Electrodynamics of pulsar magnetospheres

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The basic equations that describe quasisteady electrodynamic processes in a pulsar magnetosphere filled with an electron-positron plasma are derived. Solutions of the equations are obtained, and these make it possible to determine the features of the magnetosphere structure. A relationship is obtained between the magnitude of the longitudinal current and the electric field in the magnetosphere. It is shown that in the boundary layer near the "light surface" there is strong acceleration of particles and a current jet arises which closes the longitudinal currents flowing in the magnetosphere. The pulsar energy losses are determined. It is shown that they are determined by the longitudinal currents flowing in the magnetosphere and not by magnetic-dipole radiation as hitherto usually assumed.

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A pulsar is a rapidly rotating neutron star with very strong magnetic field $B_0 \sim 10^{12}$ G. Because of the rotation, there is also an electric field, which reaches values $E_0 \sim vB_0/c \sim 10^{11}$ V/cm near the star. It is important that the electric field has a component parallel to the magnetic field.[1] Particles entering such a strong field are accelerated and emit hard gamma rays, which, being absorbed in the magnetic field, generate electron-positron pairs.[2] In this manner is formed the pulsar magnetosphere, which consists of an electron-positron plasma rotating in the magnetic field of the star.

The presence of the plasma has a decisive influence on the structure of the magnetic field at large distances r from the star. At comparatively small r the field is of dipole type, but at $r \sim c/\Omega$ (Ω is the rotation frequency of the star) the motion of the plasma causes the lines of force to be strongly deformed and drawn out so that they extend to infinity. Therefore, the far lines of force emanating from the regions in the neighborhood of the magnetic poles are not closed in the pulsar magnetosphere, as in the magnetospheres of the Earth or other planets.

The electron-positron plasma escapes along the open lines of force. Therefore, in the pole regions it must be continuously generated. This is the reason for the occurrence of the active regions near the poles which make it possible to detect pulsars. An energy source is required to sustain the active processes. The source is the rotation energy of the star. As will be shown below, the experimentally observed[3] slowing down of the rotation is entirely due to the ponderomotive effect of the electric currents that flow on the surface of the star, enter the magnetosphere, and return to the star. Thus, the electric currents determine not only the structure of the magnetic field but also the nature of the fundamental processes taking place in the pulsar magnetosphere, and also the rotation dynamics of the star.

The theoretical description of the electric fields and currents in pulsar magnetospheres is a complicated nonlinear problem. In the magnetosphere models developed in the majority of earlier papers (see Ref. 3), this problem was in fact not solved. The most advanced studies on magnetosphere electrodynamics have been made Mestel and Wang[4,5] and Michel.[6] They succeeded in finding the magnetosphere structure in the simplest case when the axis of the magnetic dipole coincides with the rotation axis of the star, electric fields in the coordinate system rotating with the star and longitudinal currents are absent, and there is no slowing down of the pulsar.

The aim of the present paper is to investigate pulsar magnetospheres in the general case when electric fields and longitudinal currents are present, and also for arbitrary angle of inclination of the rotation axis to the axis of the magnetic dipole.

In Sec. 1 we derive from the general system of kinetic equations and Maxwell's equations the basic equations that describe quasisteady electrodynamic processes in a pulsar magnetosphere. It is important that no special model assumptions are made in the derivation. We use only certain restrictions on the efficiency of the source of the electron-positron plasma. The obtained equations are solved in Sec. 2 for the cylindrically symmetric cast when the rotation axis of the star coincides with the axis of the magnetic dipole. The structure of the magnetic lines of force is determined. A "consistency relation" between the magnitude of the longitudinal currents and the potential of the electric field is derived. It determines the energy losses. In Sec. 3, we investigate the boundary layer that arises near the "light" surface. We show that in this layer the electric currents are closed. At the same time, the electrons and positrons are strongly accelerated. In Sec. 4, we determine the magnetosphere structure in the case of an arbitrary angle of inclination of the axis of the magnetic dipole to the rotation axis. We establish a "consistency relation" in the inclined case, which determines the connection between the electric field and the longitudinal current. In the final Sec. 5 we consider the slowing down of the pulsar rotation. We show that it is completely determined by the longitudinal currents flowing in the pulsar magnetosphere, and thus fundamentally differs from the slowing down due to the magnetic-dipole radiation losses usually considered.[3] The lost energy is expended on the generatin of the electron-positron plasma, and also on the acceleration of particles in the boundary layer and emission of MHD waves in the region outside the light surface.

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§1. BASIC EQUATIONS

The dynamics of a pulsar magnetosphere is determined by the interaction of the electron-positron plasma with the magnetic field of the rotating neutron star. It is described by the system of Maxwell's equations for the electric, **E**, and magnetic, **B**, fields and the kinetic equations for the distribution functions of the elecrons, F^-, and positrons, F^+,

$$\operatorname{div} \mathbf{E}=4\pi\rho_e, \quad (1)$$

$$\operatorname{rot} \mathbf{E}=-\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \quad (2)$$

$$\operatorname{rot} \mathbf{B}=\frac{4\pi}{c}\mathbf{j}+\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \quad \operatorname{div} \mathbf{B}=0, \quad (3)$$

$$\frac{\partial F^\pm}{\partial t}+Bv_\parallel\frac{\partial}{\partial r_\parallel}\left(F^\pm\frac{1}{B}\right)\pm\frac{e}{B}(\mathbf{BE})\frac{\partial F^\pm}{\partial p_\parallel}+\mathbf{v}_\perp\frac{\partial F^\pm}{\partial \mathbf{r}_\perp}=Q;$$

$$\rho_e=e(n^+-n^-), \quad \mathbf{j}=e(n^+\mathbf{v}^+-n^-\mathbf{v}^-). \quad (4)$$

Here, $r_\parallel$ is a coordinate along a magnetic line of force, $\mathbf{r}_\perp$ is orthogonal to it, $F^\pm(p_\parallel,\mathbf{r},t)$ are the electron and positron distribution functions with respect to the longitudinal momenta $p_\parallel$, e is the electron charge, and $n^\pm$ and $v^\pm$ are the concentrations and mean velocities of the particles:

$$\frac{\partial \mathbf{p}_\perp}{\partial t}+(\mathbf{v}_\perp\nabla)\mathbf{p}_\perp\mp e\left[\mathbf{E}_\perp+\frac{1}{m_0c\gamma}[\mathbf{p}_\perp\times\mathbf{B}]\right]=0,$$

$$\mathbf{p}_\perp=m_0\gamma\mathbf{v}_\perp, \quad \gamma=[1+(p_\parallel^2+p_\perp^2)/m_0^2c^2]^{1/2}, \quad (5)$$

$$n^\pm=\int F^\pm dp_\parallel, \quad n^\pm v_\parallel^\pm=\frac{1}{m_0}\int p_\parallel F^\pm dp_\parallel/\gamma.$$

Here, we have taken into account the fact that the spread over the transverse momenta $p_\perp$ is slight, since $\varepsilon_\perp/\varepsilon_\parallel\ll 1$ by virtue of the manner in which the electron-positron pairs are generated.[1-3] As the electrons and positrons are displaced from the generation region to regions further from the star, the magnetic field rapidly decreases, and accordingly the ratio $\varepsilon_\perp/\varepsilon_\parallel$, which is proportional to B, becomes even smaller. Therefore, the velocity components $\mathbf{v}_\perp$ orthogonal to **B** are determined solely by the hydrodynamic motion in the fields **E** and **B**, and only the longitudinal component $p_\parallel$ has a kinetic spread.

Further, $Q(p_\parallel,\mathbf{r},t)$ in Eq. (4) is the source of the electrons and positrons. According to the model of Ruderman and Sutherland,[2] the electron-positron pairs are produced by photons, which are themselves produced by the curvilinear motion of the electrons and positrons in the magnetic field of the pulsar (they are called curvature photons), and also by the synchrotron radiation of the pairs. We shall not here consider in detail the generation process but merely mention some general properties of the source $Q(p_\parallel,\mathbf{r},t)$ used in what follows. First, pair is effective only in very strong magnetic fields with $B>10^{10}$ G; therefore, the action of the source Q is localized in the neighborhood of the star at distances $r-R\lesssim R$ ($R\approx 10^6$ cm is the radius of the neutron star). The mean energy $\bar\gamma$ and the mean energy spread $\Delta\gamma$ of the generated particles is in accordance with Ref. 2

$$\bar\gamma\sim\Delta\gamma\sim 10^2. \quad (6)$$

We shall assume that the integrated intensity of the source is fairly large,

$$I(\mathbf{r}_\perp)\gg I_c=B_0/eP, \quad I=\int Q(p_\parallel,\mathbf{r},t)\,dp_\parallel dr_\parallel, \quad (7)$$

but at the same time not too large, namely, is bounded by the condition

$$I(\mathbf{r}_\perp)\ll I_m=\frac{B_0^2}{2\pi m_0c\gamma}\left(\frac{\Omega R}{c}\right)^3, \quad \frac{I_m}{I_c}=\left(\frac{\Omega R}{c}\right)^3\frac{eB_0}{m_0c\Omega\gamma}. \quad (8)$$

Here, $P=2\pi/\Omega$ is the rotation period of the pulsar, and $B_0=B(R)$ is the magnetic field on the surface of the star. The condition (7) means that the source can fill the magnetosphere with plasma. It is not satisfied for extinct or expiring pulsars. Condition (8) limits the amount of plasma in the pulsar magnetosphere. It means that the energy density of the plasma in the magnetosphere is much less than the energy density of the magnetic field. In reality, this condition is evidently satisfied in the entire magnetosphere from the surface of the star at $r\approx R$ to the "light surface"[1)] $r\sim c/\Omega$.

We now simplify the basic equations. We are interested in a steady solution. Under the conditions of a uniformly rotating star, such a solutin depends on the time t and the rotation angle φ in the combination $\varphi-\Omega t$. This makes it possible to eliminate the time t from the equations by making the substitution $\varphi\to\varphi-\Omega t$ and

$$\frac{\partial \mathbf{V}}{\partial t}\to -c(\boldsymbol\beta_r\nabla)\mathbf{V}+c(\mathbf{V}\nabla)\boldsymbol\beta_r, \quad \boldsymbol\beta_r=\frac{1}{c}[\boldsymbol\Omega\times\mathbf{r}], \quad (9)$$

where **V** is an arbitrary vector. Then Eq. (2) becomes

$$\operatorname{rot}\mathbf{E}=-\operatorname{rot}[\boldsymbol\beta_r\times\mathbf{B}].$$

Here we have used the fact that $\operatorname{div}\boldsymbol\beta_r=0$. It follows from this that

$$\mathbf{E}=-[\boldsymbol\beta_r\mathbf{B}]-\nabla\Psi; \quad (10)$$

Ψ is the potential of the electric field in the coordinate system rotating with angular velocity Ω. As is clear from (10), the total electric potential in the nonrotating frame is

$$\Phi=\Phi_0+\Psi. \quad (11)$$

Here, $\Phi_0=\boldsymbol\beta_r\cdot\mathbf{A}$ is the potential of the uniformly rotating body, and **A** is the vector potential of the magnetic field: $\mathbf{B}=\operatorname{curl}\mathbf{A}$. The potential Ψ shows how the actual motion of the plama in the magnetosphere differs from corotation with the pulsar[2)]; it reflects the interaction of the magnetic field and the currents with the plkama and is a very important characteristic of the magnetosphere. In what follows, to distinguish Ψ and Φ, we shall call Ψ the potential, and Φ the total potential of the electric field.

Using (10), we obtain from Eq. (1) an expression for the charge density:

$$\rho_e=\rho_c+\frac{\boldsymbol\beta_r\operatorname{rot}\mathbf{B}}{4\pi}-\frac{\Delta\Psi}{4\pi}. \quad (12)$$

Here, ρ_c is the density of the corotation charge which arises in a plasma rotating uniformly in a magnetic field:

$$\rho_c = -\boldsymbol{\Omega}\mathbf{B}/2\pi c. \tag{13}$$

In pulsar literature, the density ρ_c is usually called the Goldreich-Julian density: $\rho_c = \rho_{GJ}$.[1] It determines the characteristic charge density ρ_e and the electric current density j_c in the pulsar magnetosphere:

$$j_c = \rho_e c = -\boldsymbol{\Omega}\mathbf{B}/2\pi. \tag{14}$$

From Eq. (3), using (9) and (10), we obtain

$$\operatorname{rot}\mathbf{B} = \frac{4\pi}{c}\mathbf{j} - [\boldsymbol{\beta}_r \times \operatorname{rot}[\boldsymbol{\beta}_r \times \mathbf{B}]] + \nabla(\boldsymbol{\beta}_r \nabla\Psi);\ \operatorname{div}\mathbf{B} = 0. \tag{15}$$

Thus, Maxwell's equations take the time-independent form (10),(15).

We now take into account the presence of the electron-positron plasma. The plasma is polarized and screens the electric field. If the concentration of particles in it is sufficiently high, the polarized plasma becomes quasineutral. This means that the electron and positron concentrations of the plasma are nearly equal, $n^- \approx n^+ = n$ ($n \gg \rho_c/e$), and the longitudinal (directed along the magnetic lines of force) electric field $E_\parallel$ is

$$E_\parallel \sim \varepsilon_\parallel / er\lambda \sim m_0 c^2 \gamma / er\lambda.$$

Here, $\varepsilon_\parallel = m_0 c^2 \gamma$ is the characteristic longitudinal energy of the particles, and λ is a dimensionless parameter that characterizes the concentration of the particles, $\lambda = en/\rho_c$. In accordance with (7), $n \approx IB/cB_0$ and

$$\lambda = I/I_c \gg 1. \tag{16}$$

Comparing the longitudinal field $E_\parallel$ with the characteristic field E (10), we find that

$$\mu_E = E_\parallel / E \approx I_E / I \ll 1, \quad I_E \approx m_0 c^3 \gamma r / 2\pi e^2 R^3. \tag{17}$$

Here, we have noted that $B(r) \approx B_0 (R/r)^3$ and that $I_E \approx 10^{12}$–$10^{16}\ cm^{-2}\cdot\sec^{-1}$ is many orders of magnitude smaller than I_c (7).[3] The solution of the Poisson equation (12) can therefore be sought in the form of a series in the parameter μ_E. In the zeroth approximation in μ_E we have $E_\parallel = 0$, i.e., the potential Ψ depends only on $\mathbf{r}_\perp$:

$$\Psi = \Psi(\mathbf{r}_\perp), \quad \mathbf{EB} = 0. \tag{18}$$

We now consider the components of the electric current orthogonal to the magnetic field $\mathbf{B}$:

$$\mathbf{j}_\perp = e(n^+ \mathbf{v}_\perp^+ - n^- \mathbf{v}_\perp^-). \tag{19}$$

The electron and positron velocities $v_\perp^\pm$ are of the order of the rotation velocities of the magnetic lines of force: $v_\perp \sim \Omega\rho$ (ρ is the distance from the rotation axis). Therefore, in the entire region from the surface of the star to the "light" surface $v_\perp < c$. Under these conditions, the drift of the electrons and positrons in the crossed fields plays the principal part in Eq. (5):

$$\mathbf{v}_{\perp 0}^+ = \mathbf{v}_{\perp 0}^- = \frac{c}{B^2}[\mathbf{E}\times\mathbf{B}]. \tag{20}$$

The corrections to $\mathbf{v}_{\perp 0}^\pm$ can be readily found from Eq. (5) by using an expansion with respect to the parameter α^{-1}:

$$\mathbf{v}_{\perp 1}^\pm = \pm\frac{c}{eB^2}\left[\mathbf{B}\times\frac{d\mathbf{p}_\perp^\pm}{dt}\right];$$

where

$$\mathbf{p}_\perp^\pm = m_0 \gamma \mathbf{v}_{\perp 0}^\pm, \quad \gamma^{-2} = 1 - (v_{\perp 0}^2 + v_\parallel^2)/c^2,$$

$$d\mathbf{p}_\perp/dt = [(\mathbf{v}_0 - c\boldsymbol{\beta}_r)\times\nabla]\mathbf{p}_\perp + c(\mathbf{p}_\perp\nabla)\boldsymbol{\beta}_r.$$

Since $dp_\perp/dt \approx \Omega p_\perp$, we have

$$\alpha^{-1} = m_0 \gamma c \Omega / eB. \tag{21}$$

It can be seen from this that the expansion parameter α^{-1} has the order of the ratio of the Larmor radius $r_L \sim m_0 c^2 \gamma / eB$ of a particle to the scale $\rho \sim c/\Omega$ of the magnetosphere. This ratio is always small: $\alpha^{-1} \ll 1$. Bearing this in mind, we represent the expression (19) for the current $\mathbf{j}_\perp$ in the form

$$\mathbf{j}_\perp = \mathbf{j}_{\perp 0} + \mathbf{j}_{\perp 1} + \ldots,$$
$$\mathbf{j}_{\perp 0} = \rho_e \mathbf{v}_{\perp 0}, \quad |\mathbf{j}_{\perp 1}|/|\mathbf{j}_{\perp 0}| = \mu_B = \lambda\alpha^{-1} \approx I/I_m. \tag{22}$$

It follows from this that the expansion (22) is in powers of the parameter $\mu_B = \lambda/\alpha$, which for pulsars is always small by virtue of the condition (8). In the zeroth approximation in μ_B, the drift approximatin (20) is valid, and the current $\mathbf{j}_\perp$ is given by the simple expression (22),(20),(10). With regard to the longitudinal component $j_\parallel$ of the current, under the conditions of a collisionless plasma and the absence (18) of a longitudinal electric field it is free, i.e., determined by the conditions at the boundaries and the continuity equation for the charge, which, with allowance for (9), is

$$\operatorname{div}(\mathbf{j} - c\rho_e \boldsymbol{\beta}_r) = 0. \tag{23}$$

Thus, in the zeroth approximation in the small parameters μ_E (17) and μ_B (22) quasisteady electrodynamic processes in the pulsar magnetosphere are described by the closed system of equations (10)–(15),(18),(20)–(23). At the same time, the electric charge ρ_e (12) and the current

$$\mathbf{j} = -\frac{c\rho_e}{B^2}[[\boldsymbol{\beta}_r\times\mathbf{B}]\times\mathbf{B}] - \frac{\rho_e c}{B^2}[\nabla\Psi\times\mathbf{B}] + i\mathbf{B} \tag{24}$$

are complicated nonlinear functions of the fields $\mathbf{B}$ and $\nabla\Psi$ [in the expression (24), the free longitudinal current $i\mathbf{B}$ is added to $\mathbf{j}_\perp$ (22)]. We find ρ_e and $\mathbf{j}$ explicitly. It follows from (15) and (24) that

$$\boldsymbol{\beta}_r \operatorname{rot}\mathbf{B} = \frac{4\pi}{c}\mathbf{j}\boldsymbol{\beta}_r + \boldsymbol{\beta}_r\nabla(\boldsymbol{\beta}_r\nabla\Psi) = 4\pi\rho_e\beta_r^2 - \frac{4\pi\rho_e c}{B^2}\boldsymbol{\beta}_r[\nabla\Psi\times\mathbf{B}]$$
$$+ \frac{4\pi}{c}i_\parallel(\boldsymbol{\beta}_r\mathbf{B}) + \boldsymbol{\beta}_r\nabla(\boldsymbol{\beta}_r\nabla\Psi), \quad i_\parallel = i - c\rho_c(\boldsymbol{\beta}_r\mathbf{B})/B^2.$$

Substituting this expression in (12), we determine ρ_e:

$$\rho_e = \left\{-\frac{\boldsymbol{\Omega}\mathbf{B}}{2\pi c} + \frac{i_\parallel}{c}(\boldsymbol{\beta}_r\mathbf{B}) - \frac{1}{4\pi}[\Delta\Psi - \boldsymbol{\beta}_r\nabla(\boldsymbol{\beta}_r\nabla\Psi)]\right\}$$
$$\times\left\{1 - \beta_r^2 + \frac{\boldsymbol{\beta}_r[\nabla\Psi\times\mathbf{B}]}{B^2}\right\}^{-1}. \tag{25}$$

Note that the electric charge ρ_e does not remain constant along the magnetic lines of force—it changes appreciably in magnitude and may even change sign. This does not contradict the condition of constancy of the potential (18), since the relation (18) is valid only in the first approximation in the small parameter (17). In the following approximation in μ_E there are corrections to the electric field, and these have longitudinal components. It can be shown that these ensure that the necessary charges remain on the lines of force. Thus, this expansion eliminates the difficulties with the electric charge that have frequently been discussed in the literature.[3,10]

The current $\mathbf{j}$ is described by the expression (24) with density ρ_e determined in accordance with (25). Substituting it in (15), we finally arrive at the nonlinear equation

$$\mathrm{rot}\{\mathbf{B}(1-\beta_r^2)+\beta_r(\beta_r\mathbf{B})+[\beta_r\times\nabla\Psi]\}=\frac{4\pi}{1-\beta_r^2+\beta_r[\nabla\Psi\mathbf{B})/B^2]}$$
$$\times\left\{\frac{i_\parallel}{c}[(1-\beta_r^2)\mathbf{B}+[\beta_r\times\nabla\Psi]]+\frac{[\nabla\Psi\times\mathbf{B}]}{B^2}\left[\frac{\Omega\mathbf{B}}{2\pi c}\right.\right.$$
$$\left.\left.+\frac{1}{4\pi}(\Delta\Psi-\beta_r\nabla(\beta_r\nabla\Psi))\right]\right\}. \qquad (26)$$

It must be augmented by the equation

$$\mathrm{div}\,\mathbf{B}=0. \qquad (27)$$

The system of equations (26),(27) describes the structure of the magnetic field in the pulsar magnetosphere with allowance for the electric field $-\nabla\Psi$ and the longitudinal current $i_\parallel$.

We note that since the charge continuity equation (23) follows from Maxwell's equations it is not independent but follows from (26). Therefore, the longitudinal current $i_\parallel$ in the magnetosphere is also determined by Eq. (26) and the boundary conditions. The potential $\Psi(\mathbf{r}_\perp)$ is constant along the magnetic lines of force. In Eq. (26), it and the longitudinal current $i_\parallel(\mathbf{r})$ play the part of sources. We emphasize that the nonlinearity of Eq. (26) consists not only of the direct dependence of the coefficients on the magnetic field but also of the requirement of constancy of the potential Ψ on the magnetic lines of force, which are themselves determined by the solution of Eqs. (26) and (27). This leads, in particular, to a complicated nonlinear dependence of the solutions of Eqs. (26) and (27) on the conditions on the boundaries of the magnetosphere.

The boundary conditions for Eqs. (26) and (27) have the following form. Near the surface of the star, on the lower boundary $S=S_0(\mathbf{r})$ of the magnetosphere, we specify the magnetic field of the pulsar, which is determined by the currents flowing within the star and on its surface:

$$\mathbf{B}|_{S_0}=\mathbf{B}_0(\mathbf{r}). \qquad (28)$$

In addition, we specify here the longitudinal currents that flow in an out of the magnetosphere:

$$\mathbf{j}_\parallel|_{S_0}=i_\parallel(\mathbf{r}_\perp)\mathbf{B}_0, \qquad (29)$$

and also the potential of the electric field:

$$\Psi|_{S_0}=\Psi(\mathbf{r}_\perp). \qquad (30)$$

At the same time, there is a great difference between the regions of closed and open (i.e., ones that go to infinity) magnetic lines of force. Regarding the conductivity of the star as infinite (see Ref. 3), we can assume that in the regin S_s of closed lines of force there are no longitudinal currents and the magnetosphere here rotates with the star (corotation). Therefore

$$i_\parallel|_{S_s}=0, \quad \Psi|_{S_s}=0. \qquad (31)$$

This condition requires a special explanation. The point is that in the closed region of the magnetosphere the electric charge ρ_e (13) is needed to produce corotation. Therefore, electric fields $\delta\Psi$ and currents $\delta i_\parallel$ are also needed to sustain it. However, $\delta\Psi\lesssim m_0c^2\bar{\gamma}/e\lesssim 6\times 10^7$ V (6), whereas potentials $\Psi\sim 10^{12}$ V are needed for continuous generation of the electron–positron plasma on the open lines.[2] It can be seen from this that

$$\delta\Psi/\Psi\leqslant 10^{-4}\ll 1. \qquad (32)$$

An estimate of the same type holds for the currents. The condition (31) is therefore satisfied everywhere in the closed magnetosphere in the zeroth approximation in the parameter (32). An exception is possible only in the neighborhood of the magnetic equator, where the magnetic lines of force enter the double-layer region at the surface of the pulsar and it is difficult to maintain the corotation charge (13) on them.

One further natural boundary condition arises on the surface S_d, which is determined by the relatin

$$1-\beta_r^2+\beta_r[\nabla\Psi\times\mathbf{B}]/B^2=0, \quad \beta_r=[\Omega\times\mathbf{r}]/c. \qquad (33)$$

On this surface, the right-hand side of Eq. (26) has a singularity. The requirement that the magnetic lines of force are able to intersect the singular surface S_d i.e., that the electric charge ρ_e (25) and the current $\mathbf{j}$ (24) remain finite on it, is a natural boundary condition of the problem. Note that the form of the singular surface is determined by the total electric field $\mathbf{E}=-\beta_r\times\mathbf{B}-\nabla\Psi$. In the absence of the potential Ψ, i.e., for $\nabla\Psi=0$, the entire magnetosphere rotates uniformly with angular frequency Ω. Then the singular surface S_d (33) has the form of a cylinder of radius $\rho=c/\Omega$. On this cylinder, the total electric field is equal in magnitude to the magnetic field, $E=B$, and the drift velocity of the motion of the particles reaches the velocity of light. Therefore, in this case the singular surface S_d coincides with the "light surface" S_c. It is called the light cylinder.[1-3] In the presence of a field $\nabla\Psi$, the magnetosphere rotates nonuniformly, so that in the general case the light surfce does not coincide with the light cylinder. It always lies further from the pulsar than the light cylinder, and further than the singular surface S_d.

We emphasize that near the light surface S_c the drift velocity of the particles $\mathbf{v}_{10}^{\pm}$ (20) approaches the velocity of light. Therefore, the energy of the particles (i.e., the value of γ) increases here and, accordingly, the expansion parameter μ_B (22) increases strongly. As a result, the conditions of applicability of Eq. (26) are violated near S_c. There is formed here a singular boundary layer, in which it is not sufficient to consider the drift approximation (20) and a more accurate solution of the equations of motin (5) of the electrons and positrons is needed.

§2. THE AXISYMMETRIC CASE

We begin by considering the simplest case. We assume that the undisturbed magnetic field of the pulsar is a dipole field, and that the dipole axis is parallel to the rotatin axis.[4)] In this case, the problem is axisymmetric—all functions depend only on two coordinates: z, along the rotation axis, and ρ, orthogonal to it.

Suppose the relation $f(\rho,z)$ = const determines the magnetic surfaces, where $f(\rho,z)$ is some scalar function. Then

$$\mathbf{B}\nabla f=0. \tag{34}$$

Using (34), we can represent the magnetic field in the form

$$\mathbf{B}=a([\nabla f\times \mathbf{e}_\varphi]+g\mathbf{e}_\varphi).$$

Here, $\mathbf{e}_\varphi$ is a unit vector in the direction of the rotation angle φ, and $a(\rho,z)$ and $g(\rho,z)$ are arbitrary scalar functions. Using the condition (27), we find that $a\equiv 1/\rho$. Thus,

$$\mathbf{B}=\{[\nabla f\times \mathbf{e}_\varphi]+g\mathbf{e}_\varphi\}/\rho. \tag{35}$$

By virtue of (18), the electric potential Ψ is constant along the magnetic lines of force, and therefore,

$$\Psi=\Psi(f). \tag{36}$$

Substituting now (35) and (36) in the basic vector equation (26), we arrive at three equations for the scalar functions f, g, and $i_\parallel$. Two of them, corresponding to the components of Eq. (26) with respect to ρ and z, can be represented in the form

$$\nabla g=\frac{1}{b}\left[\frac{4\pi i_\parallel}{c}\left(1-\frac{\Omega^2\rho^2}{c^2}-\frac{\Omega\rho^2}{c}\frac{d\Psi}{df}\right)+\frac{\rho^2 g}{g^2+(\nabla f)^2}\frac{d\Psi}{df}b_1\right]\nabla f,$$

$$b=1-\frac{\Omega^2\rho^2}{c^2}-\frac{\Omega\rho^2}{c}\frac{d\Psi}{df}\frac{(\nabla f)^2}{(\nabla f)^2+g^2},\quad b_1=\frac{2\Omega}{c\rho}\frac{\partial f}{\partial\rho}+\Delta\Psi. \tag{37}$$

It follows from this that[5)] $g=g(f)$,

$$\frac{4\pi}{c}i_\parallel=\left(b\frac{dg}{df}-b_1\frac{\rho^2 g}{g^2+(\nabla f)^2}\frac{d\Psi}{df}\right)\Big/\left(1-\frac{\Omega^2\rho^2}{c^2}-\frac{\Omega\rho^2}{c}\frac{d\Psi}{df}\right). \tag{38}$$

Writing down now the equation corresponding to the component of Eq. (26) with respect to $\mathbf{e}_\varphi$, and eliminating $i_\parallel$ in it by means of (38), we reduce it finally to the form

$$-\Delta f\left[1-\rho^2\left(1+\frac{d\Psi}{df}\right)^2\right]+\frac{2}{\rho}\frac{\partial f}{\partial\rho}-g\frac{dg}{df}+\rho^2\left(1+\frac{d\Psi}{df}\right)\frac{d^2\Psi}{df^2}(\nabla f)^2=0. \tag{39}$$

Here, we have used dimensionless functions and variables (primed):

$$\rho=\frac{c}{\Omega}\rho',\quad z=\frac{c}{\Omega}z',\quad f=\frac{M\Omega}{c}f',\quad \Psi=\frac{M\Omega^2}{c^2}\Psi',\quad g=\frac{M\Omega^2}{c^2}g',$$

$$B=\frac{M\Omega^3}{c^3}B',\quad i_\parallel=\frac{\Omega}{4\pi}i_\parallel',\quad \rho_e=\frac{M\Omega^4}{4\pi c^4}\rho_e', \tag{40}$$

where M is the magnetic moment of the dipole (in what follows, the prime is omitted).

Thus, the problem has been reduced to the single equatin (39) for the scalar function $f(\rho,z)$. The equation is nonliner and depends essentially on the functions $\Psi(f)$ and $g(f)$, which play the part of sources in it. In the absence of the sources, $\Psi=0$ and $g=0$, Eq. (39) takes the simple form[4–6]

$$-\Delta f(1-\rho^2)+\frac{2}{\rho}\frac{\partial f}{\partial\rho}=0. \tag{41}$$

The actual form of the functions $\Psi(f)$ and $g(f)$ in Eq. (39) is determined by the boundary conditions. Indeed, the function $\Psi(f)$ is given directly by the condition (30) on the lower boundary S_0 of the magnetosphere. With regard to $g(f)$, it must be borne in mind that the boundary S_0 corresponds to values $\rho\lesssim R$ and, therefore, $\rho'=\Omega\rho/c\to 0$. With allowance for (40), the relation (38) on S_0 therefore takes the form

$$i_{\parallel 0}=i_\parallel|_{S_0}=dg/df. \tag{42}$$

It follows from this that the function dg/df is determined directly by the longitudinal current on the surface S_0, which is given by the boundary condition (29). The total current flowing over the surface of the pulsar in the case of isotropic conductivity of the surface is described by the function $g(f)$:

$$I_s=M\Omega^2 g(f)/2c. \tag{43}$$

We recall that the field Ψ and the longitudinal currents in the pulsar magnetosphere are determined by the conditions (29) and (30) only on the open lines of force; on the closed lines of force we have in accordance with (31)

$$\Psi(f)=g(f)\equiv 0,\quad f\geqslant f^*. \tag{44}$$

Here, $f=f^*$ is the separatrix between the regions of closed and open lines of force. Bearing in mind that under steady conditions the total current to the surface of the star from the magnetosphere must be equal to zero, we obtain

$$g(f^*)=g(0),$$

and, using (44), we have the following condition at the pole:

$$g(f)|_{f\to 0}=0.$$

The boundary condition for Eq. (39) is determined in accordance with (28). In the limit $(\rho^2+z^2)\to 0$, the magnetic field must be a dipole field corrected by the longitudinal current $i_{\parallel 0}$ (42). In the absence of a current, this leads to the condition

$$f|_{(\rho^2+z^2)\to 0}\to\rho^2/(\rho^2+z^2)^{3/2}. \tag{45}$$

In addition, the function $f(\rho,z)$ must not have divergences on the singular surface (33),

$$\rho=\rho^*(f)=(1+d\Psi/df)^{-1}, \tag{46}$$

i.e., as follows from (39),

$$\frac{2}{\rho}\frac{\partial f}{\partial\rho}\Big|_{\rho=\rho^*}=g\frac{dg}{df}-\left(1+\frac{d\Psi}{df}\right)^{-1}\frac{d^2\Psi}{df^2}(\nabla f)^2|_{\rho=\rho^*}. \tag{47}$$

In addition, it is obvious that $f(\rho,z)$ must be symmetric about the equatorial plane:

$$f(\rho, z)=f(\rho, -z). \tag{48}$$

The relations (45)–(48) form a complete system of boundary conditions for Eq. (39).

We now turn to the solution of this equation. Note that it simplifies appreciably if the source functions $\Psi(f)$ and $g(f)$ are chosen in the form

$$\Psi(f)=\Psi_0-\beta f, \quad g(f)=i_0 f. \tag{49}$$

These relations have a simple physical meaning. According to the first of them, the total electric field (10) in the magnetosphere is $\mathbf{E} = -(1-\beta)\nabla f$. In such a field, the plasma rotates with constant angular frequency $\Omega' = \Omega(1-\beta)$ (when the influence of the longitudinal current is ignored). Therefore, the parameter β describes the slowing down of the rotation velocity of the magnetosphere plasma ($\beta > 0$ always). As is clear from (42), the second relation in (49) corresponds to the flowing of a constant longitudinal current i_0 at the surface of the pulsar. Equation (39) with allowance for (49) takes the form

$$\Delta f[1-\rho^2(1-\beta)^2]-\frac{2}{\rho}\frac{\partial f}{\partial \rho}+i_0^2 f=0. \tag{50}$$

It has the following scaling property: in the variables

$$x=\rho(1-\beta), \quad z_1=z(1-\beta) \tag{51}$$

it depends only on the single parameter $\alpha_1 = i_0^2/(1-\beta)^2$.

Equation (50) for the function f is linear. It is natural to seek its solution by an expansion in a Fourier integral with respect to z. Going over to the variables (51) and using (48), we represent the solution in the form

$$f(x, z_1)=\int_0^\infty R_\lambda(x)\cos[(\lambda^2+\alpha_1)^{1/2} z_1]\,d\lambda. \tag{52}$$

For the functions $R_\lambda(x)$, we obtain from (50)

$$\frac{d^2R_\lambda}{dx^2}-\frac{1+x^2}{x(1-x^2)}\frac{dR_\lambda}{dx}+\left(\alpha_1\frac{x^2}{1-x^2}-\lambda^2\right)R_\lambda=0. \tag{53}$$

The boundary conditions (45) and (47) for $R_\lambda(x)$ take the following form: as $x\to 0$

$$R_\lambda \to \frac{2}{\pi}(1-\beta)\lambda x K_1(\lambda x); \tag{54}$$

as $x\to 1$

$$(\alpha_1 R_\lambda - 2dR_\lambda/dx)\to 0; \tag{55}$$

$K_1(\lambda x)$ is a Macdonald function of the first order. In addition, from (53) and (55) we obtain the following asymptotic expression for the functions $R_\lambda(x)$ as $x\to 1$:

$$R_\lambda=D(\lambda)I_0[(\lambda^2-\alpha_1^2/4)^{1/2}(1-x)]\exp[-\alpha_1(1-x)/2], \tag{56}$$

where $I_0(y)$ is a Bessel function of imaginary argument and $D(\lambda)$ is a constant. We note also the asymptotic expressions for large values of $\lambda \gg 1$ that give the constant $D(\lambda)$ and $R_\lambda(x)$ in the region $\lambda^{-1} < x < 1-\lambda^{-1}$:

$$D(\lambda)=2^{1/2}\lambda(1-\beta)e^{-\lambda},$$

$$R_\lambda(x)=\left(\frac{2\lambda}{\pi}\right)^{1/2}(1-\beta)\left(\frac{x}{1-x^2}\right)^{1/2} \times e^{-\lambda x}\left[1+\frac{1}{\lambda}\left(\frac{3}{8}\frac{1}{x}+\frac{1}{4}\frac{x}{1-x^2}+\frac{1}{8}(1-2\alpha_1)\ln\frac{1-x}{1+x}-\frac{1}{2}\alpha_1 x\right)+O(\lambda^{-2})\right]. \tag{57}$$

The solution of Eq. (50) can be represented in the form of a series in powers of $1-x^2$:

$$R_\lambda(x)=D(\lambda)\sum_{n=0}^\infty a_n(1-x^2)^n, \tag{58}$$

where the coefficients a_n are connected by the recursion relations

$$a_{n+1}=\frac{(4n^2-\alpha_1)a_n+(\alpha_1+\lambda^2)a_{n-1}}{4(n+1)^2}, \quad a_0=1. \tag{59}$$

At the same time, it follows from (55) that $a_1 = -\alpha_1/4$. Using the expression (54) to find $R_\lambda(x)$ at the point $x=0$, for $D(\lambda)$ we obtain the relation

$$D(\lambda)=\frac{2}{\pi}(1-\beta)\left(\sum_{n=0}^\infty a_n\right)^{-1},$$

which takes for $\lambda \gg 1$ the asymptotic value given by (57).

The solution of Eq. (53) was found numerically by summing the series (58) and integrating (52) using the asymptotic expressions (54), (56), and (57). The results of the calculations for the case $\alpha_1 = 0, \beta = 0$re shown in Fig. 1. The dashed line in Fig. 1 shows the singular surface (46) and the light surface coincident with it—the cylinder of radius $\rho = 1$. As can be seen from the figure, the magnetic lines of force are divided into two classes: for $f > f^*$, they are closed and do not reach the singular surface; for $f < f^*$, they are open, i.e., they intersect the singular surface and go away to infinity. The separatrix dividing these two classes corresponds to the value $f = f^* = 1.592$. At the intersection of the separatrix and the

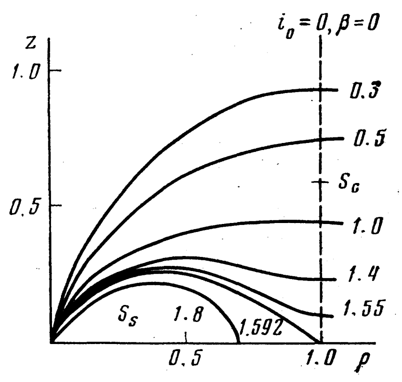

FIG. 1. Lines of force of the magnetic field $f=$ const in the axisymmetric case in the absence of a longitudinal current and accelerating potential ($i_0 = 0, \beta = 0$). The numbers are the values of f on the given line of force.

light cylinder, i.e., at $z=0, \rho=1$, there is a singular point of the magnetic field—the so-called zero point $B=0$ (in fact, of course, it is a zero line, a circle lying on the light cylinder $z=0$, $\rho=1$, $0\leqslant\varphi\leqslant 2\pi$).[11] The separatrices intersect at the zero point (see Fig. 1). The tangent $\tan\delta$ of the angle between the intersecting separatrices and the invariant

$$\Delta^0 f = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial f}{\partial\rho}+\frac{\partial^2 f}{\partial z^2}\Big|_{\rho=1,z=0},$$

which together characterize the behavior of the lines of force in the neighborhood of the zero point [see (57) and (59)],

$$f-f^* = \Delta^0 f[z^2-{}^1/_2(1-\rho)^2], \tag{60}$$

were found to be

$$\tan\delta = 2\sqrt{2},\quad \Delta^0 f = -\frac{1}{2}\int_0^\infty D(\lambda)\lambda^2 d\lambda = -4.008,$$

$$f^* = \int_0^\infty D(\lambda)\,d\lambda = 1.592. \tag{61}$$

The results shown in Fig. 1 agree with those obtained earlier by Mestel and Wang[5] and Michel[6] by numerical solution of Eq. (41).

We now turn to the solution of the main problem—the allowance for the longitudinal currents and the electric field Ψ. We note first that the expression (49) for $g(f)$ and $\Psi(f)$ when $i_0\neq 0, \beta\neq 0$ does not satisfy the condition (44). In accordance with (44), the field Ψ and longitudinal currents are absent in the region $f>$ * of closed lines of force, and there Eq. (41) is always valid. On the boundary $f=f^*$, its solution must be matched to the solution of Eq. (50). At the same time, by virtue of (49) and (44), the function $g(f)$ must change abruptly on the boundary $f=f^*$ from the value $g(f^*-0)=i_0 f^*$ to the value $g(f^*+0)=0$. Here, the derivative dg/df increases strongly in magnitude, i.e., a jet of a reverse longitudinal current arises:

$$i_{\|0} = -i_0 f^*\delta(f-f^*). \tag{62}$$

This corresponds to the following picture of the currents in the pulsar magnetosphere: In the entire magnetosphere, the current flows in one direction, and the reverse current, as will be shown in the third section, always forms an intense current jet net the boundary $f=f^*$. The electric field also has a similar rapid variation at the boundary $f=f^*$ in the case (49) that we consider: $d\Psi/df=-\beta$ for $f=f^*-0$ and $d\Psi/df=0$ for $f=f^*+0$. Accordingly, the density of the electric charge here increases sharply:

$$-(d^2\Psi/df^2)(\nabla f)^2 = -\beta\delta(f-f^*)(\nabla f)^2.$$

In contrast the potential of the electric field is always continuous as $f\to f^*$: $\Psi(f)\to 0, \Psi_0=\beta f_*$ [see (49)].

In matching the solutions in the closed and open regions, we assume in accordance with (61) that the boundary between these regions is always the separatrix $f=f^*=1.592$. In othe words, we shall ignore the change in the shape of the closed region of the magnetosphere, which strictly is valid only in the case of small values of the curent and the electric field. The separatrix $f=f^*$ ends at the singular point $z=0, \rho=1$. Therefore, in the considered approximation both the solution of Eq. (41) for the closed region as well as the solution of Eq. (39) for the open region must have a singular point at $z=0$, $\rho=1$. Bearing this in mind and also the fact that near the singular point $(\nabla f)^2\to 0$, we obtain from (39) and (60) as $\rho\to 1$, $z\to 0$ in the region of open lines of forces as $f\to f^*-0$

$$\frac{1}{\Delta^0 f}g\frac{dg}{df}\Big|_{f=f^*-0} = \left[2\frac{d\Psi}{df}+\left(\frac{d\Psi}{df}\right)^2\right]\Big|_{f=f^*-0}. \tag{63}$$

This is the condition of consistency of the solutions in the closed and open regions. It establishes a connection between the electric field and the longitudinal current in the pulsar magnetosphere. In the particular case (49) that we consider, it takes the form

$$\beta = 1-(1-ai_0^2)^{1/2},\quad a=f^*/|\Delta^0 f| = 0.40. \tag{64}$$

We emphasize that all quantities vary near the singular point most strongly. Therefore, the matching at the singular point plays a leading part and in conjunction with the fixing of the initial behavior of the separatrix near the surface of the pulsar ($f=f^*$) is sufficient for complete determination of the boundary. We illustrate this by an example. We consider the point at which the magnetic line of force $f=f_0$ reaches the light cylinder $\rho=1$. It is determined by the expression

$$\delta z_c(f_0) = \int_0^1\left(\frac{\partial z}{\partial\rho}\right)_{f_0} d\rho = \frac{1}{2\pi}\int_{\rho\leqslant 1}\delta(f-f_0)\frac{1}{\rho}\frac{\partial f}{\partial\rho}d\mathbf{r}.$$

Here, we have used the fact that the initial coordinates of the lines of force, in terms of the variables (40), are $z\to 0$ and $\rho\to 0$. Determining $\rho^{-1}\partial f/\partial\rho$ near the boundary $f=f^*$ from Eq. (41) for $f_0=f^*+0$ and from Eq. (50) for $f_0=f^*-0$, we have

$$\delta z_c^+ = \frac{1}{4\pi}\int_{\rho\leqslant 1}\Delta f(1-\rho^2)\delta[f-(f^*+0)]\,d\mathbf{r},$$

$$\delta z_c^- = \frac{1}{4\pi}\int_{\rho\leqslant 1}\Delta f[1-\rho^2(1-\beta)^2]\delta[f-(f^*-0)]\,\delta r$$
$$+\frac{i_0^2 f^*}{4\pi}\int_{\rho\leqslant 1}\delta[f-(f^*-0)]\,d\mathbf{r}. \tag{65}$$

Bearing in mind that in accordance with the solution in the closed region (Fig. 1) $\delta z_c^+=0$, we find that the condition of matching of the boundaries has the form $\delta z_c^-=0$. Then from (65), using Eq. (41)[6]

$$\Delta f = 2\frac{\partial f}{\partial\rho}\Big/\rho(1-\rho^2),$$

we have

$$i_0^2 f^* A = [1-(1-\beta)^2]C. \tag{66}$$

It is important that the constants C and A are determined here by integrating the displacement z along the complete limiting line of force:

$$A=\int_0^1 \frac{\rho d\rho}{(\partial f/\partial z)_{f^*}},\quad C=\int_0^1 \frac{2\rho^2}{1-\rho^2}\left(\frac{\partial z}{\partial \rho}\right)_{f^*} d\rho. \tag{67}$$

However, both integrals diverge logarithmically near the upper limit, i.e., near the sungular point, since here $\rho \to 1$ and $(\partial f/\partial z)_{f^*} \to 0$. Therefore, in the relation (66) only their ratio, which is determined by the behavior of the solution in the neighborhood of the singular point, is important. From (60), we have

$$(\partial z/\partial \rho)_{f^*,\, \rho\to 1}=-1/\sqrt{2},\quad (\partial f/\partial z)_{f^*,\, \rho\to 1}=\sqrt{2}\Delta^\circ f(1-\rho).$$

Substituting these values in (67), we find

$$A/C\approx -1/\Delta^\circ f.$$

With allowance for this, the relation (66) becomes identical to (64).

Thus, in the presence of a longitudinal current ($i_0 \neq 0$) and a field ($\beta \neq 0$) we must find a solution of Eq. (50) for $f<f^*$ and match it to the solution of Eq. (41) for $f<f^*$ using the relation (64). The solution constructed in this manner is shown in Fig. 2. The dashed line and dashed curve in Fig. 2 show the singular surface S_d and the light surface S_c. It can be seen that, as before, the magnetic lines of force are divided into two groups. The closed ones are situated in the region $\rho \leqslant 1$ and do not reach the singular surface. The open ones pass through the singular surface and the light surface and go away to infinity. As before, the zero point of the magnetic field is at $\rho = 1$, $z = 0$.

On the singular surface $x = 1$, the electric field is $\mathbf{E} = \mathbf{B}\times \mathbf{e}_\varphi$, but since the current is not equal to zero, and also $B_\varphi \neq 0$, we have $|\mathbf{E}| < |\mathbf{B}|$. Therefore, the drift velocity of the particles is here less than the velocity of light. The surface at which the particles reach the velocity of light and $|\mathbf{E}| = |\mathbf{B}|$ (the light surface) is given by the relation

$$x_0^2(z)=1+\frac{\alpha_1(1-\beta)^2 f^2(x_0,z)}{[\nabla f(x_0,z)]^2},\quad x_0>1. \tag{68}$$

At small currents $\alpha_1 \ll 1$, this surface is close to the singular surface $x = 1$. To find the values of the field in the neighborhood of the light surface at small currents, we use the series expansion (52),(58) of the function $f(x,z_1)$ in powers of

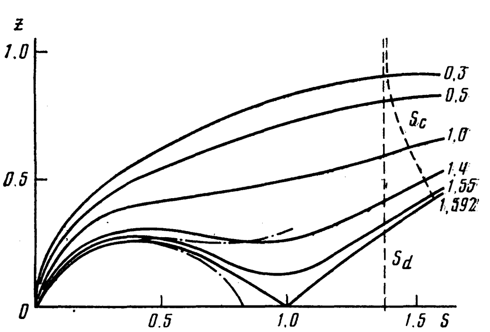

FIG. 2. Structure of the magnetic field in the presence of a longitudinal current ($i_0 = 1$). The electric field is chosen on the basis of the "consistency condition" ($\beta = 0.275$). The chain curves correspond to the position of the line $f = 1.592$ for $\beta = 0$ (upper curve) and $\beta = 0.35$ (lower).

$(1 - x^2)$. Then the fields are represented in the form

$$B_x=(1-\beta)^2\int_0^\infty D(\lambda)\lambda \sin(\lambda z_1)\, d\lambda, \tag{69}$$

$$B_z=\frac{\alpha_1(1-\beta)^2}{2}\int_0^\infty D(\lambda)\left[1+\frac{\lambda^2}{2}\frac{x^2-1}{\alpha_1}\right]\cos(\lambda z_1)\, d\lambda, \tag{70}$$

$$B_\varphi=\alpha_1^{1/2}(1-\beta)\int_0^\infty D(\lambda)\cos(\lambda z_1)\, d\lambda. \tag{71}$$

$$\mathbf{E}=x[\mathbf{B}\times\mathbf{e}_\varphi]. \tag{72}$$

It can be seen from (69)–(72) that in the considered case $B_z \ll B_\varphi \ll B_x$. To characterize the spiral nature of the magnetic field in the neighborhood of the light surface, it is convenient to introduce the parameter $\varkappa_0(z_1) = B_\varphi x_0,z_1)/B_x(x_0,z_1)$, which for small currents is equal to

$$\varkappa_0(z_1)=\frac{\alpha_1^{1/2}}{(1-\beta)}\left(\int_0^\infty D(\lambda)\cos(\lambda z_1)\, d\lambda \Big/ \int_0^\infty D(\lambda)\lambda \sin(\lambda z_1)\, d\lambda\right).$$

The coordinate $x_0(z_1)$ can be expressed in terms of $\varkappa_0(z_1)$ by the relation

$$x_0^2=1+\varkappa_0^2. \tag{73}$$

At large values $z_1 \gg 1$ all the fields decrease exponentially. This is due to the fact that all the odd derivatives of the function $D(\lambda)$ at the point $\lambda = 0$ vanish. The dependence of the field B_x on z can be expressed in the form

$$B_x(z\gg 1)\propto e^{-pz}.$$

In particular, $p = 3.0$ for the case $i_0 = 0$ and $p = 3.1$ for $i_0 = 1$. Hence and from (73) it follows that $\varkappa_0$ tends to the constant limit $\alpha_1^{1/2}/p$ for $z \gg 1$.

In a numerical calculation, one can directly verify the consistency relation (64). If it is not satisfied, then the limiting line of force $f = f^*$ in the solution of Eq. (50) either keeps close to the boundary of the closed magnetosphere or deviates strongly from it (chain curves in Fig. 2).[7] Thus, the connection between the electric field and the longitudinal current for the solution of Eq. (50) can be established on the basis of only a numerical calculation by choosing the values of β and i_0 in such a way that the boundaries of the closed and open regions coincide. The numerical dependence of β on i_0 obtained in this way agrees with sufficient accuracy with (64).

We emphasize the importance of the consistency relation (63),(64), which establishes the connection between the electric field and the longitudinal currents in the pulsar magnetosphere. It plays the part of a nonlinear Ohm's law and determines the energy losses of the pulsar. In addition, it follows from (64) that the dimensionless longitudinal current $i_0 = g(f^*)/f^*$ cannot exceed the critical value i_c. As can be seen from (64), its numerical value is $i_c = a^{-1/2} = 1.58$, but it must be borne in mind that the relation (64) itself at large $i_0 \sim i_c$ is approximate, since it does not take into account the change in the boundary f^* of the closed region of the magnetosphere, which may lead to a change in the value of the parameter a. In the dimensional quantities, using (40) and

(43), we obtain from this a restriction on the longitudinal current in the magnetosphere and on the total current on the surface of the pulsar:

$$|j_\parallel|\leqslant 0.79\frac{\Omega B_0}{2\pi},\quad |I_s|\leqslant 3.95R^3\left(\frac{\Omega}{c}\right)^2\frac{B_0c}{2\pi}. \tag{74}$$

It can be seen that the limiting value of the longitudinal current is of order $j_c = -\Omega B/2\pi$ (14). It also follows from (63) and (64) that the electric field always slows down the rotation of the magnetosphere ($d\Psi/df<0$). This slowing down increases with increasing longitudinal current. As $i_0\to i_c$, the total electric field

$$-\nabla(\Phi_0+\Psi)=-(1+d\Psi/df)\nabla f$$

vanishes, i.e., the plama in the magnetosphere is stopped. At the same time, the singular surface moves away to infinity.

We emphasize that in deriving the consistency relations we have used only general conditions connecting the open and closed regions of the magnetosphere. Simlar relations can evidently be obtained for any magnetosphere in which there are closed and open regions and the conditions (17) and (8) are satisfied (i.e., the Debye radius is small compared with the dimensions of the magnetosphere, and the plasma pressure is small compared with the magnetic field pressure). Such conditions are usually well satisfied, for example, in the magnetospheres of the Earth and the planets. The necessity as follows from (63) for the existence of a longitudinal difference of the potentials Ψ between the surface of the body and the magnetosphere in the presence of a longitudinal current dg/df is a striking and, evidently, general property of magnetospheres. In particular, in the magnetospheres of the Earth and the planets the occurrence of the longitudinal potential difference in the regions in which the longitudinal currents flow leads to the continuous acceleration of electrons and ions, i.e., it serves as the source of the aurora polaris and the radio emission of the planets.

§3. THE BOUNDARY LAYER

The solution constructed above is valid only up to the light surface (68). On the light surface, the total electric field becomes equal in magnitude to the magnetic field: $E=B$. Therefore, near the light surface the drift velocity (20) of the particles tends to the velocity of light and their energy increases sharply. At the same time, the conditions (8) and (22) cease to hold, and Eq. (26) becomes invalid. Thus, near the light surface there is formed a singular boundary layer, in which the drift approximation (20) is insufficient and a more accurate description of the motion of the electrons and positrons is required.

Under the conditions (7), the thickness of the boundary layer is always small compared with the scale of the magnetosphere—it is of order $1/\lambda$ (16) (this will be shown below). Therefore, under steady conditions all quantities in it change appreciably only in the direction ρ of the normal to the layer. Bearing this in mind, we write Maxwell's equation in the boundary layer in the form

$$\frac{\partial B_\varphi}{\partial\rho}=\frac{4\pi}{c}j_z,\quad \frac{\partial B_z}{\partial\rho}=-\frac{4\pi}{c}j_\varphi,\quad \frac{\partial}{\partial\rho}(\rho B_\rho)=0,$$
$$\frac{\partial^2\Psi}{\partial\rho^2}=-2\frac{\Omega B_z}{c}+4\pi\left(\frac{\Omega\rho}{c^2}j_\varphi-\rho_e\right). \tag{75}$$

The distribution of the electrons and positrons is described by the kinetic equation (4),(5), in which the source Q is equal to zero (plasma is generated only near the surface of the pulsar). We take into account the fact that near the light surface the particles are strongly accelerated. At the same time, their initial spread with respect to the longitudinal momenta is of little importance, so that we can set

$$F^\pm=n^\pm(\rho)\delta[p_\parallel-p_\parallel^\pm(\rho)]. \tag{76}$$

Substituting (76) in the kinetic equation and retaining only the derivatives with respect to the coordinate ρ, we obtain

$$\frac{d}{d\rho}\left(\frac{p_\rho^\pm n^\pm}{\gamma^\pm}\right)=0,$$
$$\frac{dp_\rho^\pm}{d\rho}=\frac{p_\varphi^{\pm2}}{\rho p_\rho^\pm}\pm\frac{e\gamma^\pm}{p_\rho^\pm}\left\{m_0E_\rho+\frac{1}{c\gamma^\pm}[\mathbf{pB}]_\rho\right\},$$
$$\frac{dp_\varphi^\pm}{d\rho}=-\frac{p_\varphi^\pm}{\rho}\pm\frac{e\gamma^\pm}{p_\rho^\pm}\left\{m_0E_\varphi+\frac{1}{c\gamma^\pm}[\mathbf{pB}]_\varphi\right\}, \tag{77}$$
$$\frac{dp_z^\pm}{d\rho}=\pm\frac{e\gamma^\pm}{p_\rho^\pm}\left\{m_0E_z+\frac{1}{c\gamma^\pm}[\mathbf{pB}]_z\right\}.$$

Equations (75) and (77) form a complete system describing the distribution of the plasma and the field in the thin boundary layer.

For simplicity, we shall consider the case of weak longitudinal current in (70) and (71), $\alpha_1\ll 1$, or

$$\varkappa_0^2\ll 1,\quad \varkappa_0=B_\varphi/B_\rho|_{\rho=c/\Omega(1-\beta)}.$$

In this case, the component B_z (and the light surface near it) is small, $B_z\approx B_\rho\varkappa_0^2$ (70), and it can be ignored. Using also (75), we represent the magnetic field in the boundary layer in the form

$$\mathbf{B}=B_c(z)\{1/x,\ \varkappa,\ 0\},\quad \varkappa=\varkappa(x)=B_\varphi(x)/B_c, \tag{78}$$

where x is a dimensionless coordinate: $x=\rho\Omega c^{-1}(1-\beta)$ (51), and B_c is the value of the radial magnetic field on the surface $x=1$. The dependence of B_c on z is determined by the general behavior of the solution up to the region of the boundary layer $x\approx 1$. In accordance with (72), the electric field E in the boundary layer can be expressed to the same accuracy $\sim\varkappa_0^2$ in the form

$$\mathbf{E}=B_c(z)\{0,\ 0,\ 1\}. \tag{79}$$

Introducing the dimensionless variables $\gamma=\gamma_0\gamma'$ and $p=m_0\gamma_0cp'$ and using (78) and (79), we transform the system of equations (75) and (77) into (omitting the primes)

$$\frac{d\gamma}{dx}=\alpha\frac{p_z}{p_x},\quad \frac{dp_z}{dx}=\alpha\left(\varkappa+\frac{\gamma}{p_x}-\frac{\gamma-1}{x^2p_x}\right),$$

$$p_\varphi=(\gamma-1)/x,\quad p_x^2+p_\varphi^2+p_z^2+1/\gamma_0^2=\gamma^2. \tag{80}$$

$$n=\gamma\frac{p_{x0}}{p_x}\lambda n_c,\quad \frac{\partial\varkappa}{\partial x}=4\lambda\frac{p_z}{p_x}.$$

Here, we have used the dimensionless parameters α (21) and λ (16):

$$\alpha=\frac{eB_c}{m_0c\gamma_0\Omega(1-\beta)},\quad \lambda=\frac{n}{n_c},\quad |n_c|=\frac{\Omega B_c(1-\beta)}{2\pi ec}. \tag{81}$$

Under the conditions of the pulsar magnetosphere, it follows from (7) and (21) that $\lambda \gg 1, \alpha \gg 1$ always. Of course, Eqs. (80) are valid only in a thin boundary layer Δx near the light surface:

$$\Delta x=x-x_0\ll 1,\quad x_0=(1-\varkappa_0^2)^{-1/2}\approx 1+{}^1/_2\varkappa_0^2.$$

It follows directly from (80) that

$$\varkappa(x)=\varkappa_0+\frac{4\lambda}{\alpha}(\gamma-1), \tag{82}$$

i.e., the component B^φ of the magnetic field can be expressed directly in terms of the Lorentz factor γ of the plasma particles, in accord with Ref. 13. In the drift approximation, when $\alpha/\lambda\to\infty$ (22), the solution of Eqs. (80) has the form

$$\gamma=\frac{1-|\varkappa_0|x^2[1-x^2(1-\varkappa_0^2)]^{-1/2}}{1-x^2}\approx\left(\frac{x_0}{2\varkappa_0^2}\right)^{1/2}(x_0-x)^{-1/2},$$

$$p_x=[1-x^2(1-\varkappa_0^2)]^{-1/2}\approx(x_0/2)^{1/2}(x_0-x)^{-1/2}. \tag{83}$$

In the last expressions, we have used the fact that $x_0>1$. It can be seen that as the light surface is approached, $x\to x_0$, the energy of the particles in the drift approximation becomes infinite as $(x_0-x)^{-1/2}$. At the same time, the drift approximation (22) breaks down.

To find the solution near the singularity, we introduce the new variables

$$s=\frac{4\lambda}{|\varkappa_0|}(x-x_0),\quad \Gamma=\frac{4\lambda}{\alpha|\varkappa_0|}\gamma,\quad q_x=\frac{4\lambda}{\alpha\varkappa_0^2}p_x,$$

$$q_z=\frac{4\lambda}{\alpha\varkappa_0^2}p_z.$$

Bearing in mind that $\gamma\gg 1$, we rewrite Eqs. (80) and (82) in the form

$$\frac{d\Gamma}{ds}=\frac{q_z}{q_x},\quad \frac{dq_z}{ds}=\Gamma\left(1+\frac{1}{q_x}\right)-1,\quad q_z^2+q_x^2=\Gamma^2. \tag{84}$$

It follows from (84) that

$$\frac{dq_x}{ds}=-\frac{q_z}{q_x}(\Gamma-1)=-\frac{d\Gamma}{ds}(\Gamma-1).$$

Therefore

$$q_x=\Gamma-{}^1/_2\Gamma^2,\quad q_z=\Gamma^{3/2}(1-\Gamma/4)^{1/2}.$$

Finally, we find from (84) that

$$\Gamma=2-[4-(s-s_0)^2]^{1/2}, \tag{85}$$

where s_0 is the value of s at which $\Gamma\to 0$. Matching the solutions (85) and (83) with respect to the value of Γ and the derivative $d\Gamma/ds$, we find

$$s_0=-5a,\quad s_1=-a,\quad a=2^{1/5}\lambda^{3/5}\alpha^{-2/5}|\varkappa_0|^{-1}x_0^{1/5}.$$

Here, s_1 is the matching point, which is valid for $a<\frac{1}{2}$.

The obtained expressions describe completely the distribution of the particles and the magnetic field in the boundary layer on the light surface.

It can be seen above all from (83) and (85) that in the boundary layer the energy of the electrons and positrons increases sharply:

$$\mathscr{E}=\mathscr{E}_m[1-(1-(s-s_0)^2/4)^{1/2}],$$

where

$$\mathscr{E}_m=\frac{m_0c^2\gamma_0|\varkappa_0|\alpha}{2\lambda}=\frac{ecB_{c\varphi}}{2\Omega\lambda(1-\beta)}=\frac{eI_s}{c\lambda}. \tag{86}$$

The limiting energy $\mathscr{E}_m$ is proportional to the current strength I_s (43) on the pulsar surface and is inversely proportional to the density of particles in the magnetosphere: $\lambda\sim n$ (81). Under optimal conditions, when the maximal value of the current I_s reaches the value (74) and $\lambda\sim 1$, the particles are accelerated to values $\gamma\sim 10^7 B_{0_{12}}P^{-2}$, where $B_{0_{12}}$ is the value of the field on the surface of the pulsar in units of 10^{12} G, and P is its rotation period in seconds.

The momentum p_φ of the particles in the direction of rotation increases in proportion to the energy (80). The momentum p_x in the radial direction behaves in a more distinctive manner:

$$p_x=\frac{\mathscr{E}_m}{c}|\varkappa_0|\left[1-\left(1-\frac{(s-s_0)^2}{4}\right)^{1/2}\right]\left[1-\frac{(s-s_0)^2}{4}\right]^{1/2}.$$

It increases $p_{xm}=\mathscr{E}_m|\varkappa_0|/4c$ at $s-s_0=\sqrt{3}$, and then decreases and on the boundary $s=s_c=2+s_0$ vanishes. The particles are reflected by this boundary and move in the opposite direction, continuing to be accelerated in the z direction. We emphasize that the physical situation in the region directly near the boundary S_c is quite different from the situation in the main magnetosphere. The energy density of the particles is here comparable with the magnetic field energy, and the Alfvén velocity becomes equal to the velocity of light. Therefore, magnetoacoustic waves can be generated here.[8)] Particles enter the region $s\gtrsim s_c$ together with the radiation. Their mean energy remains of order $\mathscr{E}_m$ (86). The condition (8) is violated at the boundary s_c and in the radiation region $s>s_c$, so that our original equation (26) is here invalid.

The motion of the particles in the z direction is important only in the boundary layer. The momentm p_z here increases most strongly:

$$p_z^{\pm}=\mp\frac{\mathscr{E}_m}{2c}\varkappa_0(s-s_0)\left[1-\left(1-\frac{(s-s_0)^2}{4}\right)^{1/2}\right].$$

Its main feature is that the particles with opposite sign of the charge, the electrons and positrons, move in opposite z directions. This means that in the thin boundary layer there is a strong electric current. Here there is formed a jet of surface current flowing in the z direction along the light surface. The intensity of the current jet is

$$J_j = -B_{\varphi c} c/4\pi. \tag{87}$$

Because of the presence of the intense surface current J_j (87), there is rapid variation in the boundary layer of the φ component of the magnetic field:

$$B_\varphi = B_{\varphi c}[(4-(s-s_0)^2)^{1/2}-1];$$

here, B_{φ_c} is the field in the magnetosphere on the boundary of the layer. It can be seen that already at $s = s_0 + \sqrt{3}$ the component B_φ vanishes and changes sign, and at $s = s_c = s_0 + 2$ it reaches the value $-B_{\varphi_c}$. The vanishing of B_φ corresponds to complete closing by the current jet in the boundary layer of the longitudinal currents flowing in the magnetosphere. Therefore, the currents for $s > s_0 + \sqrt{3}$ can be nominally regarded as belonging to the current system in the radiation region.

Physically, the process of closing of the current in the pulsar magnetosphere and the acceleration of the particles in the boundary layer can be readily understood. For this, let us compare the distribution of the total potential of the electric field (11) on the surface of the star,

$$\Phi_0 = E_0 R f, \quad E_0 = \frac{(\mathbf{\Omega M})\Omega}{c^2 R} = \frac{1}{2} B_0 \left(\frac{\Omega R}{c}\right)^2 \operatorname{sign}(\mathbf{\Omega M}), \tag{88}$$

and in the magnetosphere:

$$\Phi = E_0 R[f + \Psi(f)], \tag{89}$$

this being shown in Fig. 3. Concretely, in Fig. 3 we consider the case, which will also be discussed below, when the corotation charge ρ_c (13) in the polar region has positive sign ($\mathbf{\Omega}\cdot\mathbf{M} < 0$). In this case, the surface of the body on the open lines of force, $f < f^*$, has an appreciable positive potential relative to the magnetosphere (the broken line in Fig. 3). Therefore, only electrons can reach the surface of the pulsar from the magnetosphere, so that the longitudinal current in the complete region of open lines of force in the magnetosphere always has the same direction, away from the pulsar, as is shown in Fig. 4. In the boundary layer near the light surface the drift approximation (20) is violated, and the par-

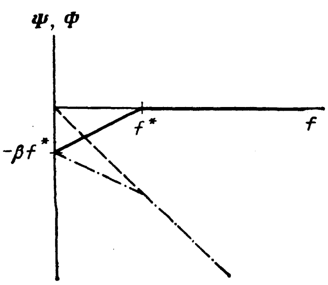

FIG. 3. Total potential Φ of the magnetosphere dash-dot line), potential of the star (dashed line), and the potential Ψ as a function of f.

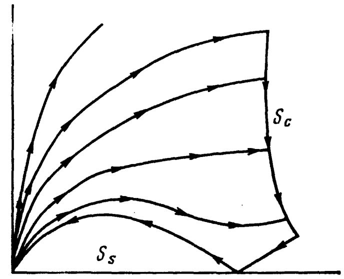

FIG. 4. Schematic representation of the currents flowing in the pulsar magnetosphere.

ticles are displaced across the equipotential surfaces. In doing so they acquire energy. As is clear from Fig. 3, the positrons, if they are to acquire energy from the electric field, must be displaced in the direction of increasing f, and the electrons in the opposite direction. As a result, there arises the intense current jet (87) in the boundary layer, which carries positive charges in the direction of increasing f (Fig. 4). As $f \to f^*$, as can be seen from Fig. 3, the potential difference between the magnetosphere and the surface of the star vanishes [see (44)]. At the same time, the positrons of the magnetosphere acquire become capable reaching the surface of the pulsar. Therefore, at the position of intersection of the light surface with the line of force $f = f^*$ the current jet bends and returns to the surface of the pulsar along the separatrix (Fig. 4). The magnitude of the reverse current jet on $f = f^*$ is

$$J_j = -\frac{\Omega}{2\pi c} I_s.$$

It cancels completely the direct longitudinal current I_s flowing in the entire magnetosphere of the pulsar.

Note that a singular boundary layer also arises near the surface of the pulsar. This is a Langmuir double layer, in which the total electric potential changes from the surface potential (88) to the magnetosphere potential (89). By virtue of the condition $\alpha \gg 1$ (21), the concentration of charged particles in this layer is negligibly small, and it is therefore called the "vacuum gap".[2,3] As follows from (12), the electric field Ψ in the layer as $\rho_e \to 0$, $\beta_r \to 0$ is described by the Laplace equation

$$\Delta\Psi = 4\pi\rho_c. \tag{90}$$

The boundary conditions for the potential Ψ are

$$\Psi(\rho, z = z_0(\rho)) = \Psi_0 = -\frac{d\Psi}{df}\left(f^* - \frac{\rho^2}{r_0^3}\right),$$

$$\Psi(z=0) = 0, \quad \frac{\partial\Psi}{\partial z}(z = z_0(\rho), \rho) = 0.$$

Here, we have used the dimensionless variables (40), r_0 is the dimensionless radius of the star, $r_0 = \Omega R/c$, and $z_0(\rho)$ is the upper boundary of the gap, on which the longitudinal electric field vanishes. In addition, since $\rho < r_0^{2/3} \ll r_0$, the normal to the surface is parallel to the z axis. The solution of (90) for $|d\Psi/df| \ll 1$ has the form

$$\Psi=[8\beta(f^*-\rho^2/r_0^3)]^{1/2}zr_0^{-3/2}-2z^2r_0^{-3}\quad(\beta=-d\Psi/df),\tag{91}$$

and the surface $z_0(\rho)$ is the ellipsoid

$$2z_0^2+\beta\rho^2=\beta f^* r_0^3.$$

The maximal height of the "gap" is $H=(\beta f^*/2)^{1/2}r_0^{3/2}$. It is in the field (91) that the particles acquire the energy needed for the generation of electron-positron pairs in the magnetosphere.

§4. ARBITRARY ANGLE BETWEEN THE ROTATION AXIS AND THE MAGNETIC AXIS OF THE PULSAR

We assumed above that the rotation axis Ω of the pulsar and the axis **M** of its magnetic dipole moment coincide. This made the problem symmetric and therefore led to a number of important simplifications in the construction of its solution. In the present section, we consider the general case of arbitrary angle χ between the rotation axis and the magnetic axis.

As before, an important part is played by the limit of complete corotation, when no longitudinal currents flow from the surface of the pulsar and in the magnetosphere there is no electric field: $\Psi=0$, $i_\parallel(S)=0$. At the same time, we note that by virtue of (23)

$$\operatorname{div}\left\{i_\parallel\mathbf{B}-\frac{c\rho_e}{B^2}[\nabla\Psi\times\mathbf{B}]\right\}=0.\tag{92}$$

In the absence of an electric field ($\Psi=0$), it follows from this relation that $\partial i_\parallel/\partial r_\parallel=0$, i.e., $i_\parallel$ remains unchanged along a line of force. Therefore, in the absence of a field and of a longitudinal current flowing from the pulsar surface, $i_\parallel(S)=0$, the longitudinal current $i_\parallel$ in the complete magnetosphere is equal to zero:

$$\Psi=0,\quad i_\parallel=0.\tag{93}$$

Thus, we consider first the case of comp[illegible] corotation (93). The system of equations (26) and (27) with allowance for (93) takes the form

$$\operatorname{rot}\mathbf{G}_0=0,\quad \mathbf{G}_0=\mathbf{B}(1-\beta_r^2)+\beta_r(\mathbf{B}\beta_r),\quad \operatorname{div}\mathbf{B}=0.\tag{94}$$

The solution of (94) can obviously be represented in the form

$$\mathbf{B}(1-\beta_r^2)+\beta_r(\mathbf{B}\beta_r)=-\mathbf{grad}\,h,\tag{A}$$

where h is some scalar function. In the dimensionless cylindrical coordinates (40), $\rho'=\Omega\rho/c$, $z'=\Omega z/c$, and φ, the magnetic field can be expressed in terms of the dimensionless effective potential h by

$$B_\rho=-B_e\frac{\partial h/\partial\rho}{1-\rho^2},\quad B_\varphi=-B_e\frac{1}{\rho}\frac{\partial h}{\partial\varphi},\quad B_z=-B_e\frac{\partial h/\partial z}{1-\rho^2},\tag{95}$$

where $B_e=M\Omega^3/c^3$. The equation for h follows from (94). We have

$$\frac{\partial^2 h}{\partial\rho^2}+\frac{1+\rho^2}{\rho(1-\rho^2)}\frac{\partial h}{\partial\rho}+\frac{\partial^2 h}{\partial z^2}+\frac{1-\rho^2}{\rho^2}\frac{\partial^2 h}{\partial\varphi^2}=0.\tag{96}$$

The boundary conditions for the effective magnetic potential h are determined in accordance with (27) and (33). It follows from the condition (27) that in the limit $\rho\to0$, $z\to0$ the magnetic field (95) must become a dipole field, i.e.,

$$h_{\rho\to0,z\to0}=\frac{z}{(\rho^2+z^2)^{3/2}}\cos\chi+\frac{\rho\cos\varphi}{(\rho^2+z^2)^{3/2}}\sin\chi.\tag{97}$$

It is natural to seek the solution of the linear equation (96) with the boundary condition (97) in the form

$$h(\rho,z,\varphi)=h_0(\rho,z)\cos\chi+h_1(\rho,z)\cos\varphi\sin\chi.$$

The potential h_0 describes the axisymmetric case ($\chi=0$), and the potential $h_1\cos\varphi$ the case of mutually orthogonal axes Ω and **M**. The potentials h_0 and h_1 satisfy the equations

$$\frac{\partial^2 h_0}{\partial\rho^2}+\frac{1+\rho^2}{\rho(1-\rho^2)}\frac{\partial h_0}{\partial\rho}+\frac{\partial^2 h_0}{\partial z^2}=0,\tag{98}$$

$$\frac{\partial^2 h_1}{\partial\rho^2}+\frac{1+\rho^2}{\rho(1-\rho^2)}\frac{\partial h_1}{\partial\rho}+\frac{\partial^2 h_1}{\partial z^2}-\frac{1-\rho^2}{\rho^2}h_1=0\tag{99}$$

with boundary conditions as $\rho\to0$, $z\to0$

$$h_0=\frac{z}{(\rho^2+z^2)^{3/2}},\quad h_1=\frac{\rho}{(\rho^2+z^2)^{3/2}}.\tag{100}$$

The second boundary condition (33) for h_0 and h_1 is the absence of singularities on the light surface:

$$\partial h_0/\partial\rho=\partial h_1/\partial\rho=0\quad\text{for }\rho=1.\tag{101}$$

The solution of the linear equations (98) and (99) is found by separation of the variables, as in Sec. 2. It is clear from symmetry considerations that h_1 is an even function of the coordinate z and h_0 an odd functin, so that

$$h_0(\rho,z)=\int_0^\infty d\lambda\,\xi_0(\rho,\lambda)\sin(\lambda z),\tag{102}$$

$$h_1(\rho,z)=\int_0^\infty d\lambda\,\xi_1(\rho,\lambda)\cos(\lambda z),\tag{103}$$

and the functions $\xi_0(\rho,\lambda)$ and $\xi_1(\rho,\lambda)$ satisfy the equations

$$\frac{d^2\xi_0}{d\rho^2}+\frac{1+\rho^2}{\rho(1-\rho^2)}\frac{d\xi_0}{d\rho}-\lambda^2\xi_0=0,$$
$$\frac{d^2\xi_1}{d\rho^2}+\frac{1+\rho^2}{\rho(1-\rho^2)}\frac{d\xi_1}{d\rho}-\left(\lambda^2+\frac{1-\rho^2}{\rho^2}\right)\xi_1=0.\tag{104}$$

As follows from (100), (102), and (103), the boundary conditions as $\rho\to0$ have the form

$$\xi_0(\rho,\lambda)=\frac{2}{\pi}\lambda K_0(\lambda\rho),\quad \xi_1(\rho,\lambda)=\frac{2}{\pi}\lambda K_1(\lambda\rho).\tag{105}$$

In the limit $\rho\to1$, the solutions of Eqs. (104) with allowance for (101) are

$$\xi_{0,1}=\frac{8D_{0,1}(\lambda)}{\lambda}(1-\rho)I_1[\lambda(1-\rho)].\tag{106}$$

It can be seen from this that on the light cylinder $\rho = 1$ not only $\partial h/\partial\rho = 0$ but also $h = 0$, i.e., the magnetic field is directed only along ρ:

$$B_z = B_\varphi = 0; \quad B_\rho \neq 0.$$

This agrees with the results of Henriksen and Norton.[14]

For arbitrary values of ρ, the eigenfunctions ξ_0 and ξ_1 can be represented in the form of the expansions

$$\xi_0(\lambda,\rho) = D_0(\lambda)\sum_{n=2}^{\infty} b_n(\lambda)(1-\rho^2)^n,$$

$$\xi_1(\lambda,\rho) = D_1(\lambda)\sum_{n=2}^{\infty} a_n(\lambda)(1-\rho^2)^n, \tag{107}$$

where $b_2 = a_2 = 1$, and b_n *and* a_n satisfy the recursion relations

$$b_{n+1} = \frac{n}{n+1}b_n + \frac{\lambda^2}{4(n^2-1)}b_{n-1} \quad (b_1 = 0),$$

$$a_{n+1} = \frac{n(2n-3)}{n^2-1}a_n - \frac{4(n-1)(n-2)-\lambda^2}{4(n^2-1)}a_{n-1} - \frac{\lambda^2-1}{4(n^2-1)}a_{n-2} \quad (a_0 = a_1 = 0).$$

Further, the functions $D_0(\lambda)$ and $D_1(\lambda)$ in (107) [see also (106)] must be chosen such that the expansion (107) satisfies the boundary conditions (105) as $\rho \to 0$. For this, using the asymptotic expressions for the functions K_0 and K_1 as $\rho \to 0$, we find

$$D_0^{-1}(\lambda) = \frac{\pi}{2\lambda \ln\rho}\sum_{n=2}^{\infty} b_n(\lambda)(1-\rho^2)^n|_{\rho\to 0},$$

$$D_1^{-1}(\lambda) = \frac{\pi}{2}\rho\sum_{n=2}^{\infty} a_n(\lambda)(1-\rho^2)^n|_{\rho\to 0}. \tag{108}$$

Graphs of these functions are shown in Fig. 5. For $\lambda \gg 1$,

$$D_{0,1}(\lambda) = \frac{\pi\sqrt{2}}{4}\lambda^2 e^{-\lambda}.$$

The obtained expressions completely determine the solution of Eq. (96). The structure of the magnetic field in the pulsar magnetosphere for different angles of inclination χ of the axes is shown in Fig. 6. For $\chi = 0°$, the solution is naturally the same as that obtained in Sec. 2. It can be seen that with increasing angle χ the region of the closed magnetosphere increases and is inclined to the rotation axis. At the position at which the closed magnetosphere touches the light cylinder there is, as in the axisymmetric case, a singular zero line. Here, the magnetic field vanishes, $B = 0$.

The form of the zero line for different angles of inclination of the axes is shown in Fig. 7. The zero line is quite close to the ellipse formed by intersection of the light cylinder with

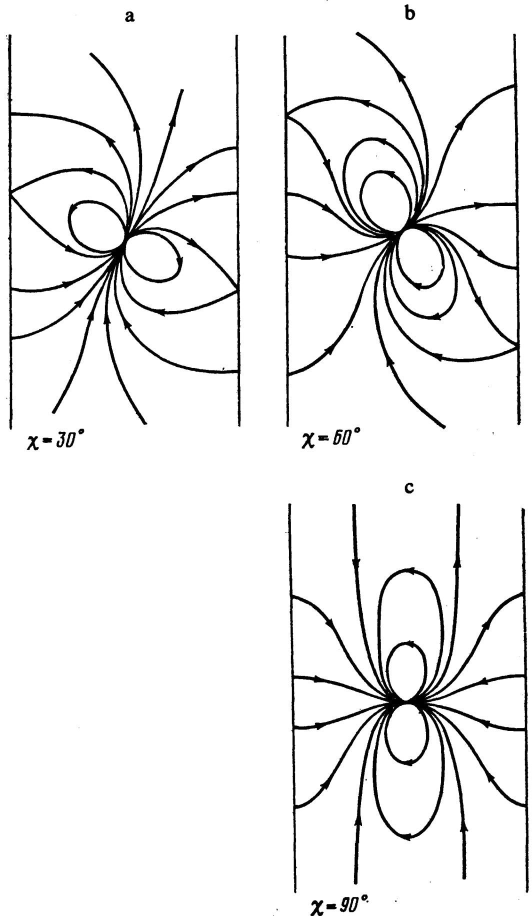

FIG. 6. Structure of the magnetic field for different angles of inclination χ. The longitudinal currents and the accelerating potential are equal to zero.

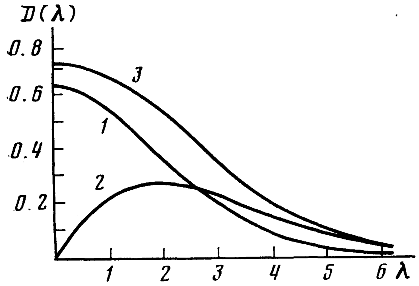

FIG. 5. Form of the functions $D(\lambda)$: curve 1) $D(\lambda)$ (58), 2) $D_0(\lambda)$, 3) $D_1(\lambda)$ (108).

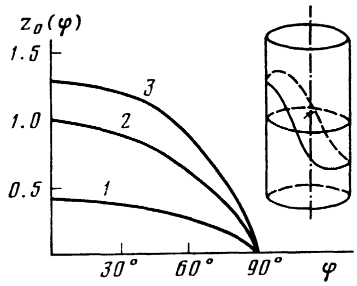

FIG. 7. Position of the zero line as a function of the cylindrical coordinate φ: 1) $\chi = 30°$, 2) $\chi = 60°$, 3) $\chi = 70°$.

the plane. In the limiting case $\chi = 90°$, the zero line degenerates into a vertical straight line. Near the singular points lying on the zero line the limiting lines of force separating the regions of the closed and open magnetosphere converge at the constant angle δ, which depends neither on the angle φ nor the angle of inclination χ: $\delta = 70.5°$ (61).

On the light cylinder, only the z component of the electric field and the ρ component of the magnetic field are nonzero, and in accordance with (10) they are equal in magnitude:

$$E_z = B_\rho = 4B_e\left[\cos\chi\int_0^\infty D_0(\lambda)\sin(\lambda z)\,d\lambda + \cos\varphi\sin\chi\int_0^\infty D_1(\lambda)\cos(\lambda z)\,d\lambda\right].$$

The charge density on the light cylinder is

$$\rho_e = \frac{\Omega B_e}{2\pi c}\left[\cos\chi\int_0^\infty D_0(\lambda)\lambda\cos(\lambda z)\,d\lambda - \cos\varphi\sin\chi\int_0^\infty D_1(\lambda)\lambda\sin(\lambda z)\,d\lambda\right] = \frac{\Omega}{8\pi c}\frac{\partial B_\rho}{\partial z}.$$

As we move away in the z direction from the zero line the magnitudes of the fields first increase and then decrease. At the point of the extremum of the magneic field, the charge density changes sign. At large z, both the magnetic field and the charge density decrease exponentially. This is due to the fact that, as can be seen from (108), $D_0(\lambda)$ is an odd function of λ and $D_1(\lambda)$ an even. For example, for the magnetic field when $z \gg 1$ we obtain in the dimensionless variables (40)

$$B_\rho(\chi=0°) \propto e^{-3.0z}, \quad B_\rho(\chi=90°) \propto e^{-4.3z}.$$

The polar cap, i.e., the region on the surface of the star on which open lines of force arrive, is in the axisymmetric case a circle of radius

$$\rho' = \rho_0' = R(\Omega R f_0^*/c)^{1/2},$$

where $f_0^* = 1.592$ is the value of f for the separatrix that divides the regions of closed and open lines of force. If there is an inclination between the axes $\boldsymbol{\Omega}$ and $\mathbf{M}$, the polar cap is deformed, retaining elliptic symmetry [to accuracy $\sim(\Omega R/c)^{3/2}$], and its boundary is

$$\rho' = (\sigma/\pi)^{1/2}\eta(\cos^2\varphi'), \quad \int_0^{2\pi}\eta^2(\cos^2\varphi')\,d\varphi' = 2\pi. \tag{109}$$

Here, ρ' and φ' are polar coordinates with respect to the magnetic pole, and σ is the area of the polar cap. As the angle of inclination changes, the area σ gradually increases from the value

$$\sigma = \sigma_0 = \pi f_0^*\Omega R^3/c$$

at $\chi = 0°$ to $\sigma = 1.23\sigma_0$ at $\chi = 90°$. The ratio of the principal scales of the ellipse changes in the same limits—from 1 to ≈ 1.2.

We now consider the general case when in the magnetosphere there is a longitudinal current $i_\parallel \neq 0$ and potential $\Psi \neq 0$. Let $f(\rho, z, \varphi) = \text{const}$ and $\Theta(\rho, z, \varphi) = \text{const}$ determine two systems of magnetic surfaces, $f(\rho, z, \varphi)$ and $\Theta(\rho, z, \varphi)$ being certain scalar functions. The intersections of the magnetic surfaces f and Θ are the magnetic lines of force. Using (27), we determine the magnetic field $\mathbf{B}$ as

$$\mathbf{B} = [\nabla f \times \nabla \Theta]. \tag{110}$$

Here and in what follows, we use dimensionless variables and the functions (40). Taking into account (18), we can assume that the potential of electric field is constant on the surfaces $f = \text{const}$, and therefore $\Psi = \Psi(f)$. Since at the same time the right-hand side of Eq. (26) is orthogonal to the vector ∇f, one of the components of the vector equation (26) takes the form

$$\text{rot}\,\mathbf{G}\nabla f = \text{div}\,[\mathbf{G}\times\nabla f] = 0. \tag{111}$$

We obtain the second component by multiplying (26) scalarly by $\mathbf{B}$:

$$i_\parallel = B^{-2}\,(\text{rot}\,\mathbf{G}\mathbf{B}). \tag{112}$$

This relation determines the longitudinal current $i_\parallel$. Finally, multiplying Eq. (26) by $\nabla\Theta$, we obtain the third component in the form

$$\nabla\Theta\,\text{rot}\,\mathbf{G} = \frac{d\Psi}{df}\{\rho B_\varphi(\mathbf{B}\,\text{rot}\,\mathbf{G}) - B^2[2B_z + \Delta\Psi - \rho^2\mathbf{e}_\varphi\nabla(\mathbf{e}_\varphi\nabla\Psi)]\} \times\{B^2(1-\rho^2) + \rho[\nabla\Psi\times\mathbf{B}]_\varphi\}^{-1}. \tag{113}$$

The current $i_\parallel$ is here eliminated in accordance with (112). Equations (111) and (113) with allowance for (110) form a complete system determining the functions $f(\rho, z, \varphi)$ and $\Theta(\rho, z, \varphi)$.

In the axisymmetric case, the function f depends on two variables $f = f(\rho, z)$, and

$$\Theta = \varphi + g(f)\int^\rho \frac{\partial z(f, \rho')}{\partial f}\frac{d\rho'}{\rho'}, \quad \Theta(\varphi + 2\pi) = \Theta(\varphi) + 2\pi.$$

For arbitrary angle of inclinatin χ, the coordinate Θ preserves its cyclic nature and natural normalization

$$\int_{f=\text{const}} d\Theta = 2\pi.$$

Bearing this in mind, we obtain from the expression (110)

$$f = \frac{1}{2\pi}\int \mathbf{B}\,d\mathbf{S}.$$

Here, the integration is over the surface of the section lying within the region $f(\rho, z, \varphi) = \text{const}$. Since the magnetic field in the polar region of the star can be assumed to be constant, $\mathbf{B} = \mathbf{B}_0$, the value of f is proportional to the area in the polar region bounded by the curve $f = \text{const}$. Therefore, in the region of the open lines of force f varies in the range

$$0 \leqslant f \leqslant f^*(\chi), \quad f^*(\chi) = 1.592\sigma(\chi)/\sigma_0,$$

where $\sigma(\chi)$ is the area of the polar cap, $\sigma_0 = \sigma(\chi = 0°)$. The value of $f^*(\chi)$ varies from 1.592 at $\chi = 0°$ to 1.95 at $\chi = 90°$.

In the axisymmetric case, the total current in the region bounded by the magnetic surface $f = \text{const}$ is constant and equal to $g(f)$ in the dimensionless variables (40). A similar current conservation relation holds for arbitrary angle of inclination χ of the axes, which can be directly seen from the expression (24) (the current cannot intersect the surface $f = \text{const}$). It takes the form

$$\frac{dg_\chi(f)}{df} = \frac{1}{2\pi} \int\limits_{f=\text{const}} \left[i_\parallel - \frac{c\rho_e}{B^2} \frac{d\Psi}{df} \left(\mathbf{B} \frac{d\mathbf{r}}{d\Theta} \right) \right] d\Theta. \quad (114)$$

Near the surface of the pulsar, the second term in (91) and (114) is small, and here the current $i_\parallel$ along a magnetic line is force is conserved. In the axisymmetric case, it does not depend on the angle φ and is equal to dg/df (42). For arbitrary angle of inclination χ of the axes, the longitudinal current near the star may depend on the coordinate Θ. In the boundary condition (29), it is therefore convenient to separate the symmetric and antisymmetric parts of the current:

$$\mathbf{j}_\parallel |_{S_0} = [i_{\parallel s}(f) + i_{\parallel a}(f, \Theta)]_{S_0} \mathbf{B}_0(f, \Theta). \quad (115)$$

By virtue of the relation (114),

$$i_{\parallel_s}(f) = dg_\chi(f)/df, \quad \int\limits_0 i_{\parallel_a}(f, \Theta)\, d\Theta = 0. \quad (116)$$

As we shall see in what follows, the asymmetric part of the current is small compared with the symmetric.

The system of equations (111), (113) for $i_\parallel \neq 0, \Psi \neq 0$ is valid only in the region of open lines of force. On the closed ones, we always have $i_\parallel = 0, \Psi = 0$ in accordance with (31). Therefore, assuming, as in the axisymmetric case, that the closed region of the magnetosphere is unchanged, we must match the solution of Eqs. (111) and (113) on the separatrix $f = f^*(\chi)$ to the solution in the closed region obtained at the beginning of this section for arbitrary angle of inclination χ of the axes. It is important that at almost all angles of inclination the region near the separatrix f^* is strongly distinguished, since it is only here that the reverse current flows, forming, as in the axisymmetric case, a narrow and strong current jet. Indeed, it follows from the expression (13) that the charge density ρ_e near the magnetic pole has constant sign (provided $\mathbf{\Omega}\cdot\mathbf{B} \neq 0$, i.e., $\chi \neq 90°$). It follows from this that in the complete region of open lines of force the potential Ψ has constant sign, and the longitudinal current constant direction. Therefore, the reverse current flows only at the separatrix, where Ψ vanishes (see Figs. 3 and 4).

To match the current region to the quiescent region of the magnetosphere, it is most important to match the magnetic surfaces near the singular points lying on the "zero" line, since there even small changes in f, which characterizes the magnetic surface, lead to appreciable deviations in the position of the surface itself. Indeed, it follows from the solution found in the first half of this section that a line of force near the singular point $z = z_0(\varphi)$ is described by the equation [cf. (60)]

$$f - f^* = \Delta_\perp{}^0 f(\varphi) \left[(z - z_0)^2 - \frac{1}{2}(1-\rho)^2 \right],$$

where

$$\Delta_\perp{}^0 f = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{\partial^2 f}{\partial z^2}$$

at the point $z = z_0(\varphi), \rho = 1$, and f^* is the value of f on the singular line of force. Hence, it can be seen that

$$\partial z/\partial f = (\Delta_\perp{}^0 f)^{-1} [2(1-\rho)^2 + 4(\Delta_\perp{}^0 f)^{-1}(f - f^*)]^{-1/2} \quad (117)$$

tends to infinity at the singular point $f = f^*, \rho \to 1$. Here, the situation is entirely analogous to the axisymmetric case, where almost the entire deviation of the magnetic surface occurs in the neighborhood of the singular point, since it is there that $(\partial f/\partial z)_{f^*} \to 0$, i.e., $\partial z/\partial f \to \infty$.

To make the matching, we first solve Eq. (111) in the neighborhood of the singular point $z = z_0(\varphi), \rho = 1$. For this, it is convenient to go over in (111) from the variables ρ, z, φ to f, ρ, φ and regard $\Theta(f, \rho, \varphi)$ as an unknown function. Then Eq. (111) takes the form

$$\frac{\partial}{\partial\varphi}\left[\frac{1-\rho^2}{\rho}\frac{(\partial z/\partial\rho)(\partial z/\partial\varphi)}{\partial z/\partial f}\frac{\partial\Theta}{\partial\rho} - \frac{1-\rho^2}{\rho}\frac{1+(\partial z/\partial\rho)^2}{\partial z/\partial f}\frac{\partial\Theta}{\partial\varphi} + \frac{d\Psi}{df}\rho\frac{1+(\partial z/\partial\rho)^2}{\partial z/\partial f}\right] = \frac{\partial}{\partial\rho}\left[\frac{\rho^{-1}(1-\rho^2)(\partial z/\partial\varphi)^2+\rho}{\partial z/\partial f}\frac{\partial\Theta}{\partial\rho} - \frac{1-\rho^2}{\rho}\frac{(\partial z/\partial\rho)(\partial z/\partial\varphi)}{\partial z/\partial f}\frac{\partial\Theta}{\partial\varphi} + \frac{d\Psi}{df}\rho\frac{(\partial z/\partial\rho)(\partial z/\partial\varphi)}{\partial z/\partial f}\right]. \quad (118)$$

Bearing in mind that $\partial z/\partial f \propto (1-\rho)^{-1}$ in the neighborhood of the singular point, we represent the solution of Eq. (118) as an expansion in powers of $\varepsilon = 1 - \rho$:

$$\frac{\partial\Theta}{\partial\rho} = \frac{\partial z}{\partial f}\left[a^{(0)} + a^{(1)}\varepsilon + a^{(2)}\varepsilon^2 \int \frac{\partial z}{\partial f} d\rho + a^{(3)}\varepsilon^2 + \ldots\right]. \quad (119)$$

Substituting the series (119) in Eq. (118) and equating the coefficients of different powers of ε, we obtain

$$\frac{\partial\Theta}{\partial\rho} = \frac{\partial z}{\partial f}\left\{g_1 + \varepsilon\left[1 - 2\left(\frac{\partial z}{\partial\varphi}\right)^2\right] g_1 - \frac{d\Psi}{df}\frac{(\partial z/\partial\varphi)(\partial z/\partial\rho)}{\partial z/\partial f} + \ldots\right\},$$
$$\frac{\partial\Theta}{\partial\varphi} = \frac{\partial}{\partial\varphi}\left[g_1 \int \frac{\partial z}{\partial f} d\rho + g_2 + \ldots\right], \quad (120)$$

where $g_1(f, \varphi)$ and $g_2(f, \varphi)$ are arbitrary functions; the analog of g_1 of the axisymmetric case is $g(f)$, and of g_2 is the angle φ. The magnetic field on the light surface near the singular point is

$$B_\varphi = g_1(f, \varphi), \quad B_z = g_1(f, \varphi)\frac{\partial z_0}{\partial\varphi}, \quad B_\rho = 0.$$

It can be seen from this that the field here is parallel to the zero line. From Eq. (117) and the condition of absence of divergences on the light surface, i.e., $\partial\Theta/\partial\varphi \neq \infty$ for $f=f^*$, $\rho=1$, it follows that

$$g_1(f,\varphi)=g_0(f)\Delta_\perp^0 f(\varphi). \qquad (121)$$

Substituting the solutions (120) and (121) in Eq. (113) and retaining there only the terms of first order in ε, we obtain the following connection between $g_0(f)$ and $d\Psi/df$:

$$\frac{\partial}{\partial f}(g_0\Delta_\perp^0 f)=2\frac{d\Psi}{df}g_0^{-1}+\frac{1}{2}\frac{d\Psi}{df}\left[4\left(\frac{dz_0}{d\varphi}\right)^3-\sqrt{2}\frac{d\ln\Delta_\perp^0 f}{d\varphi}\right]$$
$$+\left(\frac{d\Psi}{df}\right)^2 g_0^{-1}\left[1+\left(\frac{dz_0}{d\varphi}\right)^2\right]-\frac{3}{\sqrt{2}}g_0\frac{d^2z_0}{d\varphi^2}+2\frac{dz_0}{d\varphi}. \qquad (122)$$

We can express $g_0(f)$ in terms of the total electric current flowing in the ρ direction near the singular zero line of the magnetic field between two closely spaced magnetic surfaces f and f^*:

$$I(f)=\int j_\rho\,ds=\int_0^{2\pi}\int_{f^*}^{f}(\mathrm{rot}\,\mathrm{G})_\rho\frac{\partial z}{\partial f}d\varphi\,df. \qquad (123)$$

Substituting $(\mathrm{curl}\,\mathbf{G})_\rho$ with allowance for the expansion (120) in (123), we obtain

$$g_0(f)-g_0(f^*)=I(f)\bigg/\int_0^{2\pi}\Delta_\perp^0 f\,d\varphi. \qquad (124)$$

Since the electric current flowing out of the pulsar does not intersect the magnetic surfaces $f=$ const, the current $I(f)$ (123) is related to the total current $g_\chi(f)$ flowing within the complete magnetic tube f (114) by

$$I(f)=g_\chi(f^*)-g_\chi(f).$$

Hence and using (124), we obtain

$$g_0(f)=g_\chi(f)\bigg/\int_0^{2\pi}|\Delta_\perp^0 f|\,d\varphi.$$

Integrating now (122) over φ, we arrive at the final relation

$$\left(\frac{d\Psi}{df}\right)_{f^*}=-b(\chi)\left\{1-\left[1-a(\chi)g_\chi\frac{dg_\chi}{df}\bigg|_{f=f^*}\right]^{1/2}\right\},$$
$$b(\chi)=\left[1+\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{dz_0}{d\varphi}\right)^2d\varphi\right]^{-1},$$
$$a(\chi)=2\pi b^{-1}(\chi)\bigg/\int_0^{2\pi}|\Delta_\perp^0 f|\,d\varphi. \qquad (125)$$

This is the "consistency relation" connecting the total current g_χ and the electric field $d\Psi/df$ in the magnetosphere of the pulsar at the boundary f^* in the inclined case. It is entirely analogous to the relation (63) in the axisymmetric case. Only the coefficients $a(\chi)$ and $b(\chi)$, which are shown in Fig. 8, are changed.

It follows from (125) that for any angle of inclination χ, as in the axisymmetric case, a limiting current exists. In the

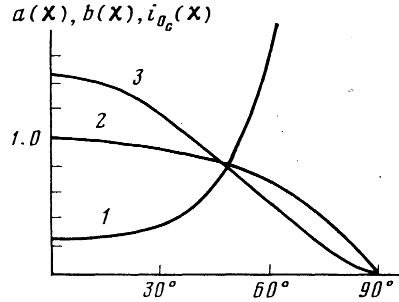

FIG. 8. Graph of the dependence of the coefficients: 1) $a(\chi)$, 2) $b(\chi)$ (125), 3) $i_{0_c}(\chi)$ (126) on the angle of inclination χ.

case of the linear dependence $g_\chi(f)=i_0 f$ [cf. (49)], the total current $g_\chi(f^*)$ is bounded:

$$g_\chi(f^*)\leqslant g_c=(f^*/a)^{1/2},\qquad i_0\leqslant i_{0c}=(af^*)^{-1/2}. \qquad (126)$$

The dependence of the limiting current i_{0_c} (126) on the angle χ is also shown in Fig. 8. It can be seen that the limiting current decreases with increasing χ. The maximal value $(d\Psi/df)_{f^*}$ of the electric field is equal to $b(\chi)$. The rotation velocity of the magnetosphere decreases in magnitude and is deflected in direction from the axis of the magnetic dipole with increasing current g_χ. However, even when the limiting current is attained the magnetosphere is completely stopped only in the axisymmetric case $\chi=0°$.

Our matching of the solutions for the closed and open magnetosphere presupposed that the presence of the longitudinal current (115) in the region of open lines of force does not perturb the fields in the region of closed lines of force. For this, the magnetic field flux produced by the longitudinal currents flowing in the magnetosphere must vanish on the singular magnetic surface $f=f^*$. In the axisymmetric case $\chi=0°$ this is obviously so, since the jet of the symmetric reverse current flowing along the separatrix f^* (62) completely screens the magnetic perturbations. In the case of an inclination $\chi\neq 0°$ of the axes, the picture is not axisymmetric, and a special analysis of the conditions of screening of the field is required. However, it is clear that if the magnetic field produced by the current flowing along the open lines of force is not to penetrate beyond the surface bounding the polar cap, its component normal to the surface must vanish. The solution of the equations for the magnetic field $\mathbf{B}_c$ of the current in the polar region,

$$\mathrm{rot}\,\mathbf{B}_c=4\pi c^{-1}i_\parallel\mathbf{B}_0,\qquad \mathrm{div}\,\mathbf{B}_c=0, \qquad (127)$$

leads under this condition to a relation between the asymmetric and symmetric currents (116):

$$i_{\parallel a}=i_{\parallel s}[L(\varphi')/\bar{L}-1]. \qquad (128)$$

Here, the function $L(\varphi')$ is determined by the shape of the polar cap (109):

$$L(\varphi')=\left[1+\left(\frac{1}{\eta}\frac{d\eta}{d\varphi'}\right)^2-\frac{1}{2}\frac{d}{d\varphi'}\left(\frac{1}{\eta}\frac{d\eta}{d\varphi'}\right)\right]\bigg/\eta^2,$$
$$\bar{L}=\frac{1}{2\pi}\int_0^{2\pi}L(\varphi')d\varphi'=\frac{1}{2\pi}\int_0^{2\pi}\frac{d\varphi'}{\eta^2(\varphi')}. \qquad (129)$$

For simplicity we have assumed here, as in (49) and (126), that the magnitude of the symmetric longitudinal current $i_{\|s}$ is constant and equal to i_0 in the complete polar region, and also that the shape of the curve $f=$ const on the surface of the star is similar to the boundary of the polar cap (109). In the axisymmetric case $\eta\equiv 1$ and, therefore, $i_{\|a}\equiv 0$. Since, as we have seen, the ellipticity of the cap is fairly weak in the general case too, the asymmetric longitudinal current is always small compared with the symmetric.

It should be emphasized that the analysis made in this section is based essentially on the circumstance that the longitudinal current at the surface of the star has one direction in the entire polar cap, and the reverse current penetrates only in a narrow region near the boundary of the cap. This is true for all angles of inclination χ of the axis except those near 90° as long as

$$\tan\chi<(\Omega R/c)^{-1/2}. \tag{130}$$

If the condition (130) is not satisfied, $90°-\chi<(\Omega R/c)^{1/2}$, the reverse current occupies an appreciable part of the polar cap, exactly half of it when $\chi=90°$.

Thus, our construction in the neighborhood of the separatrix $f=f^*$ proves completely the existence of a solution of Eqs. (26) and (27) in the entire region of open lines of force that satisfies the above requirements, so that the pulsar energy loss can be determined. The general structure of the magnetosphere and the picture of the currents flowing in it for arbitrary angle of inclination of the axes can be readily represented on the basis of the solutions obtained earlier (Figs. 1–7).

§5. PULSAR ENERGY LOSSES

We now turn to the determinatin of the loss of rotational energy of the star. The deceleration of the star is due to the currents $\mathbf{J}_s$ flowing on its surface. The moment of the forces acting on the star is

$$\mathbf{K}=\frac{1}{c}\int[\mathbf{r}\times[\mathbf{J}_s\times\mathbf{B}_0(s)]]ds. \tag{131}$$

Here, $\mathbf{B}_0(s)$ is the magnetic field on the surface of the star. The projection of $\mathbf{K}$ onto the rotation axis Ω determines the loss of kinetic energy of the star:

$$d\mathscr{E}_{\rm kin}/dt=\mathbf{K}\mathbf{\Omega}=J_r\Omega\dot\Omega.$$

Here, $\mathbf{J}_r$ is the moment of inertia of the star. The other components of the vector $\mathbf{K}$ determine how much the rotation axis turns relative to the direction of $\mathbf{M}$, i.e., it determines the change in the angle of inclination χ.

To find the surface current, it is convenient to separate its irrotational and solenoidal parts[15]:

$$\mathbf{J}_s=\mathbf{J}_s^p+\mathbf{J}_s^r,$$

$$\mathbf{J}_s^p=\operatorname{grad}\zeta,\quad \mathbf{J}_s^r=\frac{\Sigma_\wedge}{\Sigma_\perp}\left[\frac{\mathbf{B}_0}{B_0}\times\operatorname{grad}\zeta\right],$$

where ζ is the potential of the surface current, which is determined by the longitudinal currents flowing from the surface of the star:

$$\Delta\zeta=-i_\|B_0.$$

The solenoidal part $\mathbf{J}_s^r$ of the current depends on the ratio of the Hall component Σ_Λ and the Pedersen component $\Sigma_\perp$ of the conductivity tensor $\hat\Sigma$ of the star. The irrotational current $\mathbf{J}_s^p$ does not depend on the conductivity. We find it, using the expressions (128) and (129) for the magnitude of the longitudinal current $i_\|$:

$$\mathbf{J}_s^p=-\frac{B_0 i_{\|s}}{\bar L\eta^2(\varphi')}\left\{\left[1+\left(\frac{1}{\eta}\frac{d\eta}{d\varphi'}\right)^2\right]\frac{d\mathbf{r}}{df}-\frac{1}{2}\frac{1}{\eta}\frac{d\eta}{d\varphi'}\frac{d\mathbf{r}}{d\varphi'}\right\}. \tag{132}$$

Here, we have used the fact that because of the similarity the form of the curves $f=$ const on the surface of the star is determined by the relation

$$\rho'=R\left(\frac{\Omega R}{c}f\right)^{1/2}\eta(\varphi'),\qquad \int_0^{2\pi}\eta^2(\varphi')\,d\varphi'=2\pi,$$

where ρ' is the distance from the axis of the magnetic dipole, and $\eta(\varphi')$ is the function that describes the boundary (109) of the polar cap. The solenoidal current $\mathbf{J}_s^r$ does not directly contribute to the moment $\mathbf{K}$, and determines only the perturbation of the magnetic moment of the star:

$$\delta M=\frac{\Sigma_\Lambda}{\Sigma_\perp}i_0\left(\frac{\Omega R}{c}\right)^3 M.$$

Under the condition $\Sigma_\Lambda/\Sigma_\perp\ll(\Omega R/c)^{-3}$, the perturbation of the magnetic moment of the star can be ignored.

Substituting (132) in (131) and integrating, we find the moment of the forces <

$$\mathbf{K}=-\frac{f^{*2}}{8\bar L}\frac{B_0^2R^6\Omega^3}{c^3}i_0\frac{\mathbf{M}}{M}. \tag{133}$$

Here, i_0 is the dimensionless longitudinal current: $i_{\|s}=\Omega i_0/4\pi$. We emphasize that the terms of higher order [by $(c/\Omega R)^{1/2}$ times greater] that arise in the derivation of the expression (133) and are related to the asymmetric current and asymmetry of the polar cap are completely compensated under the conditions considered here.

It follows from (133) that apart from the slowing down, i.e., the decrease in the angular frequency by

$$\dot\Omega=-\frac{f^{*2}}{8\bar L}\frac{B_0^2R^6\Omega^3}{c^3J_r}i_0\cos\chi, \tag{134}$$

there will also be an increase in the angle of inclination of the axes, similar to that noted by Heintzmann[16]:

$$\sin\chi=\sin\chi_0(\Omega_0/\Omega). \tag{135}$$

Here, χ_0 and Ω_0 are the initial values of the angle of inclination and the angular velocity. We recall that in our treatment the current i_0 is bounded, $i_0\leqslant i_{0_c}$, the value of the critical current i_{0_c} depending strongly on the angle χ (Fig. 8). The quantity $(\bar{L})^{-1}$ is actually near unity, since for an ellipse it is equal to $(\bar{L})^{-1} = 2l/(l^2+1)$, where l is the ratio of its axes, and for $l = 1.2$ we have $(\bar{L})^{-1} = 0.98$.

It follows from the expressions (134) and (135) that the deceleration tends to zero when the angle χ approaches 90° and Ω approaches $\Omega_0 \sin \chi_0$. Note that at angles near 90° the condition of applicability of the obtained expressions (130) is violated. The characteristic time of variation of the angle of inclination χ of the axes and the angular velocity Ω of the star is in order of magnitude $11 P_0^2 B_{0_{12}}^{-2}$ million years.

The energy removed from the star consists of two parts—the energy carried by the electromagnetic field and the energy used to generate the plasma. The particles that generate the electron-positron plasma acquire energy at the jump of the potential Ψ in the double layer ["vacuum gap" (91)] at the surface of the pulsar. The total energy loss here is

$$W_r = \frac{B_0 R^3 \Omega^2}{2c\bar{L}} \int_0^{f^*} \int_0^{2\pi} i_{\parallel s} \Psi(f) \left[1 + \left(\frac{1}{\eta}\frac{d\eta}{d\varphi'}\right)^2\right] df\, d\varphi' = \frac{f^{*2}}{8\bar{L}} \frac{B_0^2 \Omega^4 R^6}{c^3} i_0 A,$$

$$A = \frac{1}{\pi f^{*2}} \int_0^{f^*} \int_0^{2\pi} \Psi(f) \left[1 + \left(\frac{1}{\eta}\frac{d\eta}{d\varphi'}\right)^2\right] df\, d\varphi'.$$

In the case of a linear dependence of Ψ on f (49), $A \approx \beta$. The energy transported by the electromagnetic field is

$$W_{em} = \frac{c}{4\pi} \int [\mathbf{EB}]\, ds = \frac{f^{*2}}{8\bar{L}} \frac{B_0^2 \Omega^4 R^6}{c^3} i_0 (\cos\chi - A). \quad (136)$$

Of course, Eq. (136) is valid only when $W_{em} > 0$, i.e., $\cos\chi > A$. Note that in accordance with (125) the maximal value $A = A_{max} \approx b(\chi)$ is approximately proportional to $\cos\chi$. We see that the particles in the magnetosphere carry away from the pulsar only part of the energy, and for $A \ll 1$ their fraction is small. However, in the boundary layer near the light surface the electrons and positrons are accelerated and acquire the energy $\mathscr{E}_m$ (86). Then the power expended on their acceleration is

$$W_r' = \int n c \mathscr{E}_m\, ds.$$

Here, the integration is over the light surface. Bearing in mind that in the expression (81) for n_c at nonvanishing angle of inclination χ we must add $\cos\chi$, we obtain

$$W_r' \approx W_{em}.$$

Thus, the energy lost by the pulsar is transmitted to the electrons and positrons and then, in the radiation region, partly goes over to MHD waves.

We now compare the energy loss (134) with the loss of a magnetic dipole rotating in vacuum,[17]

$$W_d = \frac{B_0^2 \Omega^4 R^6}{6c^3} \sin^2\chi,$$

which is generally used to estimate the slowing down of pulsars.[3] We see first that there is a different dependence of W on the angle of inclination χ of the axes, namely, the dipole losses are maximal for orthogonal axes and completely disappear as $\chi \to 0$. In contrast, the losses (134) (we shall call them the "current losses") are maximal in the axisymmetric case and decrease with increasing angle χ. Such a decrease of the losses is entirely natural. The point is that the origin of the losses is the need for continuous plasma generation on the open lines of force, and the amount of plasma, which is proportional to ρ_c, decreases with increasing angle χ [see (13)].

The most important difference between the current and dipole losses is that they are proportional to the longitudinal current flowing from the surface of the pulsar, which is expressed in (134) by the dimensionless factor

$$i_0 = 2\cos\chi\,(j/j_c);$$

here, j is the current density and j_c is the characteristic current determined in accordance with (14): $j_c \propto \cos\chi$. The factor i_0 cannot exceed the critical value i_{0_c} (126). In the limit $i_0 \to i_{0_c}$, the current losses are of the same order as the dipole losses; for $i_0 \ll 1$, the current losses are much smaller. In particular, in the complete absence of a current ($i_0 = 0$) a star surrounded by a plasma magnetosphere does not lose energy at all, irrespective of the angle of inclination χ of the axes. The physical significance of this can be readily understood by considering the energy flux for the complete corotation solution obtained in Sec. 4. In this case, the Poynting vector in the magnetosphere and on the light cylinder has, as can be seen from (10), only a φ component; therefore, the flux of energy leaving the star is zero. In the complete absence of a longitudinal current, the magnetosphere plasma is polarized in such a way as to completely suppress radiation.

Note that the possibility of an appreciable suppression of the dipole radiation due to a change in the value of the permittivity by the presence of plasma was pointed out by Ginzburg.[18] However, Gunn and Ostriker[19] showed that if the intensity of the radiation is very high, as can be in the case of pulsars, then by virtue of the nonlinearity a not too dense plasma is not capable of significantly changing it. In our case, the dipole radiation is suppressed for a different reason—the change in the conditions in the near zone.

It is important that the total current i_0 is related by virtue of the consistency condition (125) to the potential Ψ of the electric field. In turn, the potential is determined by the conditions of production of the electron-positron pairs. Thus, it is ultimately the generation of plasma on the open lines of force that, through the consistency relation, determines the current i_0 and, therefor, the slowing down of the pulsar. A detailed discussion of the slowing down of pulsars for specific mechanisms of generation of the electron-positron plasma and comparison of the results of theory with observational data are the subject of a separate paper.

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[1)]Indeed, taking in accordance with Ref. 3 $B_0 \approx 10^{12}$ G, $R \approx 10^6$ cm, $P = 0.66$ sec, and $\gamma = \bar{\gamma} \approx 10^2$, we have $I_c = 3 \times 10^{21}$ cm^{-2}·sec^{-1},

$I_m = 2\times 10^{27}$ cm^{-2}·sec^{-1}, $I_m/I_c = 6\times 10^5$. For the total number of particles ejected by the pulsar when $I = I_m$ we then obtain $(dN/dt)_m = 2\times 10^{36}$ sec^{-1}. This value is very large; according to the usual estimates, the outflow from pulsars is smaller: $dN/dt \ll (dN/dt)_m$ (see Ref. 7). It is also important that, as is clear from (8), $(dN/dt)_m$ increases strongly with decreasing rotation period as $\sim P^{-4}$. Therefore, although the number of ejected particles is usually greater for rapidly rotating pulsars than for slow ones, the value of $(dN/dt)_m$ for them is higher, so that the condition (8) remains valid. Thus, for the pulsar in the Crab Nebula, which is the most active and has period $P = 0.033$ sec^{-1}, we obtain $(dN/dt)_m = 3\times 10^{41}$ sec^{-1}, whereas according to Ref. 7 for this pulsar $(dN/dt) \approx 10^{40}$ sec^{-1}. The most rapid of the known pulsars has period $P = 1.5$ msec.[8] The flow of plasma from this pulsar, as can be deduced from its deceleration $\dot{P} = 1.3\times 10^{-19}$ (Ref. 9) is also much less than $(dN/dt)_m$.

2)By corotation one means motion of the plasma as a rigid body fixed firmly to the rotating pulsar, i.e., rotation of the complete plasma with angular frequency Ω.

3)Note that since the electrons and positrons move in the same direction with velocity near c, to ensure quasineutrality we require the additional condition $\lambda \gg \gamma$.

4)Note that the assumption of a dipole nature of the undisturbed magnetic field of the pulsar in the problem of the magnetosphere structure is entirely natural. Indeed, a strong perturbation of the field by the magnetosphere currents appears at distances $\rho \sim c/\Omega$, which are always much greater than the radius R of the star; usually, $c/\Omega R \sim 10^3$. The multipole fields of the currents flowing on the surface of the star or within it fall off rapidly at large distances $r \gg R$. Therefore, for $R \ll r \ll c/\Omega$ the dipole field is always predominant. This is confirmed by data of observations of the radiation of pulsars.[2,3]

5)It is easy to prove this relation by introducing a coordinate x_1 that lies on the magnetic surface and is orthogonal to $\mathbf{e}_\varphi$. By virtue of the axial symmetry $g = g(x^1, f)$; then it follows from (37) that $\partial g/\partial x^1 = 0$, i.e., $g = g(f)$.

6)Equation (41) is used here in the determinatin of Δf to simplify the calculations. Complete analysis on the basis of Eq. (50) alone with allowance for the divergence of the integrals (67) leads to the same results.

7)The attempt made in Ref. 12 to find a solution to the problem in the presence of longitudinal currents but in the absence of a field Ψ led the authors of Ref. 12 to a contradiction. This must be, since the condition (64) is not satisfied in this case.

8)In the case of rigorous axial symmetry, radiation is absent by virtue of the vanishing of the oscillating component of the electric current. Therefore, instability of the axisymmetric structure in the radiation region is important. In the case of rotation of an inclined dipole, there is no axial symmetry and radiation is generated freely.

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