Long gravitational waves in an unltrarelativistic two-component universe

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A general solution is obtained for the self-consistent linearized system of Einstein and Vlasov equations describing the evolution of long-wavelength perturbations of gravitational-wave type in a two-component Friedmann universe containing an ultrarelativistic ideal fluid and a collisionless gas. In contrast to the hydrodynamic approximation first considered by Lifshitz, the kinetic picture of the development of perturbations in a statistically nonequilibrium medium includes in the general case an infinite set of partial modes. Under the assumption of a local-equilibrium nature of the initial stage of the cosmological expansion the conditions of excitation of the partial modes at the time when the neutrinos decouple from the primordial radiation-dominated plasma are analyzed. It is shown that if the contribution of the gravitons to the space-time curvature is ignored the evolution of long gravitational waves in the ultrarelativistic two-component Friedmann universe is described by Lifshitz's classical theory, i.e., in this case the collisionless nature of the neutrinos is not manifested.

1. INTRODUCTION

The problem of the gravitational stability of cosmological models based on exact solutions of the equations of the general theory of relativity was first formulated by Lifshitz[1] and solved for a Friedmann universe with hydrodynamic energy-momentum tensor. Subsequently, various authors returned frequently to this question from different points of view and discussed the evolution of small perturbations on the background of a homogeneous and isotropic cosmological model (see, for example, the extensive survey of Lifshitz and Khalatnikov[2] and the papers of Lukash and Bardeen,[3] which contain a bibliography on this question). Despite a long history and an appreciable number of publications, the problem of describing the evolution of small perturbations in a homogeneous and isotropic universe has still not yet been completely resolved for the case when the perturbed state of the cosmological medium is not necessarily a statistically equilibrium state. In this most general case, the components of the matter energy-momentum tensor are specified as quadratures of the statistical distribution function. The development of cosmological perturbations is described by the self-consistent linearized system of Einstein equations and kinetic equations that specify the source of the gravitational field. The problem of the gravitational stability of the Friedmann cosmological model is given such a kinetic treatment by Zakharov[4] and Vishniac,[5] though their results require to be made more precise and complete.

In the general case, a statistically nonequilibrium medium is characterized by infinitely many internal degrees of freedom and corresponding perturbation modes. The partial contribution of such modes to the resulting evolution regime of the gravitational perturbations is determined not only by the initial conditions but also by the collision integral, which specifies the nature of the microscopic interactions. In this paper, to simplify the formulation of the problem, we take a model of a two-component medium with one component completely collisionless and the other "collision dominated," i.e., in local thermodynamic equilibrium. Such a model adequately describes the early universe, including weakly interacting particles (gravitons, neutrinos) and a radiation-dominated plasma.

It should be emphasized that the use of the hydrodynamic approximation and, thus Lifshitz's classical results[1] to describe the development of *arbitrary* perturbations near the cosmological singularity is not strictly speaking physically justified, since the real universe contains free gravitons already at the Planck time. The hydrodynamic approximation, which distringuishes the equilibrium degrees of freedom and freezes arbitrarily all the other degrees of freedom, essentially castrates the stability problem for the cosmological model as a whole, substituting for it the question of stability with respect to perturbations of a special form. The degeneracy of the degrees of freedom due to the hydrodynamic approximation is completely lifted in a self-consistent kinetic theory of perturbations. As one would expect, the system of equations for the perturbations in such a theory has an infinite set of fundamental solutions, the analysis of which ultimately permits a division of the complete set of admissible initial data into stability and instability regions. In other words, the solution to the problem of the gravitational stability of a given (background) cosmological model depends on the kinetic properties of the matter that produces the gravitational field.

In the present paper, we consider the evolution of long-wavelength perturbations of gravitational-wave type in a Friedmann universe containing two components: an ideal fluid (without viscosity and elasticity) with equation of state $p = \varepsilon/3$ and an ultrarelativistic collisionless gas satisfying Vlasov's kinetic equation. Analysis of this problem in the case when the universe contains just an ideal fluid is contained in the paper of Lifshitz quoted earlier.

We recall that the linearized stress tensor of a Pascal ideal fluid in a comoving frame reduces to a single scalar—the isotropic pressure. On the other hand, an anisotropic deformation due to a weak gravitational wave in a homogen-

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eous and isotropic expanding universe must be described by a doubly transverse traceless tensor, so that by virtue of Einstein's equations such a deformation takes place with constant pressure and, therefore, constant density of the medium. Physically, this means that when weak gravitational waves propagate in a homogeneous and isotropic universe filled with a Pascal ideal fluid the latter changes its shape without resistance, restoring thermodynamic equilibrium instantaneously; as a result, it has no reaction back on the gravitational wave. Therefore, in the hydrodynamic approximation the linearized Einstein equations for the (small) amplitude of an elementary doubly transvere plane wave of each of the two possible polarizations reduce to a second-order homogeneous equation with two independent solutions. As Lifshitz showed in the long-wavelength approximation these solutions correspond to an asymptotically constant (principal) mode and a singular (decreasing) mode of the gravitational waves.

The detailed picture of the propagation of gravitational waves in the two-component medium mentioned above is not so elementary by virtue of its nontrivial (compared with the case of a Pascal ideal fluid) polarization properties.[1)] Because of the existence of "intrinsic" bulk gravitational radiation, the collisionless component of the medium may interact with "incident" gravitational waves, this being reflected in the nature of the evolution of small perturbations of tensor type. In contrast to Lifshitz's hydrodynamic theory, the system of equations for the perturbations, which contains the linearized equations of the gravitational field augmented by a kinetic equation, is an integrodifferential system. Allowance for the gravitational influence of the anisotropic stresses of the collisionless component of the medium, which has infinitely many internal degrees of freedom, leads to the appearance of an infinite set of additional perturbation modes that have a kinematic nature and reflect the arbitrariness in the choice of the initial data for the distribution function. The rate of growth of these kinematic perturbation modes in the long wavelength approximation is higher the greater their anisotropy in the momentum space. We emphasize that this conclusion is obtained in the framework of linear theory and does not tell us what is the "final" fate of perturbations that evolve in accordance with the kinematic modes mentioned above.

The paper is arranged as follows. In Sec. 2, we give the necessary expressions that describe the evolution of the background cosmological model. In Sec. 3, we obtain the linearized system of Einstein-Vlasov equations for perturbations of gravitational-wave type in a Friedmann universe containing an ultrarelativistic ideal fluid and a collisionless gas. In Sec. 4, we indicate the key to the construction of a general solution of this system in the long-wavelength approximation and describe some particular solutions. In the fifth and final section, we describe the application of the theory to the early universe, including a radiation-dominated plasma and a gas of ultrarelativistic collisionless neutrinos.

2. BACKGROUND COSMOLOGICAL MODEL

In the early stage of cosmological expansion, the evolution laws for open, closed, and spatially flat Friedmann models are asymptotically the same,[6,7] and therefore the background geometry can be described by the Einstein–de Sitter metric[2)]:

$$g_{ik}=a^2(\eta)\,\mathrm{diag}\{+1;\ -1;\ -1;\ -1\} \tag{2.1}$$

($x^0=\eta>0;\ \ -\infty<x^1,\ x^2,\ x^3<+\infty$). The scale factor $a=a(\eta)$ of the cosmological model satisfies Einsteins's equations:

$$G_k^{\ i}\equiv R_k^{\ i}-\frac{1}{2}R\delta_k^{\ i}=\frac{8\pi G}{c^4}T_k^{\ i}; \tag{2.2}$$

$$T_k^{\ i}=T_k^{i(fluid)}+T_k^{i(gas)}\ ; \tag{2.3}$$

$$T_k^{i(fluid)}={}^1/_3\varepsilon_f(\eta)\,(4u^iu_k-\delta_k^{\ i}); \tag{2.4}$$

$$u^i=\{1/a;\ 0;\ 0;\ 0\}; \tag{2.5}$$

$$T_k^{i(gas)}=c\int\frac{(-g)^{1/2}d^3p}{p_0}p^ip_kF(\eta,x^\alpha;p^\beta). \tag{2.6}$$

The single-particle distribution function $F=F(\eta,\ x^a;\ p^\beta)$, describing the collisionless component of the medium (which consists of identical particles of mass m) satisfies the condition of constancy along the phase trajectories of the particles:

$$p^i\frac{\partial F}{\partial x^i}-\Gamma_{ik}^{\ \ \alpha}p^ip^k\frac{\partial F}{\partial p^\alpha}=0. \tag{2.7}$$

By virtue of the homogeneity and isotropy of the background cosmological model, the corresponding distribution function can depend only on the kinetic energy of the particles in the comoving frame[8]:

$$F=F_0[a^2p^ip^k(u_iu_k-g_{ik})]\equiv F_0(q^2), \tag{2.8}$$

where q denotes the magnitude of the Cartesian vector $\mathbf{q}=\{a^2p^\alpha\}$. For an ultrarelativistic (and sufficiently rapidly decreasing as $q\to\infty$) distribution function F_0 the background energy-momentum tensor of the collisionless gas (2.6) takes the form

$$T_k^{\ i(gas)}={}^1/_3\varepsilon_g(\eta)\,(4u^iu_k-\delta_k^{\ i}), \tag{2.9}$$

where

$$\varepsilon_g(\eta)=\frac{4\pi c}{a^4}\int_0^\infty dqq^3F_0(q^2). \tag{2.10}$$

Because the tensors (2.4) and (2.9) have the same structure, the Einstein equations (2.2) and (2.3) for the background metric (2.1) reduce to the cosmological Friedmann equations[6,7] describing the evolution of a conformally flat universe effectively containing only a single ultrarelativistic ideal fluid with energy density

$$\varepsilon_b=\varepsilon_f+\varepsilon_g \tag{2.11}$$

(the suffix b stands for background). The solution of these equations has the form

$$a=a_0\eta;\ \ \varepsilon_b=3c^4a_0^2/8\pi Ga^4, \tag{2.12}$$

where

$$a_0=\frac{4\pi}{c}\left[\frac{2G}{3c}\frac{\alpha+1}{\alpha}\int_0^\infty dqq^3F_0(q^2)\right]^{1/2}, \tag{2.13}$$

where $\alpha\equiv\varepsilon_g/\varepsilon_f=$ const is the ratio of the background energy densities of the collisionless particles and the ideal fluid.

3. SYSTEM OF EQUATIONS FOR TENSOR PERTURBATIONS. HYDRODYNAMIC APPROXIMATION

Let $\delta g_{\alpha\beta} = -h_{\alpha\beta}$ be certain small perturbations of the metric of the background space-time corresponding to weak gravitational waves in the Einstein–de Sitter universe. We represent the spatial dependence of $h_{\alpha\beta}$ in the form of a Fourier integral and thus reduce the problem of describing tensor perturbations of arbitrary shape to the consideration of an elementary doubly transverse plane wave:

$$h_\beta^\alpha = \int d^3n[\nu(\mathbf{n};\eta)\Pi_\beta^\alpha + \tilde{\nu}(\mathbf{n};\eta)\tilde{\Pi}_\beta^\alpha]e^{i\mathbf{n}\mathbf{r}} \quad (3.1)$$

(in what follows, we shall not write out the dependence of the Fourier amplitudes of the perturbations on the wave vector explicitly and we shall omit the sign of integration over d^3n). The Fourier amplitudes ν and $\tilde{\nu}$ describe gravitational waves of two independent polarizations[3]) propagating along $\mathbf{n}$:

$$\Pi_\beta^\alpha = (s^\alpha m_\beta + m^\alpha s_\beta)/sm; \quad (3.2)$$

$$\tilde{\Pi}_\beta^\alpha = (m^2 s^\alpha s_\beta - s^2 m^\alpha m_\beta)/s^2m^2. \quad (3.3)$$

It is convenient to replace the conformal time η by the new time variable $\tau \equiv n\eta$, which is proportional to the ratio of the distance to the instantaneous horizon to the wavelength of the considered perturbation ($\tau \propto t^{1/2}$). For perturbatons of tensor type with the polarization (3.2), we have

$$\delta G_\beta^\alpha = -\frac{n^2}{2a^2}\left(\nu + \frac{2}{\tau}\frac{d\nu}{d\tau} + \frac{d^2\nu}{d\tau^2}\right)\Pi_\beta^\alpha e^{i\mathbf{n}\mathbf{r}}; \quad (3.4)$$

$$\delta G_0^i = 0; \quad (3.5)$$

$$\delta T_k^{i(fluid)} = \delta T_0^{i(gas)} = 0. \quad (3.6)$$

At the same time, the deviator of the linearized kinetic stress tensor does not vanish in the general case:

$$\delta T_\beta^{\alpha(gas)} = -\frac{1}{6}\varepsilon_b \Pi_\beta^\alpha e^{i\mathbf{n}\mathbf{r}} \int_{-1}^{+1} dx(1-x^2)\Phi(\tau;x), \quad (3.7)$$

where

$$\Phi(\tau;x) \equiv \frac{8\pi G}{c^3 a_0^2}\int_0^{2\pi} d\varphi \sin 2\varphi \int_0^\infty \frac{dq q^4}{(m^2c^2a^2+q^2)^{1/2}} f(\tau;\mathbf{q}) - \frac{3\alpha(1-x^2)}{2(\alpha+1)}\nu(\tau) \quad (3.8)$$

and $f = f(\tau;\mathbf{q})$ is the Fourier amplitude of the perturbation δF of the background distribution function F_0 (the amplitude decreases sufficiently rapidly as $q \to \infty$). With regard to the function $f(\tau;\mathbf{q})$, we shall assume in all that follows that it is ultrarelativistic, i.e., the decisive contribution to the integral over dq in (3.8) is made by the region of integration for which $q \gg mca$. In accordance with this assumption, the radical $(m^2c^2a^2 + q^2)^{1/2}$ in the expression (3.8) can be replaced by the magnitude q of the conformal momentum. We emphasize that it is pecisely the presence of the shear stresses (3.7) and the bulk gravitational radiation associated with them that distinguish the collisionless gas from an ideal fluid.

Taking into account (3.4)–(3.8) and the assumptions made above, we can write the system of linearized Vlasov and Einstein equations, (2.7) and (2.2)–(2.3), for the tensor perturbations with the polarization (3.2) in the form

$$\frac{\partial\Phi}{\partial\tau} + ix\Phi + \frac{3\alpha(1-x^2)}{2(\alpha+1)}\frac{d\nu}{d\tau} = 0; \quad (3.9)$$

$$\tau^2\frac{d^2\nu}{d\tau^2} + 2\tau\frac{d\nu}{d\tau} + \tau^2\nu = \int_{-1}^{+1} dx(1-x^2)\Phi(\tau;x). \quad (3.10)$$

The equations for the gravitational waves of the second independent polarization (3.3) can be obtained from (3.4)–(3.10) by making the formal substitution

$$\nu \to \tilde{\nu};\ \sin 2\varphi \to \cos 2\varphi;\ \Pi_\beta^\alpha \to \tilde{\Pi}_\beta^\alpha. \quad (3.11)$$

Thus, the original problem has been reduced to the solution of the integro-differential system of equations (3.9) and (3.10) and the determination of the anisortopic stresses in the collisionless component of the medium by means of (3.7).

Under the conditions $\alpha = 0$, $\Phi \equiv 0$, the system of equations (3.9)–(3.10) describes the evolution of tensor perturbations in a Friedmann universe containing just an ultrarelativistic ideal fluid. In the long-wavelength approximation ($\tau \ll 1$), Lifshitz obtained the following two hydrodynamic modes of gravitational waves: an asymptotically constant (principal) mode and one decreasing in the metric proportional to $t^{-1/2}$:

$$\nu = \mathfrak{G}\frac{\sin\tau}{\tau} + \mathfrak{D}\frac{\cos\tau}{\tau}. \quad (3.12)$$

It is obvious that if at some time t_1 near the Planck time ($t_{Pl} = 10^{-43}$ sec) the dimensionless perturbations of the metric corresponding to the principal and decreasing modes were in order of magnitude equal to ε, then in accordance with (3.12) these perturbations at $t_2 = \varepsilon^{-2}t_1$ must become ε^{-1} times greater for the principal mode than for the decreasing one. In other words, beginning with the time t_2 first-order perturbation theory become inapplicable for the description of the decreasing mode of the gravitational waves. Taking $\varepsilon = 10^{-4}$, we see that the time t_2 in the evolution of the real universe occurs much earlier than the decoupling of any of the three species of cosmological neutrino (ν_e, ν_μ, and ν_τ) from the primordial plasma. Therefore, in what follows we shall set $\mathfrak{D}$ when matching the asymptotic expansions for the perturbations at the time of decoupling of the cosmological neutrinos from the radiaton-dominatd plasma.

4. GENERAL SOLUTION OF THE SYSTEM OF EQUATIONS FOR TENSOR PERTURBATIONS IN THE LONG-WAVELENGTH APPROXIMATION

We shall seek solutions of the system of equations (3.9), (3.10), and (3.7) in the form of expansions in powers of the small parameter $\tau \ll 1$. First, we write down a solution satisfying the requirement $\nu(\tau) \to \text{const} \neq 0$ as $\tau \to 0$ (here and in what follows, we retain as many terms of the expansions as are needed for the subsequent matching):

$$\nu = \mathcal{G}\left[1 - \frac{5(\alpha+1)}{2(19\alpha+15)}\tau^2 + \frac{5(\alpha+1)(103\alpha+105)}{168(19\alpha+15)(27\alpha+25)}\tau^4 + \dots\right]; \quad (4.1)$$

$$\Phi(\tau;x)=\mathscr{G}\frac{15\alpha(1-x^2)}{4(19\alpha-15)}\left[\tau^2-\frac{ix\tau^3}{3}-\frac{x^2\tau^4}{12}\right.$$
$$\left.-\frac{103\alpha+105}{84(27\alpha+25)}\tau^4+\ldots\right]; \quad (4.2)$$

$$\delta T_\beta{}^\alpha=-\mathscr{G}\varepsilon_b\Pi_\beta{}^\alpha e^{inr}\frac{2\alpha}{3(19\alpha+15)}$$
$$\times\left[\tau^2-\frac{65(\alpha+1)}{42(27\alpha+25)}\tau^4+\ldots\right]. \quad (4.3)$$

As $\alpha\to0$, the $\mathscr{g}$ mode of (4.1)–(4.3) goes over into Lifshitz's hydrodynamic $\mathfrak{G}$ mode.

The conditional $\nu(\tau)\to0$ as $\tau\to0$ leads to an infinte set of independent solutions that do not have hydrodynamic analogs. These solutions can be constructed as follows. Let $\Omega(x)$ be an arbitrary function. We form the function

$$\Psi(x)=\Omega(x)+\frac{15}{16}(x^2-1)\int_{-1}^{+1}d\xi(1-\xi^2)\Omega(\xi) \quad (4.4)$$

and regard it as the leading term of the expansion of $\Phi(\tau;x)$ in powers of τ. Specification of the function $\Psi(x)$ fixes the part of the initial data corresponding to the kinematic modes of long gravitational waves discussed in the Introduction. In the general case, this part of the initial data can be represented as an infinite set of moments $\{\mathscr{M}_k\}$ ($k=0,1,2,\ldots$), these being the coefficients in the expansion of $\Psi(x)$ with respect to some complete orthonormal system of functions, for example, the system of Legendre polynomials $\{P_k(x)\}$:

$$\mathscr{M}_k=\int_{-1}^{+1}d\xi\Psi(\xi)\frac{1}{k!\,2^k}\frac{d^k}{d\xi^k}(\xi^2-1)^k\equiv\int_{-1}^{+1}d\xi\Psi(\xi)P_k(\xi). \quad (4.5)$$

By virtue of the definition (4.4) $\mathscr{M}_0=\mathscr{M}_2$, and therefore the function $\Psi(x)$ and, by virtue of perturbation theory, $\Phi(\tau;x)$, $\nu(\tau)$, and $\delta T^\alpha_\beta(\tau)$ are completely characterized by the set of independent moments

$$\mathscr{M}_1,\ \mathscr{M}_2,\ \mathscr{M}_3,\ldots. \quad (4.6)$$

From the solutions of the basic system of equations corresponding to these moments we can form two infinite independent linear combinations that will correspond to "empty" modes, these being solutions of the main system in the calss of generalized functions. The empty modes $\mathscr{V}_1$ and $\mathscr{V}_2$ (for which $\mathscr{M}_1=\mathscr{M}_3=\ldots\mathscr{V}_1$ and $\mathscr{M}_0=\mathscr{M}_2=\ldots\equiv\mathscr{V}_2$, respectively) have the form

$$\Phi(\tau;\ x)={}^1/_2\mathscr{V}_1[\delta(1-x)-\delta(1+x)]e^{-ix\tau}; \quad (4.7)$$
$$\Phi(\tau;\ x)={}^1/_2\mathscr{V}_2[\delta(1-x)+\delta(1+x)]e^{-ix\tau} \quad (4.8)$$

(in Eqs. (4.7) and (4.8), δ denotes Dirac's singular function). The solutions (4.7) and (4.8) do not correspond to any perturbations of the gravitational field (metric) and the matter (the energy-momentum tensor) by merely characterize the arbitrariness in the choice of the function $\Phi(\tau;x)$. The linear combination of any solution of the basic system of equations with an empty mode leads to a solution physically equivalent to the original one. Therefore, the empty modes $\mathscr{V}_1$ and $\mathscr{V}_2$ [or, which is the same thing, the first two moments in the series (4.6)] can be eliminated from the following treatment.

The expansions for ν and δT^α_β corresponding to the $\mathscr{M}_k$ modes contain only even or only odd powers of τ according as the number k is even or odd. These expansions being with terms that increase in proportion to τ^{k-2} ($k\geqslant3$). Thus, $k=3$ we have

$$\nu=i\mathscr{M}_3\frac{\alpha+1}{9\alpha+5}\tau+\ldots; \quad (4.9)$$

$$\Phi(\tau;x)=\frac{7}{2}\mathscr{M}_3P_3(x)(1-ix\tau)-i\mathscr{M}_3\frac{3\alpha(1-x^2)}{2(9\alpha+5)}\tau+\ldots; \quad (4.10)$$

$$\delta T_\beta{}^\alpha=-i\mathscr{M}_3\varepsilon_b\Pi_\beta{}^\alpha e^{inr}\frac{\alpha+1}{3(9\alpha+5)}\tau+\ldots. \quad (4.11)$$

We also write down the corresponding expansions for the first two even $\mathscr{M}_k$ modes, which we shall need in what follows when we do the matching:

$$\nu=\mathscr{M}_4\frac{2(\alpha+1)}{7(19\alpha+15)}\left[\tau^2-\frac{1399\alpha+1365}{924(27\alpha+25)}\tau^4+\ldots\right]$$
$$-\mathscr{M}_6\frac{5(\alpha+1)}{1386(27\alpha+25)}\tau^4+\ldots; \quad (4.12)$$

$$\Phi(\tau;x)=\left[\frac{9}{2}\mathscr{M}_4P_4(x)+\frac{13}{2}\mathscr{M}_6P_6(x)\right]$$
$$\times\left(1-ix\tau-\frac{1}{2}x^2\tau^2+\frac{1}{6}ix^3\tau^3+\frac{1}{24}x^4\tau^4\right)$$
$$-\mathscr{M}_4\frac{3\alpha(1-x^2)}{7(19\alpha+15)}\left[\tau^2-\frac{ix\tau^3}{3}-\frac{x^2\tau^4}{12}\right.$$
$$\left.-\frac{1399\alpha+1365}{924(27\alpha+25)}\tau^4+\ldots\right]$$
$$+\mathscr{M}_6\frac{5\alpha(1-x^2)}{924(27\alpha+25)}\tau^4+\ldots; \quad (4.13)$$

$$\delta T_\beta{}^\alpha=-\varepsilon_b\Pi_\beta{}^\alpha e^{inr}\frac{2(\alpha+1)}{7(19\alpha+15)}\left\{\mathscr{M}_4\left[\tau^2-\frac{379\alpha+525}{693(27\alpha+25)}\tau^4+\ldots\right]\right.$$
$$\left.-\mathscr{M}_6\frac{25(19\alpha+15)}{594(27\alpha+25)}\tau^4+\ldots\right\}. \quad (4.14)$$

Finally, we consider solutions of the basic system of equations that are not regular at the point $\tau=0$. Then in the leading order of the expansion we can ignore the second term in the kinetic equation (3.9) compared with the first, after which integration of (3.9) gives

$$\Phi(\tau;x)=\frac{3\alpha(x^2-1)}{2(\alpha+1)}\nu(\tau). \quad (4.15)$$

Substituting (4.15) in the right-hand side of Eq. (3.10) and solving the second-order differential equation then obtained, we find the two oscillatory modes

$$\nu=\frac{1}{\tau^{1/2}}[\mathscr{D}_1\cos(\omega\ln\tau)+\mathscr{D}_2\sin(\omega\ln\tau)]+\ldots; \quad (4.16)$$

$$\delta T_\beta{}^\alpha=\frac{\varepsilon_b(4\omega^2+1)}{24\tau^{1/2}}\Pi_\beta{}^\alpha e^{inr}[\mathscr{D}_1\cos(\omega\ln\tau)$$
$$+\mathscr{D}_2\sin(\omega\ln\tau)]+\ldots,$$

where $\omega=1/2[(27\alpha-5)/(5\alpha+5)]^{1/2}$. Although the formal limit $\alpha\to0$ in (4.16) and (4.17) is impossible (the oscillatory nature of the solutions of the basic system of equations holds under the additional condition $\alpha > 5/27$), the modes $\mathscr{D}_1$ and $\mathscr{D}_2$ correspond to Lifshitz's decreasing hydrodynamic $\mathfrak{D}$ mode, which, under the influence of the anisotropic stresses of the collisionless component of the medium, changes its asymptotic behavior and is "decoupled."

It is readily seen that the perturbations modes considered above, i.e., $\mathscr{G}$, $\mathscr{M}_3$, $\mathscr{M}_4, \ldots$, $\mathscr{D}_1$, $\mathscr{D}_2$, exhaust all the independent physically significant solutions of the system of equations (3.9), (3.10), and (3.7) in the long-wavelength approximation.

5. MATCHING OF THE SYMPTOTIC EXPANSIONS FOR THE PERTURBATIONS AT THE BOUNDARY OF THE HYDRODYNAMIC AND KINETIC STAGES OF THE EXPANSION

In this section, we discuss one of the possible astrophysical applications of the kinetic theory of gravitational-wave perturbations developed above. Namely, we ask the following question: In accordance with what law do long gravitational waves evolve in a Friedmann universe after decoupling of the cosmological neutrinos from the primordial plasma if prior to the neutrino decoupling (and with neglect of the contribution of the gravitons to the space-time curvature) the perturbations evolve in accordance with Liftshitz's hydrodynamic theory? This question is of interest in connection with the current intensive development of the model of a neutrino universe. At the same time, Zakharov's original conclusion[4] that there is definitely a qualitative difference from the behavior found by Lifshitz for long-wavelength gravitational perturbations in an early Friedmann universe containing collisionless particles was obtained without analysis of the conditions of excitation of the partial modes and must therefore be re-examined.

We shall assume that the decoupling of all three species of cosmological neutrinos from the primordial plasma (it is a phase transition of second order) occurs instantaneously and simultaneously at the Hubble time $t = t_0 = 1$ sec. It is not difficult to estimate the value of the parameter α at $t = t_0 + 0$ if one knows the relative equilibrium energy densities of the photons and the electron-positron and neutrino-antineutrino pairs, which make the main contribution to the total energy density in the universe at $t = t_0 - 0$ (we do not take into account gravitons!)[9]:

$$\varepsilon_\gamma : \varepsilon_{e^\pm} : \varepsilon_{\nu_e\bar{\nu}_e} : \varepsilon_{\nu_\mu\bar{\nu}_\mu} : \varepsilon_{\nu_\tau\bar{\nu}_\tau} = 1 : {}^7/_4 : {}^7/_8 : {}^7/_8 : {}^7/_8. \tag{5.1}$$

In their monograph, Zel'dovich and Novikov[7] argue that the value of the parameter α at $t = t_0 + 0$ is close to the value α_{eq} determined by

$$\alpha_{eq} = \frac{\varepsilon_{\nu_e\bar{\nu}_e} + \varepsilon_{\nu_\mu\bar{\nu}_\mu} + \varepsilon_{\nu_\tau\bar{\nu}_\tau}}{\varepsilon_\gamma + \varepsilon_{e\pm}} = \frac{21}{22}. \tag{5.2}$$

Assuming that the space-time metric is continuous at $t = t_0$ together with all its derivatives, we conclude on the basis of Einstein's equations that the energy-momentum tensor of matter must also be infinitely differentiable at $t = t_0$. Since the perturbation of the metric at $t = t_0 - 0$ is described by Eq. (3.12) (in which $\mathfrak{D} = 0$), while for $t = t_0 + 0$ it is a linear combination of the partial modes found in the previous section, the requirement of analyticity of the metric near the phase transition point leads to an infintie system of linear algebraic equations for $\mathscr{G}$, $\mathscr{M}_3$, $\mathscr{M}_4, \ldots$, $\mathscr{D}_1$, $\mathscr{D}_2$, as functions of $\mathfrak{G}$. Without writing down this system of equations, we give its solution directly:

$$\begin{gathered}\mathscr{G}^* = \mathfrak{G};\\ \mathscr{M}_3^* = \mathscr{M}_5^* = \ldots = 0;\\ \mathscr{M}_4^* = -\mathfrak{G}\frac{7\alpha}{3(\alpha+1)} \approx -1.2\mathfrak{G};\\ \mathscr{M}_6^* = -\mathfrak{G}\frac{\alpha(1387\alpha+1095)}{25(\alpha+1)(19\alpha+15)} \approx 1.3\mathscr{M}_4^*;\\ \ldots\\ \mathscr{D}_1^* = \mathscr{D}_2^* = 0\end{gathered} \tag{5.3}$$

(in the expression for $\mathscr{M}_4^*$ and $\mathscr{M}_6^*$ the results of calculations for $\alpha = 1$ are given after the sign of approximate equality). The expressions (5.3) answer our question with regard to the regime of evolution of long gravitational waves in a two-component ultrarelativistic medium generated by Lifshitz's principal mode in the hydrodynamic stage of the evolution. As was to be expected, the oscillatory asymptotic behaviors (4.16) associated with the influence of the anisotropic stresses of the collisionless component of the medium on the damping of the "tails" of the long-wavelength gravitational perturbations do not contribute to the solution (5.3). Note also that formal analytic continuation of Lifshitz's general solution (3.12) for $\mathfrak{G} \neq 0$, $\mathfrak{D} \neq 0$ would lead to a regime of development of long-wavelength perturbations at $t = t_0 + 0$ to which all the partial modes given in the previous section would contribute.

Let us consider the structure of the solution (5.3); It is readily verified by direct calculation that the anisotropic stresses in the collisionless component of the medium corresponding to the $\mathscr{G}^*$ and $\mathscr{M}_4^*$ modes compensate each other in the leading (second) order of the expansion in τ. The resulting discrepancy between the $\mathscr{G}^*$ and $\mathscr{M}_4^*$ modes in δT_β^α of the following (fourth) order is in turn compensated by the $\mathscr{M}_6^*$ mode, etc. Overall, the solution (5.3) is such that the gravitational wave corresponding to it does not polarize the collisionless component of the medium at all, i.e., there is no perturbation of the energy-momentum tensor of the matter in all orders of the expansion in τ.

Naturally, the assertion earlier that in the above matching procedure there is an infinite number of partial perturbation modes $\mathscr{G}^*$, $\mathscr{M}_4^*$, $\mathscr{M}_6^*, \ldots$ and, thus, an infinite number of expansion terms of each such mode is not to be understood formally. Moreover, from the mathematical point of view the condition $\delta T_\beta^\alpha = 0$ for an infinite set of modes $\mathscr{G}^*$, $\mathscr{M}_4^*$, $\mathscr{M}_6^*, \ldots$ leads to a series that diverges as $N \to \infty$,

$$\Psi_N^*(x) = \sum_{k=2}^{N} \frac{4k+1}{2}\mathscr{M}_{2k}^* P_{2k}(x), \tag{5.4}$$

this representing the leading term of the expansion of $\Phi(\tau;x)$ for the combination $\mathscr{M}^*$ mode. The divergence of the series (5.4) is due to the fact that the increase in the degree of anisotropy of the partial perturbations $\mathscr{M}_4^*$, $\mathscr{M}_6^*, \ldots$ in the momentum space is accompained (in the case of equality of the initial moments of the distribution function) by a decrease in the absolute magnitudes of these perturbations in the configuration space, i.e., a decrease of the absolute magnitudes of the corresponding perturbations of the metric and the deviator of the kinetic stress tensor. In this situation, the mutual compensation of the lowest multipole configuration due to the following higher ones can occur only by virtue of a corresponding growth of the higher moments of the function (5.4). We emphasize however that if we are interested in only observable quantities (ν and $\delta T^i_{\ k}$), then the divergence of the distribution function in the momentum space will not be critical for our treatment, since an arbitrary perturbation can with any preassigned degree of accuracy be approximated by some finite (sufficiently large) set of partial modes. So as not to go beyond the accuracy of the linear perturbation theory developed in the present paper, we must restrict ourselves in the matching to some finite number of $\mathscr{M}^*_k$ modes ($k = 4, 6, \ldots, 2N$) and ignore the remaining $\mathscr{M}^*_K$ modes as introducing a vanishing (less than quadratic in the leading term of the principal mode) contribution to the pertrubation of the gravitational field. Obviously, the number N depends on the amplitude of Lifshitz's principal mode $\mathfrak{G}$. Thus, for $\mathfrak{G} = 10^{-4}$ we have $N = 4$.

Thus, with allowance for the assumptions we have made we conclude that the required regime of evolution of the long gravitational waves (5.3) in the two-component radiation-dominated Friedmann universe is described by Lifshitz's hydrodynamic $\mathfrak{G}$ mode. A similar conclusion was drawn under the same assumptions in Ref. 10 with regard to the principal mode of density perturbations. What is the physical reason for the fact that in these cases the collisionless nature of the particles is in no way manifested?

To answer this question, we recall that the kinematic modes of long-wavelength gravitational perturbations in an ultrarelativistic Friedmann universe containing collisionless particles do not have hydrodynamic analogs. Since the hydrodynamic approximation is based on the *a priori* assumption of local thermodynamic equilibrium,[11] the presence of kinematic modes can serve as an indicator—and thier partial contribution as a measure of—deviation of the perturbed state of the two-component medium from local equilibrium.

We consider an elementary doubly transverse plane wave in a radiation-dominated Friedmann universe at $t = t_0 - 0$ with characteristic spatial scale appreciably exceeding the distance to the particle horizon. The hydrodynamic nature of such a pertrubation means that in each physically small volume of the medium thermodynamic equilibrium is established instantaneously. Since there is no global statistaical equilibrium, the parameters of the local-equilibrium distributions vary appreciably over spatial scales of the order of the wavelength of the initial perturbation. Once the neutrinos become transparent at $t = t_0$, they cease to exchange energy and momentum with each other and the remaining particles (which for some time still form a single radiation-dominated plasma); however, an appreciable deviation from local thermodynamic equilibrium due to the considered long-wavelength perturbation is not to be expected immediately after the neutrino decoupling but only after a certain time interval which, appreciably exceeds the age of the universe and is needed for interaction of the initially causally decoupled regions of space-time. In other words, the statistical equilibrium within a physically small volume is disturbed only when the original long gravitational wave "goes below" the particle horizon. This is why, proceeding from the assumption of a local-equilibrium nature of the intial stage of th cosmological expansion and remaining in the long-wavelength approximation, we conclude that the partial contribution of the kinematic modes to the resulting evolution regime of the perturbations after decoupling of the neutrinos from the primordial plasma must be apropriately suppressed. The solution (5.3) confirms the general conclusion and particularizes it, namely, the degree of suppression is higher the higher the degree of anisotropy in the momentum space (thus, the $\mathscr{M}^*_4$ mode makes a contribution to the perturbations of the metric and the kinetic stress tensor of order $\mathfrak{G}$; the $\mathscr{M}^*_6$ mode makes one of order $\mathfrak{G}$, where $\tau \ll 1$, etc.).

Thus, under the assumption of a hydrodynamic nature of the initial stage of the cosmological expansion the kinetic theory of perturbations developed in the present paper establishes the self-consistency of the hydrodynamic approximation for describing long gravitational waves in a Friedmann universe after decoupling of the cosmological neutrinos from the primordial plasma. The specific features of the behavior of the system of free particles associated with the kinematic modes of long-wavelength perturbations are important for analyzing the development of perturbations in the case when we give up the *a priori* assumption of an initial local thermodynamic equilibrium. The necessity of such a step, on the one hand, and its attractiveness, on the other, which are due to the pressence near the cosmological singularity of a collisionless graviton gas and the well-known difficulties in explaining the required initial spectrum of adiabatic perturbations,[7] require a separate study.

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[1)]In the short-wavelength approximation there is a greater complexity of the dispersion relations. For perturbations with characteristic spatial scale greater than the distance to the particle horizon in the Friedmann universe the corresponding time dependence do not admit separation into an amplitude and time part of the phase in the usual eikonal sense. Therefore, in the long-wavelength approximation there is no point in speaking of dispersion or the phase velocity of propagation of perturbations.

[2)]Latin indices take values from 0 to 3, Greek (spatial) from 1 to 3.

[3)]The operations of raising and lowering of the indices of tensor quantities describing perturbations is done by means of the background spatial metric $\gamma_{\alpha\beta} = a^2\delta_{\alpha\beta}$. The Cartesian vectors **n**, **s**, and **m** form a right-handed triplet in the three-dimensional tangent space; on the transition in this space to a shperical coordinate system we choose as direction of the polar axis the direction of the vector **n**; $\mathbf{r} \equiv \{x^\alpha\}$ is the Cartesian radius vector, θ and φ are, respectively the latitude and azimuthal angles, and $x = \cos\theta$.

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