial\Phi/\partial s_1 = 0$ for $s_1 = s_1^{(0)}$. In the first approximation to (4), we find from the expansion in terms of the parameter s_1-$s_1^{(0)}$, that the first-order term in the expansion $\sqrt{g_{11}}$ is again independent of s_2'. According to the first expression in (6), $\partial\sqrt{g_{22}}/\partial s_1$ for $s_1 = s_1^{(0)}$, (i.e., the second term of the expansion) is independent of s_2'. If we now turn again to (4), we conclude that the second-order term in the expansion for $\sqrt{g_{11}}$ is again independent of s_2'. By continuing this procedure, we see that all the coefficients in the expansions for the metric tensor depend only on s_1. Hence, it follows that the curvature of all the lines of force is independent of s_2' and, consequently, they are all circles.

So far, we have confined our attention to incompressible motion and have shown that it cannot transform an axially symmetric equilibrium configuration into an equilibrium configuration that is not symmetric. In the real situation defined by (2), the potential component of the flow is small, but not zero. Let us return to (3), which describes the proposed equilibrium state. As we have said, strictly incompressible motion leads to a pressure distribution [on the right-hand side of (3)] that is in conflict with the field distribution and the form of the metric tensor. The question is whether the potential component of the velocity can redistribute the pressure so that (3) does, in fact, describe an equilibrium state. To answer this question, let us write out the equation for the energy density of the system per unit length along the z axis (see Ref. 10);

$$W = W_\perp + W_p,$$

$$W_\perp = \int \frac{H_\perp^2}{8\pi}\,dx\,dy, \quad W_p = \int \left(\frac{p}{\gamma-1} + \frac{H_z^2}{8\pi}\right) dx\,dy, \tag{7}$$

where γ is the adiabatic exponent. Incompressible deformation will change only $W_\perp$. In the new state, the quantity $W_\perp$, assumes the value $W'_\perp$, and the proposed balancing of the pressure is accomplished by potential motion of very low intensity [in the approximation given by (2)]. The magnetic configuration remains unaltered during this process, i.e., $W'_\perp$ is conserved and W_p assumes a value $W'_p < W_p$. We now return to the reverse incompressible transformation to the axially symmetric configuration, for which the energy will assume the value $W = W'_\perp + W'_p$. The asymmetric configuration is assumed to correspond to an equilibrium state and, as above, the symmetric configuration will no longer satisfy (3). The redistribution of pressure by the potential (radial) motion leads to equilibrium, $W_\perp$ is conserved, and W'_p assumes the value W''_p, $W''_p < W'_p$. We thus find that two values of energy correspond to the axially symmetric configuration for the same distribution of A and the same external pressure $P(s_1^{(0)})$, namely, $W = W_\perp + W_p$ and $W' = W_\perp + W''_p$, $W''_p < W_p$, where both states are obtained from each other by radial compression. Since, however, Eq. (3) uniquely determines the pressure distribution, the existence of two such states is impossible.

We note that known equilibrium configurations do not contradict the above conclusion. Thus, in the one-dimensional case, Eq. (1) is easily satisfied on a straight line $\mathbf{H}_\perp = 0$. However, the lines of force (straight lines) are then no longer curves inscribed into one another, and cannot be transformed into circles by compressive motion. The two-dimensional configuration with pressure $P = \exp(-A)$ has long been known.[11] A further configuration for which $P = aA^2 + b$ was proposed in Ref. 6. In these two examples, the field $\mathbf{H}_\perp$ vanishes only at points (and not on curves) and, naturally, the families of lines of force are not topologically equivalent to concentric circles. It is interesting to note that Eq. (1) is linear for $P = aA^2 + b$: it becomes the two-dimensional Schrödinger equation. It is known that a nontrivial solution may exist (if 2a is an eigenvalue of the problem) under the following boundary conditions: normal component of the vector ∇A equal to zero, or A (or the tangential component of ∇A) is zero. However, the vanishing of the resultant vector ∇A (i.e., $\mathbf{H}_\perp = 0$) is a condition that is too stringent. And provided only that the bounding curve is a circle, this requirement reduces to the single requirement that the normal component must be zero, since the condition $A = 0$ (or $A = \text{const}$) is automatically satisfied on the boundary for an axially symmetric problem. A nontrivial solution exists in the latter case, and $A \equiv 0$ is a solution of (1) for an arbitrary bounding curve.

§2. EVOLUTION OF THE FIELD IN THE TWO-DIMENSIONAL PROBLEM

Equilibrium will be lost when the symmetric field is deformed (however slightly). This does not, however, mean that we are dealing with an instability. In fact, if we take the system out of the state of equilibrium and let go, we find that Alfven-type oscillations take place, i.e., the system is stable. If, on the other hand, we compress the tube of force under consideration (for example, by magnetic "walls," i.e., a strong external transverse magnetic field), we find that equilibrium is lost. Parker[6] has examined "densely packed" cells for which the shape of the $\mathbf{H}_\perp = 0$ curve is not only noncircular but cannot, in general, by always convex. According to Ref. 6, the subsequent evolution of the field is described as follows. There is no equilibrium, so that the plasma is brought into motion which, in turn, produces a rise in the field gradient in the neighborhood of the line separating oppositely directed fields $\mathbf{H}_\perp$ in successive cells. This produces a current layer, and ohmic damping comes into play. The subsequent reconnection of the lines of force ultimately leads to the coalescence of the cells (this is referred to as the "coalescence of magnetic islands" in plasma experiments and numerical problems related to this question[12,13]). In our discussion below, we shall consider the interaction between two magnetic tubes with $\mathbf{H}_\perp = 0$ on their outer boundaries.

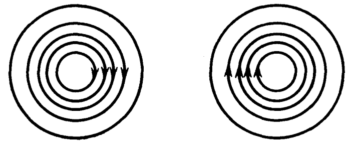

FIG. 1. Arrows show the direction of the magnetic flux. Outermost curve corresponds to the edge of the cell, where $\mathbf{H}_\perp = 0$.

In the $z = \text{const}$ plane, the lines of force of the field $\mathbf{H}_\perp$ take the form of two cells (Fig. 1), and all the lines of force are circles. Let us suppose that the left-hand cell moves toward the right-hand cell with initial velocity V. When the two collide, the lines of force become deformed (Fig. 2), and equilibrium is disturbed. In the absence of dissipation, the collision is elastic, the cells separate, and, ultimately, equilibrium is reestablished. The picture is qualitatively different when finite conductivity is taken into account. When the velocity V is high in comparison with the Alfven velocity and the velocity of sound, the rise in the field gradient at the point of contact between the two cells occurs quite rapidly, and this leads to dissipation and the reconnection of the lines of force. In the discussion given below, we shall always be interested in slow initial motion. Dissipation is then relatively ineffective, but some of the magnetic flux (if only a very small fraction) will, nevertheless, be reconnected during the collision between the cells, and will embrace both profiles (Fig. 3). Finite dissipation cannot give rise to reconnection that will return the field to the state of Fig. 2. In fact, let us suppose that $A > 0$ inside the cell and $A = 0$ everywhere else. Consequently, initially, $A = 0$ on $x = 0$, and $A > 0$ after reconnection. Diffusion is described by the equation

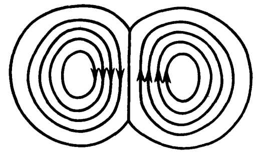

FIG. 2. Approach and compression of cells.

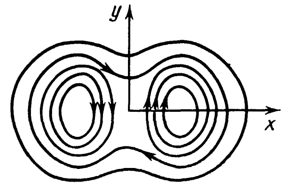

FIG. 3.

$$\partial A/\partial t = D\Delta A. \tag{8}$$

In the neighborhood of the point $x = y = 0$, we have $A = a_0 + a_2x^2$, $a_0, a_2 > 0$. Substituting this expansion into (8), we obtain $da_0/dt = D\,2a_2 > 0$. Consequently, the quantity A can only increase at $x = 0$ as a result of diffusion, and this leads to additional reconnection.

To elucidate the subsequent evolution of the fields, we must establish whether the configuration of Fig. 2 can be an equilibrium configuration. If the answer to this question is yes, the unique process of field dynamics will be determined by the very slow ohmic dissipation of the field. We shall see below that this is not an equilibrium configuration but, of course, it belongs to a class of smooth functions. The point is that field discontinuities are possible in a perfectly conducting medium. However, we shall be interested in a highly conducting (but not perfectly conducting) medium, in which such discontinuities give rise to strong field dissipation. We now turn to the outer (reconnected) lines of force in the configuration of Fig. 3. As noted in Sec. 1, the vanishing of $\mathbf{H}_\perp$ on a closed curve necessarily implies that the curve itself and the lines of force in its neighborhood are all circles. Hence, at least in the outer regions of the configuration of Fig. 3, there is no equilibrium. Tension in the outer lines of force will press inner cells against one another. The question is whether the configuration will become an equilibrium configuration when the external lines of force transform into circles in the course of the evolution process in the absence of field discontinuities. It is obvious that the answer must be no. The point is that, in the outer axially symmetric part of the configuration, the quantity $\mathbf{H}_\perp$ would rise, monotonically from zero on the boundary $(x^2 + y^2)^{1/2} = R$ to some value $H_\perp^0$ on $(x^2 + y^2)^{1/2} = R_1$, $R_1 = y_1$, where y_1 corresponds to the point of intersection between the y axis and the separatrix (between the outer and inner lines of force). In particular, on the y axis itself, the field $\mathbf{H}_\perp$ would rise from zero at $y = \pm R$ to $\mathbf{H}_\perp = H_\perp^0$ at $y = \pm R_1$, $R_1 < R$. At the same time, the symmetry of the configuration with respect to the y axis implies that $\mathbf{H}_\perp = 0$ for $> |y| < R_1$. Consequently, the field $\mathbf{H}_\perp$ would experience, a discontinuity at $y = \pm R_1$.

Thus, the configuration of Fig. 3 (with $\mathbf{H}_\perp = 0$ on the boundary) cannot be an equilibrium configuration. A nonpotential motion due to the absence of equilibrium cannot return the system to the equilibrium state. Within the framework of ideal magnetohydrodynamics, this type of motion would lead to the appearance of discontinuities, whereas the inclusion of finite conductivity results in a current layer at places where the discontinuities might otherwise appear. As we have said, the inner cells press against one another in the above example, with the result that the magnetic-field gradient grows along the y axis, and the current layer is produced. The configuration should ultimately reach a new equilibrium state with an axial symmetry, and a transition can occur only after the reconnection of some of the magnetic flux, i.e., after the dissipation is turned on.

We shall now estimate the parameter of the subsequent field evolution. The absence of equilibrium means that the electro-magnetic force [curl $\mathbf{H}\times\mathbf{H}$] contains a nonpotential component and is therefore not compensated by pressure. For a small deviation from equilibrium, corresponding to a displacement ξ, the nonpotential component is smaller than the potential component by a factor l/ξ. The resulting motion of the plasma is described by

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\nabla p + \frac{1}{4\pi\rho}[\mathrm{rot}\,\mathbf{H},\times\mathbf{H}] \tag{9}$$

$$\approx \frac{1}{4\pi\rho}[\mathrm{rot}\,\mathbf{H},\times\mathbf{H}]\frac{\xi}{l} \approx \frac{v_A^2}{l}\frac{\xi}{l},$$

where $v_A = H_\perp/(4\pi\rho)^{1/2}$. The energy of the system

$$W_\perp = \int (H_\perp^2/8\pi + {}^1/_2\rho v^2)\,dx\,dy \tag{10}$$

is conserved. When $t = 0$, $v = 0$, the increase in the kinetic energy occurs at the expense of a reduction in the magnetic energy. It was shown above that the displacement ξ can produce a reduction in magnetic energy for the configuration shown in Fig. 3. For this to happen, the displacement ξ must be chosen so that the lines of force become as closely circular as possible. It turns out that

$$\delta W' \approx \frac{\xi}{l}\int \frac{H^2}{8\pi}\,dx\,dy < 0.$$

Since the energy (10) is conserved, it follows that

$$v = v_A(\xi/l)^{1/2}. \tag{11}$$

According to (9), this velocity is established in a time

$$t_0 = (l/v_A)\,(l/\xi)^{1/2} = l/v. \tag{12}$$

The field dynamics is now described by

$$\partial A/\partial t + \mathbf{v}\nabla A = D\Delta A, \quad D = c^2/4\pi\sigma, \tag{13}$$

where σ is the conductivity [cf. (8)], and the right-hand side of this equation is negligible. It would appear that the result is the establishment of a state with stationary velocity v given by (11). In actual fact, the convective term $\mathbf{v}\cdot\nabla\mathbf{A}$ in (13) will "mix" A and will lead to an increase in ∇A, i.e., an increase in H^2. This occurs over a period amounting to several times l/v, given by (12) (see, for example, Refs. 14 and 15). We note that, prior to the time t_0, the velocity is less than v, as given by (11), and H^2 does not increase but, as we have seen above, actually decreases.

Thus, the stationary state with v independent of t cannot be established because the entire process cannot continue over a time interval much greater than t_0, given by (12). The only possible evolution of the field reduces to the following. The potential A given by (13) is transported in a time l/v from the center of each cell to the y axis. On the y axis itself, $v_x = 0$ and A remains very small (without initial reconnection, $A = 0$ at $x = 0$, and, after reconnection, $A = a_0$

$\ll \max A$). Hence the gradient ∇A rises substantially in the neighborhood of the y axis (whereas the total magnetic energy $\int(\nabla A)^2 dxdy$ falls slightly). This enhances the role of dissipation: a current layer is formed in the neighborhood of $x=0$ and, consequently, the right-hand side of (13) must be taken into account.

Appreciable field dissipation near $x=0$ produces an additional reconnection of lines of force. When a large proportion of the magnetic flux passes into the outer region, the nonpotential force is no longer small, and deformation is appreciable. This means that we can substitute $\xi = l$ in approximate calculations. Hence, it follows that the last stage of the field dissipation process will occur relatively rapidly, i.e., in a time

$$t_1 = l/v_A, \tag{14}$$

and the velocity will reach the Alfven value. A new equilibrium state is established when all the lines of force have been reconnected. The energy of the new state is $W_1 = H_1^2 l_1^2/8\pi$ and is lower by a factor of two as compared with the old energy $W_0 = 2H^2l^2/8\pi$. The point is that $l_1^2 = 2l^2$ (conservation of total area) and $H_1 l_1 = Hl$ (conservation of flux). The excess energy has been dissipated. It is important to remember that the power release rises sharply toward the end of the evolution process: it is estimated as $(v/l)H^2l^2/8\pi$ after the current layer has been formed, and as $(v_A/l)H^2l^2/8\pi$ at the final stage. Thus, we can identify four phases of the evolution of the field with the initial configuration shown in Fig. 3. Figure 4 illustrates schematically the dynamics of kinetic energy, which also reflects the rate of energy release or dissipation. The four phases mentioned above are clearly seen: (1) formation of flow and of the current layer in time $t_0' < t_0$, (2) slow (over time t_0) dissipation, (3) rapid rise in kinetic energy and dissipation of magnetic energy (in time $t_1 \ll t_0$), and (4) establishment of new equilibrium and end of motion. All four phases can readily be followed in the numerical calculation of an analogous field configuration given in Ref. 16.

We note that, in the above example, the ratio ξ/l, which is a measure of the departure from equilibrium, is determined by finite dissipation in the two-cell collision, and is therefore very small. This means that the preliminary stage that occupies the time t_0 is very long. However, the vigorous third stage (Fig. 4) will nevertheless occur, and the rate of energy release will no longer be dependent on ξ/l. A more interesting situation is that shown in Fig. 3. It occurs by itself at the initial time, before the two stages of Figs. 1 and 2 have occurred. In Parker's problem[6] (see Sec. 1), this configuration can be established by special motion on the outer boundary of the layer. The ratio ξ/l is then specified by the initial conditions, and may be of the order of unity. In the last case, the preliminary first and second phases do not occur, and dissipation occurs relatively rapidly.

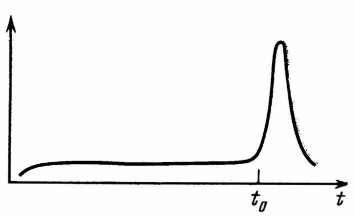

FIG. 4. Energy release as a function of time.

§3. EVOLUTION OF FORCE-FREE FIELDS

The loss of equilibrium and the formation of current layers are probably a common phenomenon in nature. In this section, we shall investigate force-free and potential configurations. Consider a low-pressure plasma with $\beta = p8\pi/H^2 \ll 1$. This situation prevails in a very tenuous medium (for example, the solar corona), and equilibrium can be established only if $[\text{curl}\,\mathbf{H}\times\mathbf{H}] = 0$ or

$$\text{rot}\,\mathbf{H} = \alpha\mathbf{H}. \tag{15}$$

Fields satisfying (15) are referred to as force-free and, when $\alpha = 0$, is irrotational or potential. We shall consider fields in a conducting layer, as mentioned in Sec. 1. Suppose that the initial field is irrotational with the following boundary conditions in terms of the cylindrical polar coordinates (r, φ, z): at $z = 0, L$, i.e., on the boundary of the layer,

$$H_r = \partial\psi/\partial r, \quad H_\varphi = 0, \quad H_z = \partial\psi/\partial z, \tag{16}$$

where the function ψ at $z=0$ is independent of φ and is nonzero only for $r<l$. The solution in the interior of the layer, where curl $\mathbf{H}=0$, is $\mathbf{H}=\nabla\psi$, where $\Delta\psi = 0$. The boundary conditions (16) uniquely determine the field in the interior of the layer, and it is easy to see that this field is axially symmetric and takes the form of lines of force leaving the lower and entering the upper "spot." The entire configuration expands toward the center of the layer, and $H_\phi = 0$ not only on the boundary, but also in its interior. To be specific, we shall suppose that $H_z > 0$ at $z = 0, L$, and it is for this reason that we say that the lines of force "leave" the lower spot. Let us now examine a further pair of spots for which $H_z < 0$ on the boundaries. It is clear that, when these two pairs are remote from one another, i.e., when the separation between them is much greater than L, the potential fields in each of the pairs have practically no interaction with one another and take the form described above. If, on the other hand, their separation is much less than L, some of the lines of force belonging to the first pair will close on the lower spot of the second, and vice versa: some of the flux associated with the second pair will transfer to the first.

Let us now suppose that, initially, the two pairs are distant from one another but, subsequently, the boundary conditions lead to their approach (this can readily be achieved by simple motion of the plasma at $z=0, L$ with the frozen-in field). When there is no plasma between the $z=0$ and $z=L$ planes, the approach of the two pairs to a distance L will be accompanied by the reconnection of the magnetic flux from the first to the second pair, and vice versa since, in this case, curl $\mathbf{H}=0$ throughout the entire process. Highly conducting plasma cannot modify the field topology, since this requires finite dissipation. It would appear at first sight that, if the two pairs under consideration approach one another to a distance, say, $L/2$, and the motion is stopped, the fields will push against each other and an equilibrium state will be established with the line of force as before leaving each lower spot for its own upper spot. However, this equilibrium will not, in fact, be established. The point is that, in this case, it will be described by (16), where the condition div $\mathbf{H} = 0$ implies that $\nabla\alpha\cdot\mathbf{H} = 0$. This means that α remains constant along the line of force. However, on $z = 0, L$

$$\mathrm{rot}_z\,\mathbf{H}=\alpha\mathbf{H}_z=0,$$

from which it follows that $\alpha = 0$, i.e., under these boundary conditions, the equilibrium field can only be irrotational and, as already noted, its topology must be different: a substantial proportion of the flux belonging to the first pair closes on the second, and vice versa.

The absence of equilibrium signifies the appearance of motion, leading to the formation of a current layer, the turning on of dissipation, and, finally, the reconnection of the lines of force (as in Sec. 2). This process was convincingly demonstrated by Syrovatskiĭ (see, for example, Ref. 8). From the point of view of applications, there is considerable interest not so much in the layer iteself as in the half-space $z > 0$ with the boundary conditions defined on $z = 0$. It is clear that each pair of spots has associated with it fields for which $H_z > 0$ for one and $H_z < 0$ for the other. The separation between them is L. Two distant pairs (separated by a distance much greater than L) are topologically unrelated; their approach must be accompanied by loss of equilibrium and reconnection.

A similar situation obtains for force-free fields. First of all, we recall the boundary conditions that are necessary to determine unambiguously the force-free field at $z = 0$. Equation (15) does not seem to be closed because α is an arbitrary function of position. In point of fact, however, and, as we have already said, α is constant along a line of force and is determined by its value on the $z = 0$ boundary. On the boundary itself, α is determined from the equation

$$\alpha H_z=\partial_x H_y-\partial_y H_x, \qquad (17)$$

where all the quantities are specified on $z = 0$. It is clear from (17) that, before α can be determined, we must know all three components of the field at $z = 0$. For $z > 0$, the field is uniquely reconstructed by using (15) and (17), for example, by expanding all the quantities in series in powers of z.[17]

We now return to the consideration of the field in a conducting layer between the $z = 0$ and $z = L$ planes. Suppose that, now, in contrast to (16), we have on $z = 0, L$

$$H_\varphi(r)\neq 0, \quad H_r\neq 0, \quad \partial H_\varphi(r=l)/\partial r=0, \quad H_z=\mathrm{const}.$$

All the functions are independent of φ, as before. In addition, we shall assume that $l \ll L$. According to (17), $\alpha(z = 0)$ vanishes for $r = l$ and changes sign for $r < l$. Its order of magnitude is $\alpha \sim 1/l$ and, since α is conserved along the line of force, $\alpha \approx 1/l$ in the interior of the layer. On the other hand, according to (15), α determines the reciprocal scale of the field, i.e., in the interior of the layer, the field varies over a characteristic length l. Along the z axis, the field varies over the much greater length L, so that the fields in the interior of the layer may be looked upon as quasi-two-dimensional (the deviation from the two-dimensional configuration is characterized by the small parameter l/L). The configuration can be described qualitatively, as follows. The lines of force are straight for $r > l'$, $l' \simeq l$ because $H_z = \mathrm{const}$, $H_\varphi = H_r = 0$, whereas, for $r < l'$, the field has a helical shape. The equilibrium state must obey (1) for $p = 0$ and the effective pressure is $H_z^2/8\pi$. It is shown in Sec. 1 that the equilibrium state must be symmetric. In fact, this conclusion is trivial because of the symmetry of the boundary conditions. However, the conclusion that axial symmetry prevails in the interior of the zone $0 < z < L$ remains valid even for nonsymmetric boundary conditions. In the latter cse, there are in the neighborhood of the boundary $z = 0, L$ two transition regions of thickness $l'' \simeq l$, in which a transition occurs between the nonsymmetric fields and the symmetric configuration was $l'' < z < L - l''$. According to (15), the latter is described by

$$H_r=0, \quad -\partial H_z/\partial r=\alpha H_\varphi, \quad (1/r)\,\partial r H_\varphi/\partial r=\alpha H_z.$$

Eliminating α, we obtain

$$\frac{H_\varphi}{r}\frac{\partial r H_\varphi}{\partial r}+\frac{1}{2}\frac{\partial H_z^2}{\partial r}=0, \qquad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right)+\frac{1}{2}\frac{\partial H_z^2}{\partial A}=0, \qquad (18)$$

i.e., we again have (1), where $P = H_z^2/8\pi$, $p = 0$.

Thus, force-free fields in low-pressure plasmas can have a quasi-two-dimensional character, and the results of Secs. 1 and 2 extend to this case as well. Toroidal fields are similarly established in plasma occupying the half-plane $z > 0$ for one of the pairs of spots and $H_z < 0$ for the other. As the force-free magnetic tubes approach one another and come into contact, equilibrium is destroyed and processes involving the formation of the current layer and dissipation of the magnetic field take place. The process can occur as follows. Suppose there are two tubes of force, similarly to the situation in the example with irrotational fields. If the separation between the tubes (or spot pairs) is greater than L, they interact weakly with one another. As they come closer, to a distance of the order of l, the characteristic dimensions of the resultant field on the $z = 0, L$ boundaries also become of the order of l and, as indicated above, the state of equilibrium should exhibit the properties of axial symmetry. This means that the force-free fields of the two tubes should merge into a single axially symmetric configuration for $l'' < z < L - l''$. However, this steady state can occur only after the reconnection of the transverse field $\mathbf{H}_\perp$.

We now proceed to the question of what happens after the breakdown of equilibrium. We begin with the example of irrotational fields. Suppose that two tubes appraoch one another with velocity V. When the separation between them is of the order of L, equilibrium is destroyed, as already pointed out. We recall that the characteristic thickness of a tube at the point of contact with the other is of the order of L. We shall be interested in the parameter ξ/L, which measures the deviation from equilibrium. It is clear that ξ varies with time as Vt. The absence of equilibrium produces a flow described by $v = t(v_A^2/L)(\xi/L)$ [cf. (9)], and the velocity reaches the value $v = v_A(\xi/L)^{1/2}$ in the time $t_0 = L/v$, and the same time is necessary for dissipation to result in a new equilibrium. Equating t_0 to the time necessary for ξ, to grow, we obtain the steady value

$$\xi/L=(V/v_A)^{2/3}.$$

When the ratio V/v_A is small, a stationary state with a current layer is established in the time

$$t_0=L/v=(L/v_A)(v_A/V)^{3/4}$$

and leads to field dissipation and to a departure from equilibrium. The entire process of dissipation occurs relatively slowly for small V/v_A and the entire "surplus" energy is released in the long time $t_2 = L/V$.

Dissipation of the force-free field occurs in a different way. Since the tube thickness is small, $l'\simeq l$, the departure from equilibrium occurs only when the two tubes approach one another to a distance l'. A new state is then established with reconnected lines of force of the transverse field (Fig. 3) in the time

$$t_0=l/v=(l/v_A)(v_A/V)^{3/4} \quad . \tag{19}$$

The subsequent evolution is practically independent of the motion of the spots, i.e., the boundary conditions, since, according to (19), the time t_0 is also the time for complete reconnection. Although this time is shorter by a factor $(L/l)(v_A/V)^{1/4}$ than the dissipation time during the approach of the irrotational fields, it is still quite long. However, the principal difference, as compared with the irrotational case, is that the case of the force-free field (as in the configuration of Fig. 3) is characterized by a fast energy-release stage, as in Fig. 4.

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