# The hydrodynamics of superfluid turbulence

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(Submitted 30 December 1982)
Zh. Eksp. Teor. Fiz. **84**, 1729–1742 (May 1983)

The equations of motion of superfluid helium, taking account of a chaotic cluster of vortex filaments (superfluid turbulence), are obtained by means of a method close to the Bekarevich-Khalatnikov theory. These equations are used for an investigation of the propagation of linear and nonlinear second sound in a supercritical helium countercurrent. A relation is obtained between the transit time of a nonlinear wave and the parameters of superfluid turbulence. This relation can be considered as a new method of probing the vortex cluster.

PACS numbers: 67.40.Vs, 67.40.Pm

Following the terminology introduced in Refs. 1 and 2, we designate as superfluid turbulence an assembly of chaotically oriented vortex filaments, i.e., a state which develops in superfluid He II when the velocities are significantly in excess of the critical values. A first attempt at a quantitative description of superfluid turbulence was made by Vinen.[1] Vinen's theory, which is based on Feynman's qualitative picture,[2] has a phenomenological character. In order to construct his theory, Vinen resorted to experiments, among them the experiment on the propagation of second sound across a supercritical flow of He II. In analyzing the experiments Vinen has assumed that the total interaction of second sound with the superfluid turbulence reduces to the damping caused by the friction of the normal component against a "frozen" system of immobile vortex filaments. Measuring the damping one can thus investigate the evolution of a vortex cluster, its equilibrium value, and other characteristics. Such an approach, i.e., considering a vortex cluster as a system of "frozen" filaments which only cause additional friction, has been in wide use recently. The authors of many papers describing supercritical helium flows limit themselves to adding a friction force to the right-hand sides of the Landau-Khalatnikov equations, which is incorrect, particularly in nonstationary situations.

It seems reasonable to assume that as soon as the cluster of vortex filaments exhibits some dynamics, as follows from Vinen's theory (*vide infra*), this should be taken into account *ab initio*, in the derivation of the equations. In other words, the hydrodynamic equations of supercritical helium must contain the equation of motion of a vortex cluster and also describe the supplementary contribution of vortices to the momentum flux tensor, to the dissipative function, etc. The Bekarevich-Khalatnikov theory[3] is a prototype of such an approach.

In the first section of the paper we investigate the Vinen-Schwarz equation, which is the basis of closing the system of fluid dynamics equations for He II. In Sec. 2 the equation of motion is derived by means of the Bekarevich-Khalatnikov method. In Secs. 3 and 4 these equations are used for an investigation of the propagation of second sound (both in the linear and nonlinear case) in a supercritical countercurrent.

## 1. THE VINEN-SCHWARZ MODEL

As already mentioned, Vinen was the first to attempt a quantitative theory of superfluid turbulence. In doing this he was guided by the following qualitative picture, developed by Feynman. Assume that the vortex cluster forms a homogeneous state characterized by the total length $L(t)$ of the filaments per unit volume. It is clear than one may talk about homogeneity only if the average distance $\delta = L^{-1/2}$ between the filaments is much smaller than the size of the system. Due to the Magnus effect the friction against the normal component causes the elements of the filaments to reduce their curvature, and this leads to an increase of the quantity $L(t)$. For large densities intersection effects come into play. As a result of intersection the filaments get fragmented, forming rings of smaller sizes, which in turn fragment, etc., until clumps are formed of a size for which the hydrodynamic description is no longer applicable, and kinetic theory effects become appreciable. Something of the nature of a diffusion in the space of sizes of vortex rings takes place. This situation is very reminiscent of classical turbulence, where an energy flow takes place from large scales to small scales, with subsequent disipation. On the basis of these considerations Vinen has derived a rate equation for the quantity $L(t)$

$$\frac{dL}{dt} = \alpha |\mathbf{V}_{ns}| L^{3/2} - \beta L^2. \tag{1}$$

Here $\mathbf{V}_{ns} = \mathbf{v}_n - \mathbf{v}_s$ is the relative velocity and $\alpha$ and $\beta$ are the empirical paramaters of the theory. The first term describes the increase of the length on account of the Magnus effect, and the second term describes the fragmentation effect.

It follows from Eq. (1) that the characteristic relation time of a vortex cluster is of the order $\tau = \beta / \alpha^2 V_{ns}^2$. This time can be large, of the order of the damping time of hydrodynamic modes, and then it becomes necessary to take into account the dynamics of the cluster itself. It is necessary here to make an important remark. In general, the quantity $L$ represents an integral characteristic (a moment) of some function (the distribution function) which carries the detailed information about the vortex cluster.[1)] There arises the problem of the damping of the other moments of this

 0038-5646/83/051009-08$04.00  

function, or of vortex-cluster characteristics different from $L$. Strictly speaking, there are no theoretical reasons to assert that the relaxation of the other degrees of freedom of the cluster occurs faster, justifying the fact that we have limited our attention to the quantity $L$. However, numerous experiments on the evolution of the vortex structure show that this evolution agrees well with the Vinen equation (1). We therefore make the natural assumption that the relaxation of the other degrees of freedom occurs at a faster rate than the relaxation of $L$.

The statistical nature of the quantity $L$ brings up another problem. Although we restrict our attention to the function $L$, it is nevertheless important to deal with quantities which are higher moments of the distribution function. Such quantities are: the mean velocity $\mathbf{v}_L$ of the cluster as a whole and the work $R'$ done by the friction forces against the frozen filament system. Continuing to assume consistently that their relaxation times are small, we make use of empirical relations obtained under static conditions for their determination. Thus, for $\mathbf{v}_L$ we make use of the following dependence determined in Ref. 5:

$$\mathbf{v}_L = b\mathbf{V}_{ns}. \tag{2}$$

Here $b = b(T)$ is a known function of the temperature.

To calculate the work of the friction forces against the "frozen" filament system we make use of the well-known Gorter-Mellink formula[1] for the mutual friction between the normal and superfluid components:

$$F^{\mathrm{GM}} = A(T)\rho_s\rho_n V_{ns}^2 \mathbf{V}_{ns}. \tag{3}$$

The work done by this force per unit time is, accordingly

$$R^{\mathrm{GM}} = A(T)\rho_s\rho_n V_{ns}^4. \tag{4}$$

This work consists of the total work $R'$ plus the energy that the vortex cluster drains in order to increase its length and then transports along the turbulence spectrum to smaller scales. As can be seen from Eq (1), this energy equals $R'' = \varepsilon_b \mathbf{V}_{ns}|L^{3/2}$ ($\varepsilon_b$ is the energy of the unit length of the filament). Here $R^{\mathrm{GM}} = R' + R''$. Hence, assuming that $R'$ is proportional to $L$ we obtain

$$R' = \alpha_1 L V_{ns}^2, \tag{5}$$

where, recognizing that $L = (\alpha/\beta)^2 \mathbf{V}_{ns}^2$, we obtain for $\alpha_1$ from Eq. (5)

$$\alpha_1 = A(T)\rho_s\rho_n \frac{\beta^2}{\alpha^2} - \varepsilon_b \frac{\alpha^2}{\beta}. \tag{5'}$$

In the sequel we shall describe the state of a vortex cluster by means of the quantity $L(\mathbf{r},t)$. In distinction from the case discussed by Vinen, we attribute to this quantity a coordinate dependence, having in mind the investigation of nonhomogeneous and nonstationary cases. Equation (1) will be the basis for closing the system of equations of motion.

## 2. THE HYDRODYNAMICS OF TURBULENT SUPERFLUIDITY

In deriving the equations of motion we make use of a method analogous to the one used in the derivation of the Bekarevich-Khalatnikov[3] equations. In distinction from these equations we cannot relate the number of vortex filaments to the curl of the average velocity, since the vortices are randomly oriented. Therefore we adopt Eq. (1) in lieu of the missing equation, rewriting it in the form

$$\frac{\partial L}{\partial t} + \operatorname{div} L\mathbf{v}_L = \alpha|\mathbf{V}_{ns}|L^{3/2} - \beta L^2. \tag{6}$$

In addition, we complement the equations of two-velocity fluid dynamics with as-yet unknown terms which may appear on account of the vortices:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\mathbf{j} = 0, \tag{7}$$

$$\frac{\partial S}{\partial t} + \operatorname{div}(S\mathbf{v}_n + \mathbf{\Sigma}) = \frac{R}{T}, \tag{8}$$

$$\frac{\partial j_i}{\partial t} + \frac{\partial \Pi_{ik}}{\partial r_k} = -\frac{\partial \pi_{ik}}{\partial r_k}, \tag{9}$$

$$\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s\nabla)\mathbf{v}_s + \nabla\mu = \mathbf{f}, \tag{10}$$

$$\frac{\partial E}{\partial t} + \operatorname{div}(\mathbf{Q} + \mathbf{q}) = 0. \tag{11}$$

Here $R$, $\pi_{ik}$, $\mathbf{f}$, $\mathbf{q}$, and $\mathbf{\Sigma}$ denote the dissipative function, the momentum flux (stress) tensor the force acting on the superfluid component, and the respective energy and entropy fluxes attributable to the presence of the vortex cluster. The remaining notation and definitions agree with Ref. 3. We also write out the expressions for $\Pi_{ik}$ $\mathbf{Q}$, and $\mathbf{\Sigma}$

$$\Pi_{ik} = \rho v_{si}v_{sk} + v_{si}j_{0k} + v_{nk}j_{0i} + p\delta_{ik}, \tag{12}$$

$$\mathbf{Q} = (\mu + v_s^2/2)\mathbf{j} + ST\mathbf{v}_n + \mathbf{v}_n(\mathbf{v}_n,\ \mathbf{j}_0), \tag{13}$$

$$E = \rho v_s^2/2 + \mathbf{v}_s\mathbf{j}_0 + E_0. \tag{14}$$

The energy $E_0$ in the superfluid system is a function of $S$, $\rho$, and $\mathbf{j}_0$, as well as of the quantity $L$. The following thermodynamic identity holds:

$$dE_0 = \mu d\rho + TdS + (\mathbf{V}_{ns},\ d\mathbf{j}_0) + \varepsilon_b dL. \tag{15}$$

Here $\varepsilon_0$ is the energy per unit length of the vortex filament, defined as $\rho_s(2\pi\hbar/m)^2(1/4\pi\ln(\delta/a_0)$. The quantity $\delta$ is a characteristic intervortex distance, and $a_0$ is an assumed core radius. Owing to the logarithm, the dependence of $\varepsilon_b$ on $L$ is weak, and for realistic conditions $(1/4\pi)\ln(\delta/a_0)$ is close to unity.

The equations (6)–(11) together with (12)–(15) are 11 relations for the 10 quantities $\rho$, $L$, $S$, $E$, $\mathbf{v}_s$, an $\mathbf{j}_0$. This overdeterminacy suffices for a unique determination of the additionally introduced terms.[2)] For this we proceed in the same manner as in Ref. 3. We differentiate Eq. (14) with respect to time and in place of the derivatives $\partial\rho/\partial t$, $\partial\mathbf{v}_s/\partial t$, $\partial L/\partial t$, and $\partial\mathbf{j}_0/\partial t$ we substitute their values from Eqs. (6)–(10). After straightforward but tedious transformations we arrive at the energy balance equation

$$\frac{\partial E}{\partial t} + \operatorname{div}[\mathbf{Q} + \pi\mathbf{v}_n + L\varepsilon_b(\mathbf{v}_L - \mathbf{v}_n)] = T\left(\frac{\partial S}{\partial t} + \operatorname{div} S\mathbf{v}_n\right)$$
$$+ (\pi_{ik} - L\varepsilon_b\delta_{ik})\frac{\partial v_{ni}}{\partial r_k} + (\mathbf{j} - \rho\mathbf{v}_n,\ \mathbf{f}) + L\nabla\varepsilon_b(\mathbf{v}_L - \mathbf{v}_n)$$
$$+ \varepsilon_b\alpha|\mathbf{V}_{ns}|L^{3/2} - \varepsilon_b\beta L^2. \tag{16}$$

In order to take into account the entropy flux we add to both sides of Eq. (16) the quantity

$$\operatorname{div}[S^L T(\mathbf{v}_L-\mathbf{v}_n)]=S^L(\mathbf{v}_L-\mathbf{v}_n)\nabla T+T\nabla S^L(\mathbf{v}_L-\mathbf{v}_n).$$

Here $S^L$ is the additional entropy due to the vortices, a quantity which cannot be determined within the framework of the proposed formalism. Its determination is an independent problem. Some considerations on the magnitude of $S^L$ can be found in Appendix I.

After these transformations we obtain

$$\frac{\partial E}{\partial t}+\operatorname{div}[\mathbf{Q}+\pi\mathbf{v}_n+L\varepsilon_b(\mathbf{v}_L-\mathbf{v}_n)+S^L(\mathbf{v}_L-\mathbf{v}_n)T] =T\left[\frac{\partial S}{\partial t}+\operatorname{div}(S\mathbf{v}_n+S^L(\mathbf{v}_L-\mathbf{v}_n))\right]$$
$$+(\pi_{ik}-L\varepsilon_b\delta_{ik})\frac{\partial v_{ni}}{\partial r_k}+(\mathbf{j}-\rho\mathbf{v}_n,\mathbf{f})+L(\mathbf{v}_L-\mathbf{v}_n)\nabla\varepsilon_b + \varepsilon_b\alpha|\mathbf{V}_{ns}|L^{3/2}-\varepsilon_b\beta L^2+(\mathbf{v}_L-\mathbf{v}_n)S^L\nabla T. \quad (17)$$

Comparing (17) with the conservation equations for energy and entropy, we obtain

$$q_i=(\pi_{ik}-L\varepsilon_b\delta_{ik})v_{nk}+L\varepsilon_b\delta_{ik}v_{Lk}+S^L(v_{Li}-v_{ni})T, \quad (18)$$

$$R=-(\pi_{ik}-L\varepsilon_b\delta_{ik})\frac{\partial v_{ni}}{\partial r_k}-(\mathbf{j}-\rho\mathbf{v}_n,\mathbf{f})-L(\mathbf{v}_L-\mathbf{v}_n)\nabla\varepsilon_b + \varepsilon_b\beta L^2-\varepsilon_b\alpha|\mathbf{V}_{ns}|L^{3/2}-(\mathbf{v}_L-\mathbf{v}_n)S^L\nabla T, \quad (19)$$

$$\mathbf{\Sigma}=(\mathbf{v}_L-\mathbf{v}_n)S^L. \quad (20)$$

The meaning of the equality (20) is as follows. The total entropy flux consists of the entropy transport by the normal fluid component plus the entropy carried by the vortices. But a thermodynamic definition of the entropy does not naturally account for this separation, and contains the total entropy (renormalized to take into account the vortices). Therefore $\Sigma$ contains the term $-S^L\mathbf{v}_n$ which subtracts from the flux $S\mathbf{v}_n$ the "extra" addition. A similar situation prevails for the expression (18) of the additional energy flux $\mathbf{q}$.

In addition, supplementary quantities make their appearance in (18). The first of these $L\varepsilon_b\mathbf{v}_L$ describes directly the energy flux of the vortices. The other, as can be seen, is related to the renormalization of the pressure.

In Eq. (19) the first term for the dissipative function $R$ has the form of a viscous stress tensor $\tau_{ik}$, hence $\pi_{ik}=\tau_{ik}\varepsilon_b L\delta_{ik}$. In the sequel we shall neglect the effect of normal viscosity: $\tau_{ik}=0$. For this reason we have not considered the irreversible entropy flux before.

The relation (19) connects two so far undefined quantities $R$ and $\mathbf{f}$. Usually (see Ref. 3) $R$ is chosen so that for certain assumptions (e.g., linearity in the grandients) $R$ should be a positve definite form. Here we construct $R$ directly, starting from the model of the vortex cluster (Sec. 1).

The dissipative function $R$ consists of the work $R'$ of the friction forces against the frozen system of vortices plus the energy transported by the cluster and released in the region of small scales. According to Eq. (1) we write

$$R=\alpha_1 L V_{ns}^2+\varepsilon_b\beta L^2.$$

Comparing this with Eq. (19) and making use of the fact that from symmetry considerations the vector $\mathbf{f}$ is collinear with $\mathbf{V}_{ns}$, we obtain

$$\mathbf{f}=\frac{\alpha_1}{\rho_s}L(\mathbf{v}_n-\mathbf{v}_s)+\frac{L(b-1)}{\rho_s}\nabla\varepsilon_b+\frac{(b-1)S^L}{\rho_s}\nabla T + \frac{\alpha}{\rho_s}\varepsilon_b\frac{\mathbf{v}_n-\mathbf{v}_s}{|\mathbf{v}_n-\mathbf{v}_s|}L^{3/2} \quad (21)$$

The quantity $\mathbf{f}$ contains $\nabla T$ and $\nabla\varepsilon_b$, the so-called reactive terms. The other two terms describe the dissipation. The dissipative parts of the equations for $\mathbf{v}_x$ and $L$ will be written in matrix form

$$\begin{pmatrix}\dfrac{\partial v_{si}}{\partial t}\\ \dfrac{\partial L}{\partial t}\end{pmatrix}=L\begin{pmatrix}\dfrac{-\alpha_1}{\rho_s}\delta_{ik} & -\dfrac{\alpha}{\rho_s}\dfrac{(\mathbf{v}_s-\mathbf{v}_n)_i}{|\mathbf{V}_{ns}|}L^{1/2}\\ \dfrac{\alpha}{\rho_s}\dfrac{(\mathbf{v}_s-\mathbf{v}_n)_k}{|\mathbf{V}_{ns}|}L^{1/2} & -\dfrac{\beta}{\varepsilon_b}L\end{pmatrix} \times\begin{pmatrix}\rho_s(\mathbf{v}_s-\mathbf{v}_n)_k\\ \varepsilon_b\end{pmatrix}. \quad (22)$$

Since $\partial E/\partial v_{sk}=\rho_s(\mathbf{v}_x-\mathbf{v}_n)_k$, $\partial E/\partial L=\varepsilon_b$, the relation (22) describes the Onsager reciprocity principle for the kinetic coefficients. The antisymmetry of the coefficients follows from the different behavior of $\mathbf{v}_s$ and $L$ under time reversal.[6]

Before writing the definitive form of the equations we express the chemical potential $\mu$ entering into them as a function of pressure and temperature. For this purpose we use the expression (15) for $E_0$ and the following expression for the pressure:

$$p=-E_0+TS+\mu\rho+(\mathbf{v}_n-\mathbf{v}_s,\mathbf{j}_0).$$

Taking account of the weak dependence of $\varepsilon_b$ on $L$ we obtain

$$\mu=\mu(p,T)+(\rho_n/2\rho)(\mathbf{v}_n-\mathbf{v}_s)^2+\varepsilon_b L. \quad (23)$$

Going over to the renormalized pressure $p_r=p+L\varepsilon_b$, one can obtain from Eq. (23)

$$\mu=\mu(p_r,T)+(\rho_n/2\rho)(\mathbf{v}_n-\mathbf{v}_s)^2. \quad (24)$$

In the sequel we shall deal only with the renormalized pressure $p$ and omit the index $r$.

The final equations of motion are:

$$\frac{\partial\rho}{\partial t}+\operatorname{div}\mathbf{j}=0, \quad (25)$$

$$\frac{\partial j_i}{\partial t}+\frac{\partial\Pi_{ik}}{\partial r_k}=0, \quad (26)$$

$$\frac{\partial S}{\partial t}+\operatorname{div}[S\mathbf{v}_n+S^L(\mathbf{v}_L-\mathbf{v}_n)]=\frac{1}{T}[\alpha_1 L(\mathbf{v}_n-\mathbf{v}_s)^2+\varepsilon_b\beta L^2], \quad (27)$$

$$\frac{\partial\mathbf{v}_s}{\partial t}+(\mathbf{v}_s\nabla)\mathbf{v}_s+\nabla\mu=\frac{L(b-1)}{\rho_s}\nabla\varepsilon_b+\frac{S^L(b-1)}{\rho_s}\nabla T +\frac{\alpha_1 L}{\rho_s}(\mathbf{v}_n-\mathbf{v}_s)+\frac{\varepsilon_b\alpha}{\rho_s}\frac{\mathbf{v}_n-\mathbf{v}_s}{|\mathbf{v}_n-\mathbf{v}_s|}L^{3/2}, \quad (28)$$

$$\frac{\partial L}{\partial t}+\operatorname{div}(L\mathbf{v}_L)=\alpha|\mathbf{v}_n-\mathbf{v}_s|L^{3/2}-\beta L^2. \quad (29)$$

We have thus obtained a system of equations of motion applicable to the case when superfluid turbulence develops in the volume of the flowing helium. The equations can be used to solve problems such as the propagation of strong pulses (in a regime which is supercritical with respect to the vortex-formation parameters), the kinetics of propagation of turbulent fronts (as observed experimentally by Peshkov and Tkachenko[7]), the motion of helium near a strongly heated wall, etc.

It is interesting that the equations (25)–(29) can be obtained by means of the method of Poisson brackets. This is shown in Appendix II. In (25)–(29) the additional reactive terms are small, and can be neglected in most cases. However, the dissipative terms can be seen not to reduce simply to the addition of mutual friction forces, and have a more complicated structure.

We discuss the domain of application of these equations. We note first that the Vinen equation (1), which we used as a basis for the closure of our system of equations, describes in principle situations which are close to equilibrium. Indeed, the balance of Magnus or Hall-Vinen forces was considered by Vinen in the stationary case. The requirement that the system should be close to equilibrium has also been used in our derivation of the dissipative function, since in a strongly nonequilibrium situation the absorption by the cluster of energy from the motion of the helium and its subsequent dissipation are a process which is nonlocal in time. On account of what was said above one should expect that the equations of motion (including Vinen's equation) in a strongly nonequilibrium case would be nonlocal in time. It is hard to estimate the influence of this nonlocality, however one may expect it not to be large. The following fact may be considered a basis for this assertion. Terms in the equations which are nonlocal in time must lead to a change in the velocity of the second sound excited in a direction perpendicular to the supercritical countercurrent. Such a phenomenon was indeed observed.[8] But at the same time the magnitude of the effect was quite small: $\Delta c_2/c_2 = 10^{-4}$.

There is another difficulty here. Being an equation balancing the growth and decay of a vortex cluster, this equation does not contain any information about the development of turbulence. In order to solve equations with initial conditions which vanish (in $L$), it seems to be necessary to introduce a preordained term, as was done by Vinen.[1]

Boundary conditions require a special discussion. In addition to the usual viscous conditions (in these cases one cannot, of course, neglect the viscosity), conditions are added which are related to the pinning of vortices, as well as possible annihilation or creation of vortices at solid interfaces.

As a result the equations obtained here are valid only for sufficiently wide channels and in cases which are close to equilibrium.

## 3. PROPAGATION OF LINEAR SECOND SOUND

We use the derived equations to investigate the problem of propagation of second sound in a supercritical countercurrent ($\mathbf{j} = 0$). As mentioned before, experiments on the damping of second sound were Vinen's experimental basis.

Assume that the main flow is along the $x$ axis and is characterized by the values of $\mathbf{V}_n$, $\mathbf{V}_s$, $L$, $T$, and $p$. We assume that $\mathbf{j} = \rho_x \mathbf{V}_s + \rho_n \mathbf{V}_n = 0$ (at the same time $\nabla p = 0$). We impose the same condition on the perturbations of the velocities. For the perturbations (deviations from the averages) $\mathbf{v}_s'$, $T'$, $L'$ we obtain the following system of equations[3)]:

$$\frac{\partial \mathbf{v}_s'}{\partial t} - \sigma \nabla T' = -\frac{\alpha_1 \rho}{\rho_s \rho_n} L \mathbf{v}_s' - \frac{\varepsilon_b \alpha L^{1/2}}{\rho_s V_s}\left(\mathbf{v}_s' - \frac{\mathbf{V}_s(\mathbf{V}_s \mathbf{v}_s')}{V_s}\right) - \left(\frac{3}{2}\frac{\varepsilon_b \alpha L^{1/2}}{\rho V_s}\mathbf{V}_s + \frac{\alpha_1 \rho}{\rho_s \rho_n}\mathbf{V}_s\right) L', \quad (30)$$

$$\left(\frac{\partial \sigma}{\partial T}\right)_L \frac{\partial T'}{\partial t} - \frac{\rho_s}{\rho_n}\sigma \nabla \mathbf{v}_s' = \frac{2\alpha_1 \rho L}{T \rho_n^2}\mathbf{V}_s \mathbf{v}_s' + \frac{\alpha_1 \rho}{\rho_n^2 T} V_s^2 L' + \frac{2\varepsilon_b \beta L}{\rho T} L', \quad (31)$$

$$\frac{\partial L'}{\partial t} - \frac{\rho b}{\rho_n} L \nabla \mathbf{v}_s' - \frac{\rho b}{\rho_n}\mathbf{V}_s \nabla L' = \frac{\alpha \rho L^{3/2}}{\rho_n V_s}\mathbf{v}_s' \mathbf{V}_s - \frac{L'}{\tau}, \quad (32)$$

Here $\tau = [2\beta L - (3/2)(\alpha\rho/\rho_n) L^{1/2} V_x]^{-1}$ is the relaxation time of a vortex cluster.

As always, we look for the solution of Eqs. (30)–(32) in the form $\exp i(\omega t - k_x x - k_y y)$ with the $y$ axis perpendicular to the countercurrent. The nontriviality condition, i.e., the vanishing of the determinant of the system (30)–(32), yields an equation of the fourth degree relating $\omega$ and $\mathbf{k}$—the dispersion law. The fourth-degree determinant stems from the vector nature of Eq. (30), thus yielding $x$- and $y$-components. One can solve the dispersion relation in the following way. We make use of the smallness (compared to the usual second sound) of the additional terms in Eqs. (30)–(32). Then the solution of the dispersion equation can obviously be written in the form

$$\omega(\mathbf{k}) = c_2 |\mathbf{k}| + \xi(\mathbf{k}), \quad c_2 = \left[\frac{\rho_s}{\rho_n}\sigma^2 \left(\frac{\partial \sigma}{\partial T}\right)_L^{-1}\right]^{1/2}.$$

Here $c_2$ is the velocity of second sound for a "frozen" cluster. Linearizing the dispersion equation with respect to $\xi(k)$ and solving it, we obtain the following result:

$$\xi = i\Gamma \frac{k_y^2}{k^2} + \frac{i}{2}\frac{k_x}{k} A_1 + i\frac{k_x^2}{k^2}\frac{\alpha_1 \rho L}{\rho_s \rho_n} + \frac{i}{2}\frac{k_x^2 A_3 - k k_x A_2}{k^2(i c_2 k \tau + 1)}. \quad (33)$$

Here we have introduced the following notations:

$$\Gamma = \frac{1}{2}\left(\frac{\alpha_1 \rho L}{\rho_s \rho_n} + \frac{\varepsilon_b \alpha L^{1/2}}{\rho_s V_s}\right), \quad A_1 = \frac{\sigma}{(\partial\sigma/\partial T)_L}\frac{2\alpha\rho L V_s}{T \rho_n^2},$$

$$A_2 = \frac{\alpha \rho L^{3/2} \sigma \tau}{T \rho_n^2}(\alpha_1 \rho V_s^2 + 2\varepsilon_b \beta L),$$

$$A_3 = \frac{\alpha \rho L^{3/2} \tau}{\rho_n}\left(\frac{3}{2}\frac{\varepsilon_b \alpha L^{1/2}}{\rho_s} + \frac{\alpha_1 \rho}{\rho_s \rho_n} V_s\right). \quad (34)$$

The following relation holds between the vectors $\mathbf{v}_s'$ and $\mathbf{k}$:

$$\frac{v_{sy}'}{v_{sx}'} = \frac{k_y}{k_x}\left[1 + i\left(\frac{\alpha_1 \rho L}{\rho_s \rho_n} + 2\Gamma + \frac{i A_3 \tau}{i c_2 k \tau + 1}\right)\frac{1}{c_2 k}\right]. \quad (35)$$

The relation (33) describes the spectrum of second sound oscillations. As can be seen, this spectrum is anisotropic.

The first three terms describe absorption and the last term describes both absorption and dispersion of the sound velocity. For small $k$ the addition to the velocity of sound depends only on the direction of the vector **k**, and estimates show that it has the order of magnitude $\Delta c_2 \approx 10^{-3} c_2$. For large $k$ this addition behaves approximately like $10^{-3} V_s^4/k^2$. In addition the sound absorption depends both on the direction of **k** and on the magnitude $k$. The source of the dispersion and of the $k$-dependent absorption is the following. The equation (32) for the quantity $L'$ has typically a relaxation form, i.e., we encounter the classical Mandelshtam-Leontovich situation.[10] A cluster which has been taken out of equilibrium by some external action relaxes to its equilibrium state during a characteristic time $\tau$. As is well known, this leads precisely to dispersion and to additional damping. It follows from the result (35) that the vector $v_s'$ is not collinear with the vector **k**. This signifies that when sound is excited in a certain direction at an angle to the flow there will also appear a sound in the $x$ direction. The reason is that the sound "latches on" to the cluster, modulating the value of $L$, and thus leading to oscillations in the $x$ direction.

The case $k_x = 0$ (sound perpendicular to the flow) requires special consideration. In this case $v_s'$ also vanishes. The interaction of sound with the flow reduces then simply to a damping with decrement $\Gamma$. This is caused by the disappearance of the "latching on" of the sound to the cluster for perpendicular excitation. Indeed, the Vinen equation for $L$ contains only the magnitude of the relative velocity, which in first approximation does not change at $\mathbf{v}_s' \perp \mathbf{V}_s$. (The coupling via temperature is negligibly small in this case.) In this sense Vinen's consideration of the cluster as a system of frozen vortex filaments is justified, one must however make one important restrictive remark. If the main flow is stationary, then $L = (\alpha/\beta)^2 |\mathbf{V}_{ns}|^2$, and the decrement equals $\frac{1}{2} A(T) \rho_n (\beta/\alpha)^2 L$, i.e., is equal to the quantity used by Vinen. However, if $L$ changes with time, then $\Gamma$ is a complicated function, and the interpretation of Vinen's experiment should be based on Eq. (34) for $\Gamma$.

## 4. THE PROPAGATION OF NONLINEAR SECOND SOUND

The nonlinear acoustics of superfluids opens up great possibilities for He II diagnostics. Thus, in Ref. 11 the turbulence which appears in the wake of an intense heat pulse is probed by a nonlinear second-sound pulse. It is known that the transit time of a nonlinear wave depends on its amplitude (for the case of helium the calculations can be found in Ref. 12), and the latter, owing to damping on vortex clusters, depends on their parameters. It is therefore of practical interest to follow the evolution of a nonlinear second sound pulse in a turbulent medium and to relate its transit time $t_{tr}$ to the parameters of the vortex cluster.

As the analysis of the preceding section shows, the main result of the intraction between the countercurrent and the perpendicular sound wave is a damping with decrement. Account of the nonlinear terms introduces a small correction both into the main motion (without consideration of the vortices) and into the damping, which is already small. Therefore the terms related to the vortex cluster will be retained only in expressions which are linear in $v_s'$. As a result we are led to the following system of equations for $\mathbf{v}_s'$ (see Ref. 13):

$$\frac{\partial \sigma}{\partial T}\frac{\partial T'}{\partial t} + \frac{\rho}{\rho_n^2}\frac{\partial \rho_n}{\partial T} v_s \frac{\partial v_s}{\partial t} - \sigma \frac{\rho_s}{\rho_n}\frac{\partial v_s}{\partial y} - \frac{\rho_s v_s}{\rho_n}\frac{\partial \sigma}{\partial T}\frac{\partial T'}{\partial y} - \sigma v_s \frac{\partial}{\partial T}\left(\frac{\rho_s}{\rho_n}\right)\frac{\partial T'}{\partial y} = 0, \quad (36)$$

$$\frac{\partial v_s}{\partial t} + v_s \frac{\partial v_s}{\partial y} - \sigma \frac{\partial T'}{\partial y} - \frac{2\rho_s}{\rho_n} v_s \frac{\partial v_s}{\partial y} - \frac{\rho}{\rho_n} v_s \frac{\partial v_s}{\partial y} = 0. \quad (37)$$

If the right-hand side were absent from Eq. (37) we would have a system of two quasilinear equations. It is known that such a system has solutions of the form of so-called simple waves, i.e., flows with a unique relation $T' = T'(v_s')$. The addition of a small damping should change this relation, but in a higher order of smallness:

$$T' = T'(v_s') + \Psi(y, t). \quad (38)$$

Since $\Psi(y,t)$ is of higher order of smallness than the first order, it must satisfy the wave equation $\partial\Psi/\partial t = c_2 \partial\Psi/\partial y$. We call the solutions (38) quasisimple waves.

Substituting further (38) into Eqs. (36) and (37), and making use of the known[13] expression $T' = T'(v_s')$ and of the fact that $\Psi t = c_2 \Psi_y$, we obtain a system of nonhomogeneous algebraic equations for $\partial v_s'/\partial t$ and $\partial v_s'/\partial y$. The requirement of compatibility of these equations allows one to express $\Psi$ in terms of $v_s'$ and to obtain the following equation for $v_s'$ by itself (see Ref. 13).

$$\frac{\partial v_s}{\partial t} + [\alpha(T) v_s + c_2]\frac{\partial v_s}{\partial y} = -\Gamma v_s. \quad (39)$$

Here $\alpha(T)$ is the nonlinearity coefficient (see Refs. 3, 12, 13).

Equation (39) can be integrated, and one can pose for it, e.g., the Cauchy problem $v_s'|_{t=0} = \omega(y)$. We show how this is done. First we go over to a comoving (velocity $c_2$) coordinate system and introduce the new variable $u = v_s' e^{\Gamma t}$. In the comoving frame we obtain for $u$ the equation

$$\frac{\partial u}{\partial t} + \alpha(T) e^{-\Gamma t} u \frac{\partial u}{\partial y} = 0. \quad (40)$$

We note that for a signal of finite extent (compact support)

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} u\,dy = -\alpha(T) e^{-\Gamma t}\int_{-\infty}^{\infty} u \frac{\partial u}{\partial y}\,\partial y = 0, \quad (41)$$

i.e., the total area under the signal in the coordinates $u, y$ is conserved in time.

The Cauchy problem with the initial condition $u = \omega(y)$ is solved by means of the method of characteristics. The characteristics in the $y, L$ space satisfy the following equations with their respective solutions

$$\frac{dy}{dt} = \alpha(T) e^{-\Gamma t} \rightarrow y = -\frac{\alpha u e^{-\Gamma t}}{\Gamma} + C_1, \qquad \frac{du}{dt} = 0 \rightarrow u = C_2. \quad (42)$$

We determine the characteristic curve passing through the point $t_0, y_0, u_0$:

$$y = -\frac{\alpha(T) u e^{-\Gamma t}}{\Gamma} + y_0 + \frac{\alpha(T) u e^{-\Gamma t_0}}{\Gamma}, \quad u = u_0. \quad (43)$$

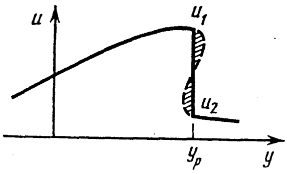


FIG. 1

Let us relate the points $t_0$, $y_0$, $u_0$ and the initial condition $t_0 = 0, u_0 = (y_0)$. Then Eq. (43) gives a parametric solution of the problem. Eliminating the parameter $y_0$ we obtain the solution in implicit form

$$u=\omega\left(y-\frac{\alpha(T)u}{\Gamma}+\frac{\alpha(T)ue^{-\Gamma t}}{\Gamma}\right). \tag{44}$$

We note that for $\Gamma \to 0$, as expected, the solution goes over into the usual Riemann solution for a simple wave.

The result (44) is however valid only for smooth solutions. It is known that the nonlinear term $\alpha(T)v_s'\partial v_x'/\partial y$ may lead to a steepening of the wave profile and ultimately to a discontinuity, when the solution (44) is no longer valid, if only because it gives an ambiguous answer (Fig. 1). In order to find the subsequent evolution of the signal one must make use of the area conservation law (41). Equation (41) implies that in a virtual "overspill" (Fig. 1) the profile has to be cut off by a shock front so that the two shaded regions should have the same areas:

$$\frac{\partial}{\partial t}\int_{u_1}^{u_2}(y(u)-y_p)\,du=0. \tag{45}$$

This yields

$$\frac{\partial}{\partial t}\int_{u_1}^{u_2}y(u)\,du=(u_2-u_1)\frac{\partial y_p}{\partial t}. \tag{46}$$

Recognizing that according to the solution (44) $\partial y(u)/\partial t = \alpha(T)ue^{-\Gamma t}$, and combining this with Eq. (46), we obtain

$$\frac{\partial y_p}{\partial t}=\alpha(T)\frac{(u_1+u_2)}{2}e^{-\Gamma t}. \tag{47}$$

We have thus derived an equation for the motion of the discontinuity. The solution is smooth to the right and the left of the discontinuity, and one can use Eq. (44)

$$y_p=\omega^{-1}(u_1)-\frac{\alpha(T)u_1}{\Gamma}(e^{-\Gamma t}-1), \tag{48}$$

$$y_p=\omega^{-1}(u_2)-\frac{\alpha(T)u_2}{\Gamma}(e^{-\Gamma t}-1). \tag{49}$$

The equations (47)–(49) are a complete set of equations for $y_p$, $u_1$, and $u_2$ (compare this derivation with Ref. 14).

As an example we calculate the evolution of a triangular profile (Fig. 2). In this case the Cauchy condition $\omega(y)$ is:

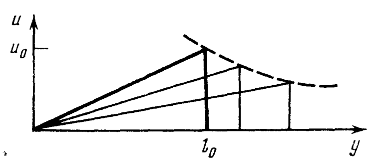


FIG. 2

$$\omega(y)=\begin{cases}0, & y<0,\\ (u_0/l_0)y, & 0<y<l_0,\\ 0, & l_0<y.\end{cases} \tag{50}$$

We obtain the following equations for the coordinate of the discontinuity $y_p$ and the velocity at the crest $u_1$:

$$\frac{\partial y_p}{\partial t}=\frac{\alpha(T)u_p}{2}e^{-\Gamma t},\quad y_p=\left(\frac{l_0}{u_0}u_1-l_0\right)-\frac{\alpha(T)u_1}{\Gamma}(e^{-\Gamma t}-1). \tag{51}$$

Solving this system we obtain, e.g., for $u_1(t)$ the following expression:

$$u_1(t)=(l_0u_0)^{1/2}[l_0u_0^{-1}-\alpha(T)\Gamma^{-1}(e^{-\Gamma t}-1)]^{-1/2}. \tag{52}$$

Making use of the result (52) one can determine the transit time of a pulse for a given distance $B_0$ according to the following equation:

$$\int_0^{t_{tr}}\left[c_2+\frac{\alpha(T)u_1(t)}{2}e^{-\Gamma t}\right]dt=B. \tag{53}$$

Equation (53) relates the transit time of a nonlinear signal with quantities characteristic for superfluid turbulence, and thus solves the problem we have posed. One can similarly calculate $t_{tr}$ for more complicated cases.

In conclusion the authors thank A. F. Andreev, S. V. Iordanskiĭ, Yu. G. Mamaladze, L. P. Pitaevskiĭ, V. L. Pokrovskiĭ, A. A. Sobyanin, E. B. Sonin, and I. M. Khalatnikov, for discussions of this work and a series of remarks. One of the authors (S.N.) is grateful to Academician S. S. Kutateladze for his attention to this work.

## APPENDIX I

In Appendix I we estimate the entropy $L^L$ transported by the vortices. That part which is related to the vortex component of the superfluid must be determined from the identity

$$dp=\rho d\mu+SdT+(\mathbf{j}_0, d\mathbf{V}_{ns}). \tag{I.1}$$

The dependence of the pressure on the relative velocity has the form

$$p=p_0(\mu,T)+\frac{\rho_n}{2}V_{ns}^2.$$

Taking into account the fact that in view of the hydrodynamic equations the quantities $T, \mu+v_s^2/2$, and $\mathbf{v}_s$ cannot have singularities (as well as $\partial p_0/\partial\mu=\rho$), we obtain for the singular part of the pressure

$$p_{sing}=-{}^1/{}_2\rho_sV_s^2.$$

In agreement with Eq. (I.1) the singular part of the entropy density is now equal to

$$S_{sing}=-\frac{1}{2}\left(\frac{\partial\rho_s}{\partial T}\right)_\mu v_s^2.$$

This involves the partial derivative for constant chemical potential

$$\left(\frac{\partial\rho_s}{\partial T}\right)_\mu=\left(\frac{\partial\rho_s}{\partial T}\right)_p-S\left(\frac{\partial\rho_s}{\partial p}\right)_T.$$

Integrating this expression near the vortex, one can obtain the entropy $S^L$ carried away by them:

$$S_L=-\frac{\partial\rho_s}{\partial T}\left(\frac{2\pi\hbar}{m}\right)^2\frac{L}{4\pi}\ln\frac{\delta}{l}. \qquad (I.2)$$

The lower cutoff is, however, not the radius of the vortex, but $l$, the mean free path of excitations, i.e., the limit under which the hydrodynamic discussion is no longer justifiable. In the region $\sim l$ the distribution function of the excitations differs from an equilibrium distribution, and this region contributes to $S^L$ an addition which, on account of the smallness of the volume of the region, can hardly exceed the contribution (I.2). The same can apparently be said about the proper entropy of the vortices, although it is extremely difficult to estimate it, without having an idea on the structure of the vortex cluster. Consequently, in the hydrodynamic equations one must consider $S^L$ as a prescribed function (which should be obtained from microscopic considerations), transported with the velocity $\mathbf{v}_L$, and estimated by Eq. (I.2).

## APPENDIX II

It is convenient to formulate the nondissipative hydrodynamic equations in the language of Poisson brackets (see the reviews[15,16]).*) There arises a problem related to the fact that the Poisson brackets must satisfy the Jacobi identity. However, it is by far not always possible to construct adequate hydrodynamic equations by means of a system of Poisson brackets satisfying the Jacobi identities. For this reason Volovik and Kats,[17] who considered the hydrodynamics of liquid crystals, were forced to introduce the auxiliary quantity $L$, which later had to be eliminated from the final equations.

The reason for this is that the hydrodynamic degrees of freedom are slowly relaxing, surviving after one excludes from the complete set degrees of freedom the rapidly relaxing variables. Since the latter are not constants of the motion, such an exclusion from the expressions for the Poisson brackets (which for a complete set of observables satisfy the Jacobi identities) inevitably leads to Poisson brackets for the remaining quantities not satisfying the Jacobi identities. The Jacobi identities may accidentally still be valid**) (for example in the case of a classical fluid, or for superfluid He II), owing to limited number of hydrodynamic variables. In the sequel, formulating the hydrodynamic equations, we do not impose the Jacobi identities, but leave in force the requirement of antisymmetry of the Poisson brackets.

The hydrodynamic variables split into the densities of conserved (or slowly relaxing) quantities, and into variables describing the order parameter degrees of freedom. The expressions for the Poisson brackets involving densities generally do not depend on the concrete system under consideration and have a universal character determined by symmetry considerations.[18] These universal considerations guarantee that the corresponding conservation laws are satisfied for any system. The arbitrariness related to the violation of the Jacobi identities appears only in the expressions of the Poisson brackets for quantities related to the order parameter. We note that the antisymmetry of the Poisson brackets suffices for automatic validity of the energy conservation law.

In the case considered here we have densities of mass $\rho$ and momentum $\mathbf{j}$, and in addition there is the entropy density $S$. Related to the order parameter***) are the superfluid velocity $\mathbf{v}_s$, and the length $L$ per unit volume of the vortex filaments. The enumerated quantities form a complete set of hydrodynamic variables for turbulent He II. The Poisson brackets for the densities $\rho$ and $\mathbf{j}$ have the following universal form:

$$\{\mathbf{j}(\mathbf{r}_1),\ \rho(\mathbf{r}_2)\}=\rho(\mathbf{r}_1)\nabla\delta(\mathbf{r}_1-\mathbf{r}_2), \qquad (II.1)$$

$$\{j_k(\mathbf{r}_1),\ j_i(\mathbf{r}_2)\}=j_i(\mathbf{r}_1)\nabla_k\delta(\mathbf{r}_1-\mathbf{r}_2)+j_k(\mathbf{r}_2)\nabla_i\delta(\mathbf{r}_1-\mathbf{r}_2). \qquad (II.2)$$

Also universal are the brackets

$$\{\rho(\mathbf{r}_1),\ \mathbf{v}_s(\mathbf{r}_2)\}=\nabla\delta(\mathbf{r}_1-\mathbf{r}_2), \qquad (II.3)$$

$$\{j_i(\mathbf{r}_1),\ v_{sj}(\mathbf{r}_2)\}=v_{si}(\mathbf{r}_2)\nabla_j\delta(\mathbf{r}_1-\mathbf{r}_2)-\nabla_i v_{sj}\delta(\mathbf{r}_1-\mathbf{r}_2). \qquad (II.4)$$

The Poisson brackets between the momentum density $\mathbf{j}$ and the densities $S$ and $L$ are similar to (II.1)

$$\{j(\mathbf{r}_1),\ S(\mathbf{r}_2)\}=S(\mathbf{r}_1)\nabla\delta(\mathbf{r}_1-\mathbf{r}_2), \qquad (II.5)$$

$$\{j(\mathbf{r}_1),\ L(\mathbf{r}_2)\}=L(\mathbf{r}_1)\nabla\delta(\mathbf{r}_1-\mathbf{r}_2). \qquad (II.6)$$

The expressions for the Poisson brackets involving the superfluid velocity can be derived on the basis of the fact that the quantities $L$ and the entropy $S^L$ are convected with the velocity $\mathbf{v}_L$. This fixes the following expressions:

$$\{v_{sk}(\mathbf{r}_1),v_{si}(\mathbf{r}_2)\}=\frac{1}{\rho_s}(\nabla_k v_{sk}-\nabla_k v_{si})\delta(\mathbf{r}_1-\mathbf{r}_2), \qquad (II.7)$$

$$\{\mathbf{v}_s(\mathbf{r}_1),L(\mathbf{r}_2)\}=-\frac{(b-1)L}{\rho_s}(\mathbf{r}_1)\nabla\delta(\mathbf{r}_1-\mathbf{r}_2), \qquad (II.8)$$

$$\{\mathbf{v}_s(\mathbf{r}_1),S(\mathbf{r}_2)\}=-\frac{(b-1)S^L}{\rho_s}(\mathbf{r}_1)\nabla\delta(\mathbf{r}_1-\mathbf{r}_2). \qquad (II.9)$$

The nondissipative hydrodynamic equations for turbulent He II can now be formulated by means of the Hamiltonian

$$H=\int d^3\mathbf{r}\left(\frac{\rho v_s^2}{2}+\mathbf{v}_s\mathbf{j}_0+E_0(\rho,S,L,\mathbf{j}_0)\right). \qquad (II.10)$$

These equations have the following form[4]:

$$\frac{\partial\rho}{\partial t}=\{H,\rho\}=-\nabla\mathbf{j}, \qquad (II.11)$$

$$\frac{\partial S}{\partial t}=\{H,S\}=-\nabla(S\mathbf{v}_n+S^L(\mathbf{v}_L-\mathbf{v}_n)), \qquad (II.12)$$

$$\frac{\partial L}{\partial t}=\{H,L\}=-\nabla(L\mathbf{v}_L), \qquad (II.13)$$

$$\frac{\partial v_{si}}{\partial t}=\{H,v_{si}\}=-\nabla_i\mu-(v_s\nabla)v_{si}+\frac{(b-1)S^L}{\rho_s}\nabla_i T+\frac{(b-1)L}{\rho_s}\nabla\varepsilon_b, \qquad (II.14)$$

$$\frac{\partial j_i}{\partial t}=\{H,j_i\}=-\nabla_i p-\nabla_k(\rho_s v_{si}v_{sk}+\rho_n v_{ni}v_{nk}). \qquad (II.15)$$

Here the pressure is

$$p=-E_0+\mu\rho+TS+(\mathbf{V}_{ns},\mathbf{j}_0)+\varepsilon_b L. \qquad (II.16)$$

Equations (II.11)–(II.15) lead to the following form of the energy conservation law

$$\frac{\partial E}{\partial t} = -\nabla\left[ST\mathbf{v}_n + S^L T(\mathbf{v}_L - \mathbf{v}_n) + \varepsilon_L \mathbf{v}_L L + \mathbf{j}(\mu + v_s^2/2) + (\mathbf{v}_n, \mathbf{j}_0)\mathbf{v}_n\right]. \quad \text{(II.17)}$$

1)Schwarz[4] has introduced the function $\lambda(\mathbf{v}_l t)$ representing the length of the filaments per unit volume, such that the selfinduced velocities (i.e., the velocities caused by the local curvature of the filament) are within the interval $(\mathbf{v}_l, \mathbf{v}_l + d\mathbf{v}_l)$. However, his attempt at deriving a kinetic equation for $\lambda$ directly from the dynamical equations of motion is incorrect. The Bogolyubov chain (BBGKY hierarchy) does not contain a small parameter in this case and its artificial truncation, as proposed by Schwarz, is illegitimate.

2)One does, of course, appeal to additional considerations or empirical facts for the construction of the dissipative function (cf. Ref. 3).

3)As is well known, in a countercurrent one encounters a "drag" of the second sound, i.e., $c_2 = c_{20} + \Delta c d_{\rm Khal}$.[9] In order not to complicate the already long formulas we omit from the equations all terms leading to this effect. In view of the fact that both the Khalatnikov correction $\Delta c_{\rm Khal}$, as well as the corrections we have found, are small they will simply contribute additively to final effect.

4)In calculations one should substitute $\mathbf{j}_0 = \mathbf{j} - \rho \mathbf{v}_s$.

*)[Translator's note] The use of Poisson brackets and Poisson structures in continum mechanics has become widespread during recent years. In addition to the reviews,[15,16] the following references written in better English (albeit in the less familiar language of modern differential geometry) could be consulted by the reader: R. Abraham, J. E. Masden, and T. Ratiu, Manifolds, Tensor Analysis, and Applications (Sec. 8.2), Addison-Wesley, Reading, MA, 1983; V. Guillemin and S. Sternberg, Ann. Phys (NY) **127**, 220–253 (1980); J. E. Marsden and A. Weinstein, Coadjoint Orbits, Vortices, and Clebsch Variables for Incompressible Fluids, Physics **D**, in the press; M. E. Mayer, Plasma Kinetic Theory and Differential Geometry, Proc. XI Intern. Conf. on Differential-Geometric Methods in Physics. S. Sternberg, Ed., Reidel, Dordrecht, 1983, A Poisson Structure for Realistivistic Superfluids, to appear in Proc. of AMS Summer Conference, 1983. Additional references can be found in these books and papers.

**)[Translator's note] In this connection, see the papers quoted in the preceding footnote. The existence of "good" Poisson brackets is not quite accidental, but is related to some form of "gauge invariance," and can be understood in terms of the geometric formulations (the "momentum map") quoted in the previous footnote.

***)[Translator's note] For superfluids the order parameters are the phase of the Bose-Einstein condensate (of which $\mathbf{v}_s$ is the gradient) and a Clebsch potential(s) describing the vortex filaments. (Cf. the corresponding discussion of a relativistic superfluid in the translator's paper quoted in footnote*).)


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