Equation for irreversible response in spin glasses

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Explicit equations for four quantities that describe spin glasses are obtained in the molecular-field approximation, namely, for the susceptibility of and the Edwards-Anderson parameter at macroscopic (χ_0 and q_0) and microscopic (χ and q) times. The differences of these quantities ($\chi_0 - \chi$ and $q_0 - q$) are measures of the irreversible phenomena in spin glasses which, in turn, are due to degeneracy and non-ergodicity in these systems. The problem is solved for the "soft" model of spin glass for the case of weak anharmonicity. It is shown that the system contains an infinite number of zero-gap modes. In our approximation, χ_0 and q_0 do not depend on the temperature T. Near the critical temperature in a zero magnetic field we have $T(\chi_0 - \chi)\alpha\tau^2$, and in a finite field τ is obtained in place of τ^2, in agreement with experiment.

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1. INTRODUCTION

Failure to understand phenomena that occur in the region of the Alameida-Thouless (AT) instability[1] have recently raised difficulties in the theory of the spin-glass molecular field. An attempt to solve the problem formally within the framework of broken replica symmetry has led Parisi to develop a theory in which the order parameter is a function $q(x)$ that is a generalization of the Edwards-Anderson (EA) parameter q and depends on a continuous parameter x (see, e.g., Refs. 2–5). This theory, described by an infinite number of order parameters $q(x)$, has many interesting physical properties. In particular, the negative-gap mode, which is in fact the cause of the AT instability, becomes a zero-gap mode in the Parisi theory.[6] The physical meaning of the order parameter $q(x)$ in this theory, however, remained utterly incomprehensible. Nor was the meaning of the mathematical procedure in this paper understood. It was not completely clear how the physical quantities are expressed in terms of $q(x)$. Thus, although it was felt on the one hand that the theory leads to reasonable physical conclusions, more than enough questions remained.

At the same time, analysis of the equations of Thouless, Anderson, and Palmer[7] and of numerical modeling by various workers[8–10] led to the conclusion that spin glasses are non-ergodic systems, in view of the degeneracy due to the presence in them of infinite barriers. The usual thermodynamic averaging and averaging over time lead in this case to different results. In particular, two different susceptibilities are obtained (see, e.g., Ref. 11). If this is the case, the usual methods of calculating thermodynamic quantities become unsuitable. In particular, it is impossible to apply to these systems the replica method, at least in its usual form. A hypothesis was advanced[12] that the Parisi theory describes correctly the non-ergodic behavior of spin glasses. If this hypothesis is correct, it becomes clear that the mathematical contrivances in the Parisi theory are connected simply with an attempt to adapt the replica method to a description of non-ergodic systems. We shall see presently that this is indeed the case.

It is known that if a system is not ergodic, only a temporal description is possible for it. The statistical quantities are calculated by taking the limit as $\omega \to 0$, where ω is the frequency. For example, the static susceptibility is defined as the limit $\chi = \chi(\omega \to 0)$. If, however, we calculate the quantity $\chi_0 = \chi(\omega \to 0)$ by direct statistical averaging, we get $\chi_0 \neq \chi$. We denote their difference by

$$\Delta/T = \chi_0 - \chi, \tag{1}$$

where T is the temperature. We shall name the parameter Δ, following a number of papers (see, e.g., Ref. 13), the irreversible response. Obviously $\Delta \neq 0$ only in non-ergodic systems. The parameter Δ is a more important characteristic of spin glasses than the EA parameter. In particular, the phase transition due to the appearance of degeneracy and non-ergodicity can take place also in an external magnetic field,[13] although the EA parameter differs from zero in this case also in the paraphase.

Once it became clear that spin glass are non-ergodic systems amenable not to a statistical description but only to a temporal one, papers were published[14–17] based on precisely the latter description.

The first result of these papers was an elucidation of the physical meaning of the T singularity. It turned out that at the AT-singularity point the damping in the system vanishes. This means in fact the onset of infinite barriers and of non-ergodicity. An attempt to describe the system below the AT singularity by introducing an irreversible response Δ in the simplest manner[17] led to the same equations for the EA parameters q and Δ that were obtained earlier by Sommers[18,19] by an entirely different method. It is known on the other hand that the AT singularity cannot be removed by this method, for a mode with negative gap remains in the system as before.[20] The reasons for this are discussed in detail in Refs. 14–16. It was shown in these papers that there is some uncertainty in the theory and that introduction of the irreversible response Δ in the simplest manner does not eliminate this uncertainty. This is why the AT singularity likewise remains. Sompolinsky[14] proposed a simple method of solving this problem. He suggested the existence of an infinite hierarchy of infinite relaxation times τ_i. This hierar-

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chy corresponds to a hierarchy of infinite barriers, and it is the latter that lead to the non-ergodicity. If this is so, there should exist also a hierarchy of the EA parameters q_i and of the irreversible responses Δ_i $(i = 1,2,...,k)$. The contribution of each is, of course, infinitely small ($\sim 1/k$). It turns out that if such a picture is accepted, the uncertainty of the theory is completely eliminated (of course, as $k \to \infty$). Introduction of a continuous parameter $x = i/k$ leads to an infinite number of order parameters $q(x)$ and $\Delta(x)$. It is very interesting that the Parisi theory was found to be a particular case of the Sompolinsky theory. Moreover, it is clear at present how to obtain the results of Ref. 14 from the concept of broken symmetry of the replicas.[21,22] Sompolinsky's approach, however, is of course much more physical and illustrative.

On the other hand, no explicit equations whatever were obtained for $\Delta(x)$ and $q(x)$ in this approach, and the theory exists only in functional form. The reason is that in the general case the problem is a typical field-theory problem, and solution of field problems calls for some parameter. We study in this paper the so-called "soft" spin-glass model, which is the one mainly used in the study of spin-glass dynamics (see, e.g., Refs. 15–17), since the usual model of a fixed spin does not lend itself to theoretical investigation. This model contains an anharmonicity parameter u. We obtain explicit expressions for $\Delta(x)$ and $q(x)$ by standard perturbation theory in u, and use the smallness of a certain parameter ε which we call the supercriticality parameter. It will be defined below. The expansion in terms of ε, however, can be obtained formally with the aid of the perturbation theory in u.

No AT singularity arises in first order in u. We use therefore perturbation theory up to u^2. It is in this order that we study the phase transition connected with the appearance of degeneracy and non-ergodicity in the system. We obtain and investigate explicit equations for the parameters $q(x)$ and $\Delta(x)$, which are, generally speaking, integral equations. In our approximation, however, these equations reduce to algebraic and are easily solved.

2. BASIC EQUATIONS AND THE SUPERSYMMETRIC DIAGRAM TECHNIQUE

We choose the Hamiltonian in the form[15–17]

$$H=-\sum_{ik} J_{ik}m_i m_k+\sum_i U(m_i)-\sum_i h_i m_i, \qquad U(m)=m^2/2b+um^4/8. \tag{2}$$

Here m_i are classical fields, j_{ik} exchange integrals, and h_i local magnetic fields. By choosing $U(m) = \delta(m^2 - 1)$ we obtain the usual Ising model, whose dynamics is very difficult to study. In the soft model $U(m)$ takes the form given in (2). In this model the spin modulus fluctuates and not only its direction. We consider the case of spin glass with $\langle J \rangle = 0$ and with random magnetic fields h_i. We shall assume that J_{ik} and h_i have Gaussian distributions and

$$\langle J_{ik}^2\rangle=I_{ik}, \qquad \langle h_i^2\rangle=h^2. \tag{3}$$

It is easily shown[9] that the random magnetic fields and the constant magnetic field are gauge-invariant if $\langle J \rangle = 0$, but random magnetic fields are easier to analyze.

The dynamic equations are in our case Langevin equations with random forces $\varepsilon_i(t)$, i.e.,

$$\frac{1}{\Gamma T}\frac{\partial m_i}{\partial t}=-\frac{1}{T}\frac{\partial H}{\partial m_i}+\varepsilon_i(t), \qquad \langle \varepsilon_i(t)\,\varepsilon_j(t')\rangle=\frac{2}{\Gamma T}\delta_{ij}\delta(t-t'), \tag{4}$$

where T is the temperature and Γ^{-1} the unrenormalized relaxation time.

Generally speaking, one can use the diagram technique developed for averaging equations of type (4) (see, e.g., Ref. 23). This technique, however does not have an effective Hamiltonian in explicit form, and this is frequently inconvenient. On the other hand, in Ref. 24 was indicated a general method of constructing an effective Hamiltonian in problems with random forces, by using the supersymmetry ideas. This Hamiltonian was used for dynamic problems, in particular, in Ref. 25. We shall describe briefly its derivation.

The complete stochastic functional Ψ for the functions $m_i(t)$ (the analog of the distribution function in probability theory) is of the form

$$\Psi\{m_i(t)\}=\prod_{i,t}\delta\left\{\frac{1}{\Gamma T}\frac{\partial m_i}{\partial t}+\frac{1}{T}\frac{\partial H}{\partial m_i}-\varepsilon_i(t)\right\} \times \mathrm{Det}\left\{\frac{1}{\Gamma T}\delta_{ij}\left(\frac{\partial}{\partial t}\right)_{t,t'}+\frac{1}{T}\frac{\partial^2 H}{\partial m_i\,\partial m_j}\delta(t-t')\right\}. \tag{5}$$

The symbol $(\partial/\partial t)_{t,t'}$ is used for the differentiation operator in matrix form. Using next the identities

$$\delta(x)=\int_{-i\infty}^{i\infty}\frac{d\varphi}{2\pi i}e^{-\varphi x}, \qquad \mathrm{Det}(A_{ik})=\int\prod_i d\eta_i^* d\eta_i \exp\left\{-\sum_{ik}A_{ik}\eta_i^*\eta_k\right\}, \tag{6}$$

where η_i and η_i^* are Grassmann variables having the following anti-commutation and integration properties[26]:

$$\{\eta_i\eta_j\}=\{\eta_i^*\eta_j\}=\{\eta_i^*\eta_j^*\}=0, \quad \eta_i^2=(\eta_i^*)^2=0, \quad \int d\eta=\int d\eta^*=0, \quad \int \eta\, d\eta=\int \eta^* d\eta^*=1, \tag{7}$$

and averaging over ε_i, we obtain the following expression for Ψ:

$$\Psi=\int\prod_{i,t}\frac{d\varphi_i(t)}{2\pi i}d\eta_i^*(t)\,d\eta_i(t)\,e^{-L},$$

$$L=\frac{1}{T}\int dt\left\{\sum_i\left[\varphi_i(t)\left(\frac{1}{\Gamma}\frac{\partial m_i}{\partial t}+\frac{\partial H}{\partial m_i}\right)+\frac{1}{\Gamma}\eta_i^*\frac{\partial\eta_i}{\partial t}-\frac{1}{\Gamma}\varphi_i^2(t)\right]+\sum_{ik}\frac{\partial^2 H}{\partial m_i\,\partial m_k}\eta_i^*\eta_k\right\}. \tag{8}$$

We introduce now the superfield[27,28]

$$\Phi_i(t)=m_i(t)+[\theta^*\eta_i(t)+\eta_i^*(t)\theta]-\theta^*\theta\varphi_i(t), \tag{9}$$

where θ and θ^* are the supersymmetric coordinates. L takes then the form

$$L=-\frac{1}{2\Gamma T}\int dt\int d\theta^\cdot\, d\theta\, d\xi^\cdot\, d\xi\sum_i \Phi_i(\theta,t)$$
$$\times\left\{P(\theta,\xi)\frac{\partial\Phi_i(\xi,t)}{\partial t}+2\Phi_i(\xi,t)\right\}$$
$$+\frac{1}{T}\int dt\int d\theta^\cdot\, d\theta H(\Phi), \tag{10}$$
$$P(\theta,\xi)=-(\theta^\cdot-\xi^\cdot)(\theta+\xi).$$

Equation (10) solves in the most general form the problem of the effective Hamiltonian (more accurately, Lagrangian) in the dissipative-dynamics problem. We note the operator $P(\theta,\xi)$ is antisymmetric, since the operator of differentiation with respect to time is also antisymmetric, and the complete expression must be symmetric.

To obtain an effective Hamiltonian that is specific for our problem we must substitute in (10) the Hamiltonian $H(\Phi)$ from (2) and average the stochastic functional over J_{ik} and h_i, using (3). We then obtain

$$\Psi=\int\prod_{i,t}\frac{d\varphi_i(t)}{2\pi i}d\eta_i^\cdot(t)\,d\eta_i(t)\,e^{-L_0},$$
$$L_0=\int dt\,dt'\int d\theta^\cdot\, d\theta\, d\xi^\cdot\, d\xi L_R(t,t',\theta,\xi),$$
$$L_R(t,t',\theta,\xi)=-\frac{1}{2\Gamma T}\sum_i\left\{P(\theta,\xi)\Phi_i(\theta,t)\frac{\partial\Phi_i(\xi,t)}{\partial t}\right.$$
$$\left.+2\Phi_i(\theta,t)\Phi_i(\xi,t)\right\}\delta(t-t') \tag{11}$$
$$+\frac{1}{T}\sum_i U[\Phi_i(t)]\delta(t-t')\delta(\theta-\xi)$$
$$-\frac{h^2}{2T^2}\sum_i\Phi_i(\theta,t)\Phi_i(\xi,t')$$
$$-\frac{1}{T^2}\sum_{ik}I_{ik}\Phi_i(\theta,t)\Phi_i(\xi,t')\Phi_k(\theta,t)\Phi_k(\xi,t'),$$
$$\delta(\theta-\xi)=-(\theta^\cdot-\xi^\cdot)(\theta-\xi)\equiv E(\theta,\xi).$$

We shall solve the problem in the molecular-field approximation. In the language of the effective Hamiltonian (11) this reduces to factoring the last term as follows:

$$\sum_{i,k}I_{ik}\Phi_i(\theta,t)\Phi_i(\xi,t')\Phi_k(\theta,t)\Phi_k(\xi,t')$$
$$\to 2I_0G(\theta,\xi,t-t')\sum_i\Phi_i(\theta,t)\Phi_i(\xi,t'), \tag{12}$$
$$G(\theta,\xi,t-t')=\langle\Phi(\theta,t)\Phi(\xi,t')\rangle,\quad I_0=\sum_k I_{ik}.$$

The averaging in (12) for the function G is over the Hamiltonian (11), which becomes single-node after the factoring (12). Equations (11) and (12) yield a self-consistent system of equations, which we shall name in fact the theory of the molecular field in spin glasses, since it is a generalization of the usual molecular-field theory to include the dynamic case.

After factoring in (12), all the terms in the Hamiltonian (11) except the quaternary term in $U(\Phi)$ are quadratic in Φ. It is therefore easy to develop a standard perturbation theory in this quaternary term. We shall assume for simplicity that $h=0$. Generalization to the case of finite h is quite simple and will be done at the appropriate place.

It is easy to show that Wick's theorem holds in the supersymmetric diagram technique, so that we can use standard methods. Dyson's equation has in this case the following form in the ω-representation:

$$\hat G^{-1}(\omega)=\hat G_{00}^{-1}(\omega)-\hat\Sigma(\omega)-4I_0\hat G(\omega)/T^2,$$
$$\hat G_{00}^{-1}(\omega)=\frac{1}{T}\left(\frac{1}{b}\hat E+\frac{i\omega}{\Gamma}\hat P-\frac{2}{\Gamma}\hat\Lambda\right). \tag{13}$$

The quantities with the carets in (13) are operators in the superfield space, $\hat E$ is the unit operator, $\hat P$ was introduced in (19), and $\Lambda(\theta,\xi)\equiv 1$. It is easy to show that

$$\hat P^2=\hat E,\quad \hat\Lambda^2=0,\quad \hat\Lambda\hat P=-\hat P\hat\Lambda=\hat\Lambda. \tag{14}$$

It is seen from (14) that products of the operators $\hat E$, $\hat P$, and $\hat\Lambda$ are expressed in terms of the operators themselves, so that the correlators in (13) can always be expanded in terms of these three operators. The operator $\hat\Sigma$ in (13) is the mass operator. We put

$$\hat G(\omega)=G_0(\omega)\hat E+G_1(\omega)\hat P+D(\omega)\hat\Lambda,$$
$$\hat\Sigma(\omega)=\Sigma_0(\omega)\hat E+\Sigma_1(\omega)\hat P+\sigma(\omega)\hat\Lambda, \tag{15}$$
$$\hat S(\omega)=\hat G_{00}^{-1}(\omega)-\hat\Sigma(\omega)=S_0(\omega)\hat E+S_1(\omega)\hat P+B(\omega)\hat\Lambda,$$
$$G_\pm=G_0\pm G_1,\quad S_\pm=S_0\pm S_1,\quad \Sigma_\pm=\Sigma_0\pm\Sigma_1.$$

We then easily obtain from (13) and (14) the following equations:

$$G_\pm^{-1}-S_\pm+\frac{4I_0}{T^2}G_\pm=0,\qquad D=-\frac{BG_+G_-}{1-(4I_0/T^2)G_+G_-},$$
$$G_0=\frac{S_0G_+G_-}{1+(4I_0/T^2)G_+G_-},\qquad G_1=-\frac{S_1G_+G_-}{1-(4I_0/T^2)G_+G_-}, \tag{16}$$
$$S_\pm=\frac{1}{T}\left(\frac{1}{b}\pm\frac{i\omega}{\Gamma}\right)-\Sigma_\pm,\qquad B=-\frac{2}{\Gamma T}-\sigma.$$

Equations (16) are in fact the basic equations of our theory. Let us note first the physical meaning of the quantities D and $G_\pm$. Using the definitions of the superfield (9) and of the Green function (12) it can be easily shown that $D(\omega)$ and $G_-(\omega)$ are respectively the Fourier transforms of the correlator $\langle m(t)m(t')\rangle$ and of the retarded correlator $\langle m(t)\varphi(t')\rangle\vartheta(t-t')$. It is easy to show that $TG_-(\omega)$ is the dynamic susceptibility of our system. If $\hat\Sigma=0$ and $I_0=0$ we obtain, naturally, equations for the dissipative dynamics of a harmonic oscillator:

$$G_\pm(\omega)=\frac{bT}{1\pm i\omega b/\Gamma},\qquad D(\omega)=\frac{2T}{\Gamma}\frac{b^2}{1+(\omega b/\Gamma)^2}. \tag{17}$$

Since it can be easily shown that G_+G_- is real and is a maximum at $\omega=0$, it can be seen from (16) that when the conditions

$$(4I_0/T^2)g^2=1,\quad g=G_\pm(\omega=0) \tag{18}$$

are satisfied the correlators $D(\omega)$ and $G_1(\omega)$ have singularities. This means that the condition (18) determines the phase-transition point. We note that if the renormalized damping time is defined as

$$\tau = i\left[\partial G_-^{-1}(\omega)/\partial\omega\right]_{\omega\to 0}, \tag{19}$$

then $\tau \to \infty$ at the phase-transition point.

3. PERTURBATION THEORY FOR $\hat{\Sigma}(\omega)$

It can be easily shown that the contributions of the first- and second-order diagrams yields the following expression for $\hat{\Sigma}(t)$:

$$\Sigma_\pm(t) = -\frac{3u}{2T}D(t=0)\delta(t) + \frac{9u^2}{2T^2}G_\pm(t)D^2(t), \qquad \sigma(t) = \frac{3u^2}{2T^2}D^3(t). \tag{20}$$

In the ω representation we get from (20)

$$\Sigma_\pm(\omega) = -\frac{3u}{2T}g + \frac{9u^2}{2T^2}\int\frac{d\omega_1 d\omega_2}{(2\pi)^2}G_\pm(\omega-\omega_1-\omega_2)D(\omega_1)D(\omega_2),$$
$$\sigma(\omega) = \frac{3u^2}{2T^2}\int\frac{d\omega_1 d\omega_2}{(2\pi)^2}D(\omega-\omega_1-\omega_2)D(\omega_1)D(\omega_2). \tag{21}$$

Recognizing that $D(\omega)$ and $G_-(\omega)$ satisfy the fluctuation-dissipation theorem (FDT), we easily obtain a number of useful relations:

$$D(\omega) = \frac{2}{\omega}\,\mathrm{Im}\,G_-(\omega), \qquad D(t=0) = G_-(\omega=0) = g,$$
$$\sigma(\omega) = \frac{2}{\omega}\,\mathrm{Im}\,\Sigma_-(\omega), \qquad \Sigma_-(\omega=0) = -\frac{3u}{2T}g + \frac{3u^2}{2T^2}g^3. \tag{22}$$

We have already used the second of these relations on going from (20) to (21). The last relation of (22) expresses in the usual manner the static mass operator in terms of the static correlator. We note that it follows from (22) that $\sigma(\omega)$ and $\Sigma_-(\omega)$ satisfy a relation similar to the FDT. This is the necessary condition under which D and G_- in (16) also satisfy the FDT.

We introduce now a random magnetic field. It can be easily shown by using (11) that the following changes will occur in the perturbation theory: $D(t)$, $\sigma(t)$, and $B(t)$ acquire time-independent terms which we introduce in the following manner:

$$D(t) = q + D_0(t), \qquad \sigma(t) = \xi + \sigma_0(t),$$
$$B(t) = -\frac{2}{\Gamma T}\delta(t) - \sigma_0(t) - \frac{h^2}{T^2} - \xi. \tag{23}$$

Obviously, q in (23) is the EA parameter. The term h^2/T^2 in $B(t)$ stems from the effective Hamiltonian, and ξ from perturbation theory. In the ω representation we have

$$D(\omega) = 2\pi q\delta(\omega) + D_c(\omega), \qquad \sigma(\omega) = 2\pi\xi\delta(\omega) + \sigma_0(\omega),$$
$$B(\omega) = -2\pi\left(\frac{h^2}{T^2} + \xi\right)\delta(\omega) - \frac{2}{\Gamma T} - \sigma_0(\omega). \tag{24}$$

The following must be noted here. A natural generalization of the FDT to include the case of a magnetic field, constant or random, is the FDT for the time-dependent quantities, i.e., for $D_0(t)$ and $\sigma_0(t)$. Therefore it is necessary to replace D and σ in (22) by D_0 and σ_0.

Introduction of the random magnetic field added a new parameter to our theory, the EA parameter q, which is to be determined. We must therefore obtain an equation for q. We assume (and this is one of the basic assumptions of the present paper) that Eqs. (16) are valid also in this case. Then, substituting (24) in the equation for $D(\omega)$ in (16) and equating the coefficients for the δ function, we obtain for q the equation

$$\left(1 - \frac{4I_0}{T^2}g^2\right)q = \left(\frac{h^2}{T^2} + \xi\right)g^2. \tag{25}$$

Equations for $G_\pm$ and D_0 are obtained from (16) by simply replacing D and σ by D_0 and σ_0. These, however are not the only changes in the equations, since $\Sigma_\pm$ and σ_0 must be calculated with q taken into account. To this end we must substitute the expression for D from (23) or (24) into (20) or (21). We then obtain

$$\Sigma_-(\omega) = \frac{9u^2}{2T^2}q^2G_-(\omega) + \Sigma_{-00}(\omega),$$
$$\sigma_0(\omega) = \frac{9u^2}{2T^2}q^2D_0(\omega) + \sigma_{00}(\omega),$$
$$\Sigma_{-00}(\omega) = -\frac{3u}{2T}(g+q) + \frac{9u^2}{2T^2}\left\{2q\int\frac{d\omega_1}{2\pi}G_-(\omega-\omega_1)D_0(\omega_1) + \int\frac{d\omega_1 d\omega_2}{(2\pi)^2}G_-(\omega-\omega_1-\omega_2)D_0(\omega_1)D_0(\omega_2)\right\},$$
$$\sigma_{00}(\omega) = \frac{9u^2}{2T^2}\left\{q\int\frac{d\omega_1}{2\pi}D_0(\omega-\omega_1)D_0(\omega_1) + \frac{1}{3}\int\frac{d\omega_1 d\omega_2}{(2\pi)^2}D_0(\omega-\omega_1-\omega_2)D_0(\omega_1)D_0(\omega_2)\right\}, \tag{26}$$
$$\xi = \frac{3u^2}{2T^2}q^3.$$

We have separated in (26) the terms Σ and σ_0, which are proportional to G and D_0, inasmuch as Σ and σ in Dyson's equation should not contain these terms. An equation for ξ was also obtained from (20). We then obtain from (16)

$$G_\pm^{-1} - S_{\pm 0} + \left(\frac{4I_0}{T^2} + \frac{9u^2}{2T^2}q^2\right)G_\pm = 0,$$
$$\left\{1 + \left(\frac{4I_0}{T^2} + \frac{9u^2}{2T^2}q^2\right)G_+G_-\right\}G_0 = S_{00}G_+G_-,$$
$$\left\{1 - \left(\frac{4I_0}{T^2} + \frac{9u^2}{2T^2}q^2\right)G_+G_-\right\}G_1 = -S_{10}G_+G_-, \tag{27}$$
$$\left\{1 - \left(\frac{4I_0}{T^2} + \frac{9u^2}{2T^2}q^2\right)G_+G_-\right\}D_0 = -B_{00}G_+G_-,$$
$$S_{\pm 0} = \frac{1}{T}\left(\frac{1}{b} \pm \frac{i\omega}{\Gamma}\right) - \Sigma_{\pm 00}, \qquad B_{00} = -\frac{2}{\Gamma T} - \sigma_{00}.$$

It can be seen from (27) that the singularity in $D_0(\omega)$ at $\omega = 0$ did not vanish and is determined now by the condition

$$\left(\frac{4I_0}{T^2} + \frac{9u^2}{2T^2}q^2\right)g^2 = 1. \tag{28}$$

Taking into account Eq. (25) for q and the expression for ξ from (26), we obtain from (28) the following condition for the singularity in D_0:

$$q^3 = (h/\sqrt{3}u)^2. \tag{29}$$

We note that Eq. (25) together with Eq. (27) for $G_\pm$ yield at $\omega = 0$ equations for two thermodynamic quantities, the EA parameter q and the static susceptibility g:

$$q\left\{1-\left(\frac{4I_0}{T^2}+\frac{3u^2}{2T^2}q^2\right)g^2\right\}=\frac{h^2}{T^2}g^2, \quad (30a)$$

$$\left(\frac{1}{bT}-\alpha\right)g-\frac{4I_0}{T^2}g^2-1=0, \quad (30b)$$

$$\alpha=-\frac{3u}{2T}(q+g)+\frac{9u^2}{2T^2}\left(q^2g+qg^2+\frac{1}{3}g^3\right).$$

Equations (30) are the Sherrington-Kirkpatrick equations[29] of second order in u for our model. This can be verified by using the standard replica method. The condition (28) is an equation for the AT singularity in the same approximation. Thus, as expected, the AT singularity means divergence of $D_0(\omega)$, or in analogy with (19), that the relaxation time becomes infinite. The relaxation time usually becomes infinite when degeneracy, and hence non-ergodicity, appears in the system.

4. SIMPLEST EQUATION FOR THE IRREVERSIBLE RESPONSE

As already mentioned in the Introduction, onset of nonergodicity brings about a difference between two susceptibilities or, more conveniently, between two correlators, which we designate as

$$g=\lim_{\omega\to 0}G_\pm(\omega), \quad g+\Delta=G_\pm(\omega=0). \quad (31)$$

Just as a δ function can be regarded as the result of a limiting transition, Δ can be regarded as the function:

$$\Delta(\omega)=\Delta\lim_{\gamma\to 0}\frac{i\gamma}{\omega+i\gamma}, \quad \Delta(t)=\Delta\lim_{\gamma\to 0}\gamma e^{-\gamma t}\vartheta(t). \quad (32a)$$

It is then obvious that the limiting values are $\Delta(\omega)=0$ or $\Delta(\omega)=\Delta$ at $\omega=0$, depending on the sequence in which the limit is taken as ω and γ go to zero. We shall distinguish between these limiting transitions by using the following notation:

$$\Delta(\omega\to 0)=0, \quad \Delta(\omega=0)=\Delta. \quad (32b)$$

We then obtain Equation (31). We will use for quantity Δ the term "irreversible respond." Formally we can introduce Δ in the following way. We put in an analogy with Equation (23)

$$G_-(t)=\Delta(t)+G_{-0}(t), \quad \Sigma_-(t)=\rho(t)+\Sigma_{-0}(t),$$
$$D(t)=q(t)+D_0(t), \quad \sigma(t)=\xi(t)+\sigma_0(t), \quad (33)$$

where $q(t)$ is the Fourier transform of $2\pi q\delta(\omega)$. Here, of course, $\Delta(t)$ and $q(t)$ do not change over physical times, but we shall write them as time dependent, since this will become important later. The quantities $\rho(t)$ and $\xi(t)$ are also constant in physical time. Obviously, products of the type $\Delta(t)D_0(t)$ yield zero, and products of the type yield $q(t)D_0(t)=qD_0(t)$. Therefore Eqs. (26) and (27) remain in force, but it is necessary to replace in them $G_\pm(\omega)$ by $G_{\pm 0}(\omega)$. This means that Eqs. (30b) are also preserved if it is assumed that $g=G_{\pm 0}(\omega=0)$.

We must now write equations for Δ and q. It can be easily shown that

$$\xi(t)=\frac{3u^2}{2T^2}q^3(t), \quad \rho(t)=\frac{9u^2}{2T^2}\Delta(t)q^2(t). \quad (34)$$

If we put

$$q(t)=q\lim_{\Gamma\to 0}e^{-\Gamma|t|}. \quad (35)$$

It can be seen from (32), (34), and (35) that the product $\Delta(t)q^2(t)$ depends substantially on the sequence of the limiting transitions with respect to γ and Γ in (32) and (35). The definition of $\rho(t)$ thus becomes somewhat ambiguous, as first pointed out by Sompolinsky.[14] Obviously,

$$\rho(t)=\frac{9u^2}{2T^2}\Delta(t)q^2, \quad \Gamma\ll\gamma,$$
$$\rho(t)=0, \quad \gamma\ll\Gamma. \quad (36)$$

In the second case we again obtain the equations of the preceding section. We consider therefore the first case.

Just as in the preceding section, we assume that Eqs. (16) are valid also at $\omega\sim\Gamma,\gamma$, which we shall call the macroscopic frequencies. Then, considering in succession in the regions $\omega\ll\Gamma$ and $\Gamma\ll\omega\ll\gamma$, we readily obtain the following equations for q and Δ:

$$\frac{1}{\Delta+g}-\left(\frac{1}{bT}-\alpha\right)+\left(\frac{4I_0}{T^2}+\frac{9u^2}{2T^2}q^2\right)\Delta+\frac{4I_0}{T^2}g=0,$$
$$q\left\{1-\frac{4I_0}{T^2}(\Delta+g)^2\right\}=\left(\frac{h^2}{T^2}+\frac{3u^2}{2T^2}q^3\right)(\Delta+g)^2. \quad (37)$$

Equations (37) together with (30b) are the complete system of equations for three thermodynamic quantities: q, g, and Δ. From (30b) and from the first equation of (37) we have

$$\frac{\Delta}{g(\Delta+g)}=\Delta\left(\frac{4I_0}{T^2}+\frac{9u^2}{2T^2}q^2\right), \quad (38)$$

from which it can be seen that Δ becomes different from zero when the AT condition (28) is violated. It can be easily shown, for example, by using the method of Ref. 20, that Eqs. (37) and (38) are the Sommers equations[18,19] for our problem. As aleady noted in the Introduction, the Sommers solution is unsatisfactory. This is easily understood also from our analysis. Indeed,the first equation of (37) is similar to (30b), but is written for a Green function $G_-(\omega)$ in the range $\Gamma\ll\omega\ll\gamma$, while (30b) is applicable at $\omega\gg\gamma$. However, a detailed analysis in the region $\Gamma\ll\omega\ll\gamma$ shows that a new AT singularity appears there and must again be removed. If this is done, we remove it, but a succeeding one arises, etc. We see thus that an infinite sequence of AT singularities appears, and can be removed only by introducing an infinite number of parameters q_i and Δ_i. As already mentioned in the Introduction, this is exactly what Sompolinsky proposed to do. We shall show in the next section how this can be done.

5. EQUATIONS FOR $\Delta(x)$ AND $q(x)$

To implement this program we proceed as follows. We introduce a dual hierarchy of frequencies γ_i and Γ_i (with all γ_i, $\Gamma_i\to 0$) and of the functions $\delta_i(\omega)$ and $\mu_i(\omega)$, using the formulas (getting ahead of ourselves, we note that in the limit as $k\to\infty$ the equations cease to depend on the chosen relations between γ_i and Γ_i):

$$\Gamma_0\ll\gamma_0\ll\Gamma_1\ll\gamma_1\ll\ldots\ll\Gamma_k\ll\gamma_k,$$
$$\delta_i(\omega)=\frac{1}{\pi}\frac{\Gamma_i}{\omega^2+\Gamma_i^2}, \quad \mu_i(\omega)=\frac{i\gamma_i}{\omega+i\gamma_i}. \quad (39)$$

We carry out here, in effect, an iteration of the procedure considered in the preceding section. This means physically introduction of the infinite hierarchy of barriers, which was proposed by Sompolinsky and was referred to in the Introduction. We put

$$q(\omega)=2\pi\sum_{i=0}^{k} q_i'\delta_i(\omega),\quad \Delta(\omega)=-\sum_{i=0}^{k}\Delta_i'\mu_i(\omega),$$
$$\Delta_i=-\sum_{j=i}^{k}\Delta_i',\quad q_i=\sum_{j=0}^{i} q_i'. \tag{40}$$

Just as in the preceding sections, we assume that Eqs. (16) are valid also in the region of macroscopic frequencies. All the conclusions of the preceding sections, as already mentioned, can be reproduced by the replica method. Since the procedure of the present section is an interaction of that in the preceding one, and since it is known from the replica method[21,22] that such an interaction corresponds to the Sompolinsky scheme, our assumption is fully confirmed.

We note first that Eqs. (26), (37), and (30b) remain again in force, subject to the stipulations made in the preceding section. To derive $2k$ equations for the $2k$ quantities q_i' and Δ_i' we must consider in succession in (16) the $2k$ regions in (39) and take (33), (34), and (36) into account. We then obtain after straightforward but laborious transformations

$$\left\{1-\left(\frac{4I_0}{T^2}+\frac{3u^2}{2T^2}q_0^2\right)(\Delta+g)^2\right\}q_0=\frac{h^2}{T^2}(\Delta+g)^2.$$
$$\left\{1-\left(\frac{4I_0}{T^2}+\frac{9u^2}{2T^2}q_{i-1}^2\right)(\Delta_i+g)^2\right\}q_i'=0,$$
$$\left\{1-\left(\frac{4I_0}{T^2}+\frac{9u^2}{2T^2}q_i^2\right)(\Delta_i+g)(\Delta_{i+1}+g)\right\}\Delta_i'=0, \tag{41}$$
$$\Delta=\Delta_0,\quad q_0=q_0'.$$

The term with q_0 in (41) is separated at $h\neq 0$, since it is clear that in the language of Eq. (40) the term with h^2 is of the form $(h^2/T^2)\delta_0(\omega)$. It can be seen from (41) that generally speaking the coefficients of q_i' and Δ_i' are not equal. On the other hand, the coefficients of Δ_i' are AT singularities in the corresponding region. It is therefore easy to show that at a finite number of iterations of k the AT singularity does not vanish. The only way out is to go to the limit as $k\to\infty$. Then $q_i',\Delta_i'\sim 1/k$ (except for q_0 at finite h), and we obtain, by introducing the continuous variable $x=1/k$, $dx=1/k$,

$$q_0\left\{1-\left(\frac{4I_0}{T^2}+\frac{3u^2}{2T^2}q_0^2\right)(\Delta+g)^2\right\}=\frac{h^2}{T^2}(\Delta+g)^2,$$
$$\left\{1-\left[\frac{4I_0}{T^2}+\frac{9u^2}{2T^2}q^2(x)\right][\Delta(x)+g]^2\right\}q'(x)=0,$$
$$\left\{1-\left[\frac{4I_0}{T^2}+\frac{9u^2}{2T^2}q^2(x)\right][\Delta(x)+g]^2\right\}\Delta'(x)=0,$$
$$q_0=q(x=0),\quad q=q(x=1),\quad \Delta=\Delta(x=0),\quad \Delta(x=1)=0. \tag{42}$$

We note first that in the region where q' and Δ' differ from zero, i.e., in the region of the AT singularity, two equations in (42) coincide. This means physically that in all the frequency-hierarchy intervals there are zero-gap modes, i.e., we have an infinite number of zero-gap modes. Next, Eqs. (30b) and (42) are simple algebraic equations. We note that $q'(x)$ and $\Delta'(x)$ themselves are not determined from the equations. Only the relations between $q(x)$ and $\Delta(x)$ are determined, and equations are also obtained for the four physical quantities q_0, q, Δ, and g. We note that q_0 and $\Delta+g$ are respectively the EA parameter and the response of the system at macroscopic frequencies, while q and g are the same for microscopic ones. If we introduce in analogy with the susceptibility the Fourier transform $q(\omega)$ of the magnetization correlator $q(t)=\langle m(t)m(0)\rangle$, we obviously get in analogy with (31)

$$q_0=q(t=\infty),\quad q=\lim_{t\to\infty}q(t). \tag{43}$$

We note also that q_0 and $\Delta+g$ can be defined as the correlator and response obtained by Gibbs averaging, while q and g are obtained as time averages. As already mentioned in the Introduction, these quantities do not coincide in non-ergodic systems. From (30b) and (42) we obtain for these four quantities the following four equations:

$$q_0\left\{1-\left(\frac{4I_0}{T^2}+q_0^2\frac{3u^2}{2T^2}\right)(\Delta+g)^2\right\}=\frac{h^2}{T^2}(\Delta+g)^2,$$
$$\left(\frac{4I_0}{T^2}+\frac{9u^2}{2T^2}q_0^2\right)(\Delta+g)^2=1,\quad \left(\frac{4I_0}{T^2}+\frac{9u^2}{2T^2}q^2\right)g^2=1,$$
$$\left(\frac{1}{bT}-\alpha\right)g-\frac{4I_0}{T^2}g^2=1. \tag{44}$$

Equations (44) hold in the degenerate phase, in the nondegenerate phase however, $q=q_0$, and $\Delta=0$, and we have Eqs. (30). The boundary between these phases is determined by the condition (28). We note that the second and third equations of (44) define zero-gap modes for macroscopic and microscopic times, respectively. We note also that the two susceptibilities χ_0 and χ, which we defined in the Introduction, are equal to

$$\chi_0=(\Delta+g)/T,\quad \chi=g/T, \tag{45}$$

and it is this which leads to (1).

6. SOLUTION OF THE EQUATIONS

It can be seen from (44) and (45) that q_0 and χ_0 are independent of temperature, while q and χ are independent of h. We note that it has long been known that two different susceptibilities are observed below a definite temperature (see, e.g., Ref. 30), and that the larger of them is practically independent of temperature. It is natural to identify these two susceptibilities with χ_0 and χ (Ref. 13). If this is done, it follows indeed from the experimental data that χ_0 is independent of temperature.

It follows from the foregoing that

$$q(T,h)=q_c(T),\quad g(T,h)=g_c(T), \tag{46}$$

where q_c and g_c are the values of q and g on the $h_c(T)$ phase curve. These three functions of temperature, q_c, g_c, and h_c, are determined by solving simultaneously the three equations (18) and (30). We shall not determine $h_c(T)$, but regard it as a known function of temperature. We then obtain directly the following expressions for q_0, χ_0, q, and χ:

$$q_0^3=\frac{h^2}{3u^2},\quad \chi_0^2=\left[4I_0+\frac{9}{2}u^2q_0^2\right]^{-1},$$
$$q^3=q_c^3=\frac{h_c^2}{3u^2},\quad \chi^2=\frac{g_c^2}{T^2}=\left[4I_0+\frac{9}{2}u^2q_c^2\right]^{-1}. \tag{47}$$

Equations (47), in principle, solve the problem completely. We note now the following. We have used in this paper an expansion in the parameter u. We have noted in the Introduction that an expansion in u is an expansion in a certain supercriticality parameter ε. To introduce this parameter we consider the case of low temperatures $T\ll T_c$, where T_c is determined from the formula $h_c(T_c)=0$. At $T\ll T_c$ it is easily seen that $g_c\propto T$ and q_c is a constant. The equation for α in (30b) is then simplified and we obtain from (28) and (30) at $T=0$

$$\left(\frac{h_c^2}{3u^2}\right)^{1/3}=\frac{2}{21ub}[1-(1-7\varepsilon)^{1/2}],\quad \varepsilon=16I_0b^2-1. \tag{48}$$

It can be seen from (48) that at $\varepsilon<0$ the critical field is $h_c=0$ even at $T=0$. Thus, in our model the spin-glass phase exists only at $\varepsilon>0$. We shall assume that u, $\sqrt{I_0}$, and $1/b$ are energies of the same order and are of atomic scale. The physical quantities h_c and T_c are then located in the energy region of interest only if $\varepsilon\ll1$. The situation here is thus perfectly analogous to an ordinary phase transition in ferroelectrics of the displacive type,[31] when T_c is determined by cancellation of the atomic constants. We shall in fact assume that $\varepsilon\ll1$. The anharmonicity is then small (but not the anharmonicity constant u), and perturbation theory in terms of ε can be formally obtained by expansion in terms of u. If all this is taken into account we easily obtain near T_c

$$T_c=\frac{2\sqrt{I_0}}{3bu}\varepsilon,\quad h_c=\frac{u}{3}\left(\frac{\varepsilon|\tau|}{bu}\right)^{3/2},\quad \tau=\frac{T-T_c}{T_c},$$
$$q_0=\left(\frac{h^2}{3u^2}\right)^{1/3},\quad q=q_c=\left(\frac{h_c^2}{3u^2}\right)^{1/3}=\frac{\varepsilon|\tau|}{3bu},$$
$$\chi_0=\frac{1}{2\sqrt{I_0}}\left(1-\frac{9}{16}\frac{u^2}{I_0}q_0^2\right),\quad \chi=\frac{1}{2\sqrt{I_0}}\left(1-\frac{9}{16}\frac{u^2}{I_0}q_c^2\right),$$
$$\frac{\Delta}{T}=\chi_0-\chi=\frac{9u^2q_c^2}{32I_0^{3/2}}\left[1-\left(\frac{h}{h_c}\right)^{4/3}\right]. \tag{49}$$

It can be seen from (49) that at $h=0$ we have $\Delta\propto\tau^2$, and one can easily show that at $h\neq0$

$$\Delta\infty|\tau(h)|=|T-T_c(h)|/T_c(h),$$

which agrees with experiment.[13] It follows also from experiment that $q(h=0)\neq0$ and $q_0(h=0)=0$. This means that in each of the valleys that are separated by infinite barriers there are local frozen-in magnetizations m_i, but they are oriented in different directions in different valleys, and in the calculation of q_0 and m_i they therefore vanish after averaging over the valleys.

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