Model of random phase screen in the problem of thermal self-action of light

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The two-dimensional problem of thermal self-action of light in a fixed medium is considered in the geometric-optics approximation. An exact solution of the Cauchy problem is obtained for the given approximation. Exact expressions for the mean intensity and phase gradient of the beam behind the random phase screen are obtained on the basis of the solution of the Cauchy problem, i.e., the average over a random initial phase. The same model describes also the thermal self-action of a spatially incoherent beam. It is demonstrated that to obtain finite expressions for the mean square intensity it is necessary to go beyond the framework of the geometric optics approximation. Some concrete examples are considered.

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I. INTRODUCTION

Propagation of high-power light beams in a medium changes the characteristics of the latter, particularly its temperature. This changes the refractive index of the medium and as a consequence alters the light beam itself. The ensuing self-action problems are difficult to treat analytically. They are frequently made complicated by the fact that the characteristics of the light beam as it enters the medium can be random functions. The investigation of such problems, called the random-phase-screen model, is the subject of an extensive literature in radiophysics (we mention only Refs. 1 and 2 and the bibliography therein). Within the framework of this model one describes the passage of the wave through a thin layer of a randomly inhomogeneous medium, such that the phase of the wave becomes randomly modulated, while the amplitude remains a definite function. The phase-screen model is used to describe the propagation of microwaves through the ionosphere and the flickering of stars in the randomly inhomogeneous interstellar and interplanetary medium.[3] In addition, the propagation of a beam whose initial phase is a random function is included among the studies of the propagation of partially coherent beams.[4]

Different moments of a field are described in linear statistical radiophysics either by averaging of the model equations and solving the obtained linear equations for the field moments, or by averaging the exact solutions. In the nonlinear case, such as thermal self-action of light, averaging the model equations leads, by virtue of their nonlinearity, to the appearance of a coupled series of equations for the moments. To uncouple this system and to obtain a final closed system equation for the first, most interesting moments of the field, it is necessary either to terminate the chain at the required step and close the system, by introducing additional assumptions, or introduce additional notions, which do not follow from the formulation of the problem, concerning the statistical character of the solution.

Averaging of the exact solutions of the model nonlinear equations seems to be more promising. The number of self-action models for which an exact solution for a sufficiently wide class of initial conditions is known is, however, rather small. In addition, in those cases when such a solution is known, for example in the case of cubic Kerr nonlinearity described by an equation of quasi-optic type with local cubic nonlinearity, there is no known averaging of the solution over the random initial data. The model solution of the Kerr effect was described by the inverse problem method,[5] and the complexity of the problem has not permitted so far an averaging, say, over the random initial phase.

It is of interest in this connection to consider a simple model of thermal self-action of light, for which it is possible not only to obtain exact solutions, but also to average them in an exact manner over a random initial phase, and thus describe the average intensity and average phase, as well as the second moments and their spectra, for a sufficiently large class of initial conditions.

We shall consider a two-dimensional model of stationary thermal self-action of light in the geometric-optics approximation. The random phase screen will be assumed to be Gaussian and statistically homogeneous, i.e., we shall assume that the initial phase of the beam is a statistically homogeneous random function with a Gaussian distribution. We assume for simplicity that the initial average phase is a constant. In Sec. II we formulate the model equations of the problem and obtain exact solutions. In Sec. III we find expressions for the average intensity and average gradient of the beam phase, of the second moment and of the intensity and its spectrum. In Sec. IV, using the general expressions of Sec. III, we consider the behavior of concrete beams and compare the cases of a definite and random initial phase. An interesting feature of the considered model is that it is possible to obtain in it moments of arbitrary order:

$$\left\langle \prod_{i=1}^{n} u_i \prod_{j=1}^{m} q_j \right\rangle,$$

where $u_i = u(z_i,x_i)$ is the beam intensity and $q_j = q(z_j,x_j)$ is the gradient of the beam phase, but the expressions for the moments become much more complicated with increasing n and m. Nonetheless, the operation $\langle ... \rangle$ of averaging over a random initial phase can be carried out in explicit form for any moment of the field.

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II. MODEL EQUATIONS. EXACT SOLUTIONS

To obtain the geometric-optics equations we shall use the quasi-optics equations and the stationary heat conduction equation; these equations describe the effect of stationary thermal self-action in an immobile medium.[6] We neglect the damping of the beam and take absorption into account only in the heat-conduction equation:

$$2i\frac{\partial E}{\partial z}+\frac{\partial^2 E}{\partial x^2}+2\varepsilon TE=0,\qquad \frac{\partial^2 T}{\partial x^2}=-|E|^2. \tag{1}$$

Here $\varepsilon=\pm 1$, where $\varepsilon=1$ describe self-focusing and $\varepsilon=-1$ the self-defocusing of the beam E; T is the temperature of the medium. Representing the field E in the form

$$E=u^{1/2}e^{i\varphi},\qquad |E|^2=u$$

and neglecting the term $\partial^2 u^{1/2}/\partial x^2$, we obtain the geometric-optics equations that serve as the basis for the analysis that follows ($q=\partial\varphi/\partial x$):

$$\frac{\partial u}{\partial z}+\frac{\partial}{\partial x}(uq)=0,\qquad \frac{\partial q}{\partial z}+q\frac{\partial q}{\partial x}=\varepsilon\frac{\partial T}{\partial x},\qquad \frac{\partial^2 T}{\partial x^2}=-u. \tag{2}$$

Equations of this type were obtained in Ref. 7, which starts out with a cylindrically-symmetric analog of the system (1), and in which the damping of the beam is taken into account in the quasi-optics equation. In Ref. 7, however, is given an exact solution only for an initial beam with a planar phase front. To describe the model of a random phase screen it is necessary to know the solution at an arbitrary initial phase. We proceed now to the plan of the solution.

An exact solution of the system (2) is obtained by using a nonlinear transformation of the coordinates (z,x) to reduce the nonlinear system (2) to a linear form. This transformation was obtained by applying the Estabrook-Wahlquist method to the system (2), but it can be "guessed" directly. The gist of the method[8] is an algorithm for constructing the system, of the form

$$\frac{\partial U}{\partial x}=A,\qquad \frac{\partial U}{\partial z}=B, \tag{3}$$

such that the conditions of consistency of the system (3) coincide with the equations of system (2). For the sake of brevity we present only the result of the use of the Estabrook-Wahlquist method, namely the conditions for the consistency of the system

$$\begin{aligned}\frac{\partial U}{\partial x}&=\gamma\frac{\partial q}{\partial x}+\varepsilon\gamma uU,\\ \frac{\partial U}{\partial z}&=1-\gamma q\frac{\partial q}{\partial x}-\varepsilon\gamma uqU,\end{aligned} \tag{4}$$

where γ is an arbitrary constant, coincide with the system (2). Solving the first equation of the system (4) with respect to U, we obtain

$$U(z,x)=(K_0(z)-\gamma q_-)F(z,x)+\gamma q+\varepsilon\gamma^2F(z,x)G(z,x).$$

We have put here

$$F(z,x)=\exp\left\{\varepsilon\gamma\int_{-\infty}^{x}u(z,y)\,dy\right\},$$

$$G(z,x)=\int_{-\infty}^{x}q(z,y)u(z,y)\exp\left\{-\varepsilon\gamma\int_{-\infty}^{y}u(z,\zeta)\,d\zeta\right\}dy.$$

The initial conditions are assumed to satisfy the relations

$$u(0,x)\to 0,\qquad x\to\pm\infty,$$
$$q(z,x)\to q_-=r_1z+\psi_1,\qquad x\to-\infty,\qquad r_1,\ \psi_1-\text{const},$$
$$q(z,x)\to q_+=r_2z+\psi_2,\qquad x\to+\infty,\qquad r_2,\ \psi_2-\text{const};$$

r_1 and r_2 are determined from the initial conditions

$$|r_1-r_2|=\int_{-\infty}^{\infty}u(0,x)\,dx,$$

$K_0(z)$ is the constant of the integration with respect to x.

We elucidate the behavior of U as $x\to\pm\infty$. As $x\to-\infty$ we have

$$U(z,x)\to K_0(z).$$

As $x\to+\infty$,

$$U(z,x)\to(K_0(z)-\gamma q_-)F(z,+\infty)+\gamma q_++\varepsilon\gamma^2F(z,+\infty)G(z,+\infty).$$

From the first equation of the system (2), which is simply a conservation law, it follows that

$$F(z,+\infty)=\exp\left\{\varepsilon\gamma\int_{-\infty}^{\infty}u(0,x)\,dx\right\}=F_0=\text{const}.$$

Substituting the asymptotic forms of U as $x\to\pm\infty$ in the second equation of the system (4), we obtain

$$\frac{\partial}{\partial z}K_0(z)=1,\qquad (1-\gamma r_1)F_0+\gamma r_2+\varepsilon\gamma^2F_0\frac{\partial G(z,+\infty)}{\partial z}=1.$$

We find now the expression for the function $G(z,+\infty)$

$$G(z,+\infty)=\frac{1-\gamma r_2-(1-\gamma r_1)F_0}{\varepsilon\gamma^2F_0}z+G(0,+\infty),$$

$$G(z,+\infty)=\int_{-\infty}^{\infty}q(0,y)u(0,y)\exp\left\{-\varepsilon\gamma\int_{-\infty}^{y}u(0,\zeta)\,d\zeta\right\}dy.$$

If we now put $\gamma=i\lambda$ and take the Fourier transform of $G(z,+\infty)$ with respect to λ, we obtain

$$\begin{aligned}&\int_{-\infty}^{\infty}G(z,+\infty)e^{-i\lambda r}\,d\lambda\\ &=\int_{-\infty}^{\infty}d\lambda\int_{-\infty}^{\infty}dy\,q(z,y)u(z,y)\exp\left\{-i\lambda\left[r+\varepsilon\int_{-\infty}u(z,\zeta)\,d\zeta\right]\right\}\\ &=\int_{-\infty}^{\infty}dy\,|T_{xx}(z,y)|\,\delta(\varepsilon T_x(z,y)-r)\,q(z,y)=q(z,x)|_{x:\varepsilon T_x=r}.\end{aligned} \tag{5}$$

From a careful examination of (5) it is clear that this is in fact the sought nonlinear transformation. In fact, let us make a nonlinear transformation of the coordinates:

$$(z,x)\to(z,r), \tag{6}$$

where r is defined as

$$r(z,x)=\varepsilon T_x\equiv\varepsilon\frac{\partial T}{\partial x}=r_1-\varepsilon\int\limits_{-\infty}^{x}u(z,y)\,dy. \qquad (7)$$

It is easy to show that

$$\frac{\partial r}{\partial x}=-\varepsilon u,\quad \frac{\partial r}{\partial z}=\varepsilon uq. \qquad (8)$$

Using (6) and (8) we easily obtain from the system (2) the following system in terms of the variables z and r:

$$\frac{\partial}{\partial z}(u^{-1})=-\varepsilon\frac{\partial q}{\partial r},\quad \frac{\partial q}{\partial z}=r,\quad \frac{\partial x}{\partial z}=q. \qquad (9)$$

The initial conditions $u(0,r)$, $q(0,r)$, $x(0,r)$ are determined from the initial conditions $u(0,x)$ and $q(0,x)$ with the aid of the transformation (5). If the initial intensity profile $u(0,x)$ satisfies

$$u(0,x)>0,\quad X_1<x<X_2;\qquad u(0,x)=0,\quad x\leqslant X_1,\quad x\geqslant X_2, \qquad (10)$$

where X_1 and X_2 can be either finite or infinite, we get

$$u(0,r)=\int\limits_{X_1}^{X_2}u^2(0,y)\,\delta\Big[\varepsilon\int\limits_{X_1}^{y}u(0,\zeta)\,d\zeta+r\Big]dy,$$

$$q(0,r)=\int\limits_{X_1}^{X_2}dy\,q(0,y)\,u(0,y)\,\delta\Big[\varepsilon\int\limits_{X_1}^{y}u(0,\zeta)\,d\zeta+r\Big], \qquad (11)$$

$$x(0,r)=\int\limits_{X_1}^{X_2}dy\,y\,u(0,y)\,\delta\Big[\varepsilon\int\limits_{X_1}^{y}u(0,\zeta)\,d\zeta+r\Big].$$

The solution of the system (9) for the initial conditions (11) is of the form

$$q(z,r)=rz+q(0,r),$$

$$u(z,r)=\Big[u^{-1}(0,r)-\varepsilon z\frac{\partial q(0,r)}{\partial r}-\varepsilon\frac{z^2}{2}\Big]^{-1}, \qquad (12)$$

$$x(z,r)=\frac{rz^2}{2}+zq(0,r)+x(0,r).$$

Relations (12) describe the solution of the system (2) in terms of the variables z and r, and to obtain a solution in terms of the initial variables z and x it is necessary to carry out a nonlinear transformation inverse to (6), namely

$$u(z,x)=\int\limits_{r_1}^{r_2}dr\,\delta(x(z,r)-x), \qquad (13)$$

$$q(z,x)=\int\limits_{r_1}^{r_2}dr\,q(z,r)\,u^{-1}(z,r)\,\delta(x(z,r)-x). \qquad (14)$$

Expressions (13) and (14) yield indeed the solution of the system (2) for the initial conditions $u(0,x)$ and $q(0,x)$, where the functions $u(z,r)$, $q(z,r)$, and $x(z,r)$ are defined in (11) and (12). We emphasize once more that (13) and (14) are exact solutions of the two-dimensional model of stationary thermal self-action in the geometric-optics approximation. We proceed now to use the obtained exact solutions in the model of the random phase screen.

III. PHASE-SCREEN MODEL

We assume now that at the entrance into the nonlinear medium in which the thermal self-action takes place the light beam has a definite intensity profile $u(0,x)$ and a random phase. Since the equations of the system (2) contain not the phase itself but its gradient $q=\partial\varphi/\partial x$, we shall assume that the random function $q(0,x)$ has a Gaussian distribution, a zero mean value, and a correlation function

$$\langle q(0,x)\,q(0,y)\rangle=Q(x-y)=Q_0b(x-y),\quad b(0)=1; \qquad (15)$$

Q_0 is the variance and $\langle\ldots\rangle$ denotes averaging. Using the results of the preceding section, we find the average intensity $\langle u(z,x)\rangle$, the average phase gradient $\langle q(z,x)\rangle$, the second moment of the intensity $\langle u(z_1x_1)u(z_2x_2)\rangle$, and its spectrum. It will be made clear below that no basic difficulties are encountered when finding $\langle q(z,x)q(z,y)\rangle$ and other moments, but the expressions become extremely cumbersome and will not be given here.

To find the average intensity we use (13):

$$\langle u(z,x)\rangle=(2\pi)^{-1/2}\Big\langle\int\limits_{r_1}^{r_2}dr\int\limits_{-\infty}^{\infty}d\lambda\,\exp\{i\lambda[x(z,r)-x]\}\Big\rangle.$$

We substitute for $x(r,z)$ its expression from (12) and take the definite terms outside the averaging sign:

$$\langle u(z,x)\rangle=(2\pi)^{-1/2}\int\limits_{r_1}dr\int\limits_{-\infty}^{\infty}d\lambda\,\exp\Big\{i\lambda\Big[\frac{rz^2}{2}+x(0,r)-x\Big]\Big\}$$

$$\times\langle\exp\{i\lambda zq(0,r)\}\rangle. \qquad (16)$$

We consider separately the averaged term, using (22):

$$\langle\exp\{i\lambda zq(0,r)\}\rangle$$

$$=\Big\langle\exp\Big\{i\lambda z\int\limits_{X_1}^{X_2}dy\,q(0,y)\,u(0,y)\,\delta\Big[\varepsilon\int\limits_{X_1}^{y}u(0,\zeta)\,d\zeta+r\Big]\Big\}\Big\rangle$$

$$=\Big\langle\exp\Big\{i\int\limits_{-\infty}^{\infty}q(0,y)\,\eta(y,z,r,\lambda)\,dy\Big\}\Big\rangle,$$

where we have put

$$\eta(y,z,r,\lambda)=\lambda zu(0,y)\,\delta\Big[\varepsilon\int\limits_{-\infty}^{y}u(0,\zeta)\,d\zeta+r\Big]. \qquad (17)$$

We recall that $u(0,x)$ satisfies (10). In this case we obtain directly

$$\langle\exp\{i\lambda zq(0,r)\}\rangle=\exp\{-{}^1\!/_2\lambda^2z^2Q(x(0,r)-y(0,r))\}$$

$$=\exp\{-{}^1\!/_2Q_0z^2\lambda^2\}. \qquad (18)$$

We substitute this expression in (16) and integrate with respect to λ. We obtain an expression for the average intensity

$$\langle u(z,x)\rangle=\frac{(2\pi)^{-1/2}}{zQ_0^{1/2}}\int\limits_{r_1}^{r_2}\exp\Big\{-\frac{[rz^2/2+x(0,r)-x]^2}{2z^2Q_0}\Big\}dr. \qquad (19)$$

Transforming in (19) to integration with respect to the initial variable $y=x(0,r)$ we obtain the final expression for the average intensity

$$\langle u(z,x)\rangle = \frac{(2\pi)^{-1/2}}{zQ_0^{1/2}}\int_{x_1}^{x_2} u(0,y)\exp\left\{-\left[\frac{z^2}{2}\left(r_1-\varepsilon\int_{x_1}^{y}u(0,\zeta)\,d\zeta\right)+y-x\right]^2(2z^2Q_0)^{-1}\right\}dy. \tag{20}$$

The function

$$G_1(z,x;0,y) = \frac{(2\pi)^{-1/2}}{zQ_0^{1/2}}\exp\left\{-\left[\frac{z^2}{2}\left(r_1-\varepsilon\int_{x_1}^{y}u(0,\zeta)\,d\zeta\right)+y-x\right]^2(2z^2Q_0)^{-1}\right\} \tag{21}$$

can be interpreted as the "probability Green function" of the system for the average intensity

$$\langle u(z,x)\rangle = \int u(0,y)\,G_1(z,x;0,y)\,dy.$$

It follows from (19) and (20) that the average intensity $\langle u(z,x)\rangle$ is everywhere a finite smooth function. The spectrum of the average intensity can be easily shown to be given by the expression

$$S_1(z,k)=\exp\left\{-\frac{Q_0k^2z^2}{2}\right\}\int_{x_1}^{x_2} dx\,u(0,x) \times\exp\left\{ik\left[x+\frac{z^2}{2}\left(r_1-\varepsilon\int_{x_1}^{x}u(0,y)\,dy\right)\right]\right\}.$$

We obtain now an expression for the average gradient of the phase:

$$\langle q(z,x)\rangle =(2\pi)^{-1/2}\left\langle\int_{r_1}^{r_2}dr\int d\lambda\, q(z,r)\,u^{-1}(z,r)\exp\{i\lambda[x(z,r)-x]\}\right\rangle.$$

Using (12), we get

$$\langle q(z,r)\rangle=\sum_{n=1}^{4}P_n,$$

$$P_1 =(2\pi)^{-1/2}z\int_{r_1}^{r_2}dr\int d\lambda r\left(u^{-1}(0,r)-\frac{\varepsilon z^2}{2}\right)e^{i\lambda N}\langle\exp\{i\lambda zq(0,r)\}\rangle,$$

$$P_2=-(2\pi)^{-1/2}\varepsilon z^2\int_{r_1}^{r_2}dr\int d\lambda r e^{i\lambda N}\left\langle\frac{\partial q(0,r)}{\partial r}\exp\{i\lambda zq(0,r)\}\right\rangle,$$

$$P_3 =(2\pi)^{-1/2}\int_{r_1}^{r_2}dr\int d\lambda\left(u^{-1}(0,r)-\frac{\varepsilon z^2}{2}\right)e^{i\lambda N}\langle q(0,r)\exp\{i\lambda zq(0,r)\}\rangle,$$

$$P_4 =-(2\pi)^{-1/2}\varepsilon z\int_{r_1}^{r_2}dr\int d\lambda e^{i\lambda N}\left\langle q(0,r)\frac{\partial q(0,r)}{\partial r}\exp\{i\lambda zq(0,r)\}\right\rangle.$$

We have put here

$$N\equiv N(z,r,x)=rz^2/2+x(0,r)-x.$$

Using (18), we can easily show that

$$\langle q(0,r)\exp\{i\lambda zq(0,r)\}\rangle=i\lambda zQ_0\exp\{-{}^1/{}_2Q_0\lambda^2z^2\},$$

$$\left\langle\frac{\partial q(0,r)}{\partial r}\exp\{i\lambda zq(0,r)\}\right\rangle=0,$$

$$\left\langle q(0,r)\frac{\partial q(0,r)}{\partial r}\exp\{i\lambda zq(0,r)\}\right\rangle=0.$$

Thus, $P_2 = P_4 = 0$, and for the average phase gradient we obtain

$$\langle q(z,x)\rangle =\frac{(2\pi)^{-1/2}}{z^2Q_0^{1/2}}\int_{r_1}^{r_2}dr\left(u^{-1}(0,r)-\frac{\varepsilon z^2}{2}\right)(rz^2-N)\exp\left\{-\frac{N^2}{2z^2Q_0}\right\}.$$

Transforming, as in the case of the average intensity, to integration with respect to the variable $y = x(0,r)$, we obtain for the average phase gradient

$$\langle q(z,x)\rangle=\frac{(2\pi)^{-1/2}}{z^2Q_0^{1/2}}\int_{x_1}^{x_2}\left[1-\frac{\varepsilon z^2}{2}u(0,y)\right] \times\left[\frac{z^2}{2}\left(r_1-\varepsilon\int_{x_1}^{y}u(0,\zeta)\,d\zeta\right)-y+x\right] \times\exp\left\{-\left[\frac{z^2}{2}\left(r_1-\varepsilon\int_{x_1}^{y}u(0,\zeta)\,d\zeta\right)+y-x\right]^2(2z^2Q_0)^{-1}\right\}dy. \tag{22}$$

We find now an expression for the second moment of the intensity

$$\langle u(z_1x_1)\,u(z_2x_2)\rangle =(2\pi)^{-1}\left\langle\iint_{r_1}^{r_2}dr\,d\rho\iint_{-\infty}^{\infty}d\lambda\,d\mu\exp\{i\lambda[x(z_1,r)-x_1] +i\mu[x(z,\rho)-x_2]\}\right\rangle$$

$$=(2\pi)^{-1}\iint_{r_1}^{r_2}dr\,d\rho\iint_{-\infty}^{\infty}d\lambda\,d\mu e^{i\lambda N_1+i\mu N_2}\langle\exp\{i\lambda z_1q(0,r) +i\mu z_2q(0,\rho)\}\rangle. \tag{23}$$

We have put here

$$N_1\equiv N_1(z_1,r,x_1)=rz_1^2/2+x(0,r)-x_1,\quad N_2\equiv N_2(z_2,\rho,x_2) =\rho z_2^2/2+x(0,\rho)-x_2.$$

Considering the averaged term, we can find

$$\langle \exp\{i\lambda z_1 q(0,r)+i\mu z_2 q(0,\rho)\}\rangle$$
$$=\exp\{-{}^1/_2 Q_0[\lambda^2 z_1^2+2b\lambda\mu z_1 z_2+\mu^2 z_2^2]\},$$
$$b\equiv b(x(0,r)-x(0,\rho)).$$

Substituting the last expression in (23), integrating with respect to λ and μ, and changing to the integration variables $y_1 = x(0,r)$ and $y_2 = x(0,\rho)$, we obtain for the second moment of the intensity

$$\langle u(z_1,x_1)u(z_2,x_2)\rangle=\frac{(2\pi)^{-1}}{z_1 z_2 Q_0}\iint_{x_1}^{x_2} dy_1\,dy_2\frac{u(0,y_1)u(0,y_2)}{[1-b^2(y_1-y_2)]^{1/2}}$$
$$\times\exp\left\{-\frac{1}{2Q_0(1-b^2)}\left[\left(\frac{N_1}{z_1}\right)^2-2b\frac{N_1N_2}{z_1z_2}+\left(\frac{N_2}{z_2}\right)^2\right]\right\}. \quad (24)$$

In this case N_1 and N_2 take the form

$$N_1=\frac{z_1^2}{2}\left(r_1-\varepsilon\int_{x_1}^{y_1}u(0,\zeta)\,d\zeta\right)+y_1-x_1,$$
$$N_2=\frac{z_2^2}{2}\left(r_1-\varepsilon\int_{x_1}^{y_2}u(0,\zeta)\,d\zeta\right)+y_2-x_2;\quad b\equiv b(y_1-y_2).$$

At $z_1 = z_2 = z$ the expression (24) takes the form

$$\langle u(z,x_1)u(z,x_2)\rangle$$
$$=\frac{(2\pi)^{-1}}{z^2Q_0}\iint_{x_1}^{x_2}dy_1\,dy_2\frac{u(0,y_1)u(0,y_2)}{[1-b^2]^{1/2}}$$
$$\times\exp\left\{-\frac{N_1^2-2bN_1N_2+N_2^2}{2z^2Q_0(1-b^2)}\right\}. \quad (25)$$

A function of the form

$$G_2(z_1,x_1,z_2,x_2;0,y_1,0,y_2)=\frac{(2\pi)^{-1}}{z_1z_2Q_0}[1-b^2]^{-1/2}$$
$$\times\exp\left\{-\frac{1}{2Q_0(1-b^2)}\times\left[\left(\frac{N_1}{z_1}\right)^2-2b\frac{N_1N_2}{z_1z_2}+\left(\frac{N_2}{z_2}\right)^2\right]\right\}$$

can be interpreted in analogy with the function (21), but now already as a "two-point probability Green function" of the second moment of the intensity, in which case we have, naturally,

$$G_1(z,x;0,y)=\int G_2(z,x,z_2,x_2;0,y,0,y_2)\,dx_2,$$

If $\int u(0,x)dx = 1$. The spectrum of the second moment of the intensity is equal to

$$S_2(z_1,z_2,k_1,k_2)$$
$$=\exp\left\{-\frac{Q_0}{2}(k_1^2z_1^2+k_2^2z_2^2)\right\}\iint_{x_1}^{x_2}dy_1\,dy_2 u(0,y_1)u(0,y_2)$$
$$\times\exp\{ik_1M_1+ik_2M_2-Q(y_1-y_2)k_1k_2z_1z_2\},$$
$$M_1=y_1+\frac{z_1^2}{2}\left(r_1-\varepsilon\int_{x_1}^{y_1}u(0,\zeta)\,d\zeta\right),$$
$$M_2=y_2+\frac{z_2^2}{2}\left(r_2-\varepsilon\int_{x_1}^{y_2}u(0,\zeta)\,d\zeta\right).$$

If, using (25), we consider the mean square of the intensity, we obtain

$$\langle u^2(z,x)\rangle$$
$$=\frac{(2\pi)^{-1}}{z^2Q_0}\iint_{x_1}^{x_2}dy_1\,dy_2\frac{u(0,y_1)u(0,y_2)}{[1-b^2]^{1/2}}$$
$$\times\exp\left\{-\frac{(N_1-N_2)^2+2N_1N_2(1-b)}{2z^2Q_0(1-b^2)}\right\}.$$

We make the change of variables:

$$y=y_1-y_2,\quad Y={}^1/_2(y_1+y_2).$$

We obtain then the following estimate for the mean square of the intensity

$$\langle u^2(z,x)\rangle\sim\int dY u^2(0,Y)g(z,Y)\int dy(1-b^2(y))^{-1/2}$$
$$\times\exp\left\{-\frac{y^2}{2z^2Q_0(1-b^2(y))}\right\},$$
$$g(z,Y)\sim\exp\{-N_1N_2/z^2Q_0(1+b)\}.$$

We recall that $b(0) = 1$. For the correlation function, which behaves near zero like

$$b(y)\approx1-y^2,\quad y^2\ll1, \quad (26)$$

we obtain

$$\int_0 dy(1-b^2)^{-1/2}\exp\left\{-\frac{y^2}{2z^2Q_0(1-b^2)}\right\}$$
$$\approx\exp\left\{-\frac{1}{4z^2Q_0}\right\}\int_0\frac{dy}{y}.$$

This means that the exact geometric-optics value of the mean squared intensity behind a random phase screen with a correlation function of the type (26) has in the case of thermal self-action a logarithmic singularity everywhere at $z > 0$ for all x and for any admissible profile of the initial beam intensity. To obtain the final value of the mean square intensity we must go outside the framework of geometric optics. We shall not deal here with these questions. We refer only to an investigation[9] of a linear medium, where an expression was found for the mean squared intensity in a randomly inhomogeneous medium in the quasi-geometric-optics approximation. The result contains a term the form $\ln d_0$, is a small quantity, and can be interpreted as the minimum admissible area of the light tube. In the linear case[9] we have $d_0 = 2L_0/kl^2$, where k is the wave number, l is the microscale of the inhomogeneity of the medium, and L_0 is the length of the route. In our problem, apparently, to obtain the final value of the mean square of the intensity we can introduce a similar small parameter.

We note also that if the correlation function b behaves near zero not like (26), but is of the form

$$b(y)\approx1-y^{2n},\quad n\geqslant2,$$

this corresponds to the random function $q(0,x)$ being a random quantity:

$$\left\langle\left[\frac{\partial q(0,x)}{\partial x}\right]^2\right\rangle=0,$$

and we obtain a situation that coincides fully with the definite case, when $q(0,x) = 0$.

IV. EXAMPLES OF CONCRETE BEAMS

In this section we consider briefly the behavior of concrete beams in the phase-screen model for thermal self-action, and compare them with the case of a definite initial phase.

Consider a rectangular beam with intensity profile

$$u(0,x)=\theta(1-|x|);\qquad \theta(x)=\begin{cases}1, & x>0\\ 0, & x\leq 0\end{cases}. \tag{27}$$

Everywhere in this section we consider the case of symmetrical temperature, i.e., we assume that $r_1 = -r_2$. This means that we disregard deflection of the beam in the nonlinear medium. In the definite case, i.e., when $q(0,x) = 0$, we obtain, using the results of Sec. II,

$$u(z,x)=\frac{2}{2-\varepsilon z^2}\theta(2-\varepsilon z^2-2|x|),\qquad q(z,x)=\frac{2zx}{z^2-2\varepsilon}. \tag{28}$$

The beam, as can be seen from (28), retains its rectangular form, and in the case of self-focusing ($\varepsilon = 1$) the intensity becomes infinite at $z = \sqrt{2}$.

We see now what happens when $q(0,x)$ is a random function with a Gaussian distribution, zero mean value, and correlation function (15). Using (19) or (20) we can obtain

$$\langle u(z,x)\rangle=\frac{(2\pi)^{-1/2}}{zQ_0^{1/2}}\int_{-1}^{1}\exp\left\{-\frac{[(1-\varepsilon z^2/2)y-x]^2}{2z^2Q_0}\right\}dy.$$

In the case of self-focusing at $z = \sqrt{2}$ we find

$$\langle u(z,x)\rangle|_{z=\sqrt{2}}=(\pi Q_0)^{-1/2}\exp\{-x^2/4Q_0\}.$$

As $Q_0\to\infty$ this expression tends to a δ-function. The average intensity on the beam axis is

$$\langle u(z,x)\rangle|_{x=0}=\frac{2}{|2-\varepsilon z^2|}\operatorname{erf}\frac{|2-\varepsilon z^2|}{2z(2Q_0)^{1/2}}.$$

The spectrum of the average intensity is of the form

$$S_1(z,k)=2\exp\left\{-\frac{Q_0k^2z^2}{2}\right\}\frac{\sin k(1-\varepsilon z^2/2)}{k(1-\varepsilon z^2/2)}.$$

Investigation of the integrals in (19) and (20) for more general beams is a rather difficult task, and the use of a computer is preferable. For a beam whose initial profile is of the form

$$u(0,x)=(1-x^2)\theta(1-|x|),$$

a computer was used to calculate the integrals (20) and to construct the beam-intensity profiles for different values of z. Both in the case of self-focusing ($\varepsilon = 1$) and in the case of self-defocusing ($\varepsilon = -1$) a stratification of the beam was observed, similar to that noted in Ref. 10. A numerical investigation of the two-dimensional problem of thermal self-action was carried out in Ref. 10 on the basis of Eqs. (1), with the initial field a random function with a Gaussian distribution.

V. CONCLUSION

The reported investigation of the model of a random phase screen in the problem of thermal self-action of a light beam was based on solution of the Cauchy problem of the system (2)—a two-dimensional geometric-optics approximation of the thermal self-action problem in an immobile medium. It turns out that the system (2) is used, in addition, to describe entirely different physical effects.[11,12] In Ref. 11 it is considered as a one-dimensional Newtonian approximation of general relativity theory, and in Ref. 12 almost the same system is used to describe one-dimensional motion of a plasma with cold ions. Consequently the results of the present paper can be used not only in nonlinear optics.

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