

Quantum error limit of Doppler measurements of momentum

Yu. I. Vorontsov and T. P. Bocharova

Moscow State University

(Submitted October 29, 1982)

Zh. Eksp. Teor. Fiz. **84**, 1601–1604 (May 1983)

It is shown that any analysis of the quantum error limit of Doppler measurements of momentum must take into account the wave-packet frequency modulation due to acceleration of the body and due to reaction of the radiation. The error limit of momentum measurements by means of this method is $\Delta p = (\hbar m/\tau)^{1/2}$ as in the case of momentum measurements based on observations of the coordinate of the body (m is the mass of the body and τ is the duration of measurement).

PACS numbers: 06.30.Ft, 06.30.Gv

This problem was first discussed by von Neumann¹ in his search for a body-momentum measurement method that would differ in principle from the method used for determining momentum through coordinate measurements performed at different moments of time. Since coordinate measurements are associated with random disturbances of the momentum, the coordinate at subsequent moments of time is determined not only by the initial velocity of the body, but also by random variations in its velocity during the measurement process. With a designated observation period τ_m , we may associate an optimal value of the coordinate measurement error at which the momentum measurement error is a minimum. In the case of a free-falling mass, the minimal error of such a momentum measurement² is

$$\Delta p = (\hbar m/\tau_m)^{1/2}. \quad (1)$$

A measurement of momentum based on the Doppler effect does not explicitly involve a measurement of the coordinate of the body. Therefore, it might be expected that this method would not be subject to the limit (1). Von Neumann found¹ that the error in a momentum measurement which uses only a single photon is determined in this case by the relation

$$\Delta p_H \approx mc/\omega_0\tau, \quad (2)$$

where c is the speed of light, ω_0 the mean photon frequency, and τ the duration of the wave packet. The unknown disturbance in the momentum is thus determined solely by the indeterminacy of the photon frequency $\Delta\omega = 1/\tau$ and is equal to $\Delta p_d = \hbar/c\tau$. Hence it follows that the measurement error can be reduced by increasing the frequency ω_0 until $mc/\omega_0\tau \leq \hbar/c\tau$, i.e., $\hbar\omega_0/mc^2$. Since it is possible that $\hbar/\tau \ll mc^2$, i.e., $\hbar/c\tau \ll (\hbar m/\tau)^{1/2}$, the Doppler measurement method could have an error $\Delta p_d \ll (\hbar m/\tau)$. However, a more careful analysis shows that Eq. (2) is not applicable at $\Delta p_d \lesssim \Delta p$.

The possibility of reducing the error in momentum measurements based on the Doppler effect below the level $(\hbar m/\tau)^{1/2}$ is open to question, for in this case the error ΔE in the measurement of the energy of the body may be less than \hbar/τ . For this purpose, immediately after making a measurement lasting $\tau_k \lesssim \tau$, we need only compensate (with error p_H) by means of a second photon for the momentum transferred to the body by the first photon. In principle, the inequality $\Delta E < \hbar/\tau_m$ does not contradict the foundations of

quantum mechanics; further, conditions are known under which this relation may hold.³ But in our case these conditions do not hold.

In order to produce a clearer picture of the phenomenon that von Neumann did not take into account, we present his proof of (2).¹ The error in measurements of the momentum of a body is $\Delta p_m \approx mc\Delta\omega/\omega_0$, while the momentum indeterminacy is equal to the indeterminacy in the photon momentum $\hbar\Delta\omega/c$. If $\hbar\omega_0 \ll mc^2$, we have $\hbar\Delta\omega/c \ll mc\Delta\omega/\omega_0$. If a wave train has duration τ , the frequency indeterminacy will be $1/\tau$. The frequency over a time τ can also be measured to within $1/\tau$. Replacing $\Delta\omega$ by $1/\tau$ in the expression $\Delta p_m = mc\Delta\omega/\omega_0$ yields the relation (2). This line of reasoning is not entirely accurate, however, since: (1) the indeterminacy in the frequency of the wave train, i.e., its spectral width, depends not only on the duration, but also on the shape of the train. (2) The shape of a reflected wave train is not the same as the shape of the incident train, as the body will accelerate during the wave-train-reflection time as a result of recoil the reflected wave train acquires a frequency modulation. (3) The error in the computed mean frequency of the wave train depends not only on the resolution of the spectral device, but also on the spectral width of the train. If an initial wave train has a spectral width $1/\tau$, relation (2) will be valid only if the spectrum broadening upon reflection is less than $1/\tau$.

Let us find the spectrum broadening at which the initial wave packet has the form of a train $a \sin(\omega_0 t - kx)$ of duration τ . This train has a spectrum width $2/\tau$ at the level e^{-1} . If we suppose that the initial velocity of the body is much less than the speed of light and if $\hbar\omega_0 \ll mc^2$, it can be approximately assumed that the body is acted upon by a constant recoil during the photon-reflection time and moves with an acceleration $2\hbar\omega_0/mc\tau$. Then the reflected train will have a pulse-modulated frequency

$$\omega(t) = \omega_0 \left(1 - \frac{2v}{c} - \frac{2\hbar\omega_0}{mc^2} t \right),$$

where v is the initial velocity of the body, $0 \leq t \leq \tau$. Results of a previous computation⁴ can be used to compute the spectrum of such a frequency-modulated impulse. It may be shown that, at the e^{-1} level, the spectral half-width of such a train is

$$\delta\omega = [(2\hbar\omega_0^2/mc^2)^2 + (1/\tau)^2]^{1/2}. \quad (3)$$

The first term in (3) will exceed the second term once $mc/\omega_0\tau < (\hbar m/\tau)^{1/2}$, i.e., once $\Delta p_m < (\hbar m/\tau)^{1/2}$. Since von Neumann did not take account of spectral broadening upon reflection in his computations, the latter comparison shows that (2) is valid only if $\Delta p_m > (\hbar m/\tau)^{1/2}$.

Let us find the true limit of error of the momentum measurements. If only a single quantum is used, the spectrum analyzer will yield a frequency reading that may correspond to a definite probability of an arbitrary frequency in the spectrum of the reflected wave packet. The reading accuracy is determined by the resolution of the analyzer, which is related to the indeterminacy of the photon delay time τ_a in the analyzer as $(\Delta\omega)_a \sim 1/\tau_a$. To compute the initial velocity of the body v , it is necessary to determine the frequency ω_1 corresponding to the maximal spectral density of the wave packet. If we have a single random reading accurate to $1/\tau_a$, the frequency ω_1 can be determined only with error

$$(\Delta\omega_1)^2 = (1/\tau_a)^2 + \alpha(\delta\omega/2)^2, \quad (4)$$

where α is a numerical coefficient that depends on the confidence level of the result. (A coefficient $\alpha > 1$ means confidence level greater than 0.8). The error in the measurement of the body momentum is then

$$\Delta p_D = mc\Delta\omega_1/\omega_0 = mc[(1/\omega_0\tau_a)^2 + \alpha(\delta\omega/2\omega_0)^2]^{1/2}. \quad (5)$$

The quantity $\delta\omega/2\omega_0 = (\hbar\omega_0/mc^2)^2 + (1/2\omega_0\tau)^2$ has a minimum $\hbar/mc^2\tau$ when $(\omega_0)_{\text{opt}}^2 = mc^2/\hbar\tau$. Consequently, the error in the measurement of the momentum of a body by the Doppler method is bounded by the inequality

$$\Delta p_{\text{conf}} > (\hbar m/\tau)^{1/2}. \quad (6)$$

If the incident wave were to contain not one but N photons, the acceleration of the body would be N times greater. But in place of a single reading there would be N frequency readings, and the momentum measurement error would then be

$$(\Delta p_D)_N = mc \left\{ \frac{1}{N(\omega_0\tau_a)^2} + \frac{\alpha}{N(2\omega_0)^2} \left[\left(\frac{2N\hbar\omega_0^2}{mc^2} \right)^2 + \left(\frac{1}{\tau} \right)^2 \right] \right\}^{1/2}. \quad (7)$$

The second term in (7) has a minimum $\hbar/mc^2\tau$, as in the case of a single photon, though now when $(\omega_0)_{\text{opt}}^2 = mc^2/N\hbar\tau$, i.e., at a radiation power

$$P_{\text{opt}} = N\hbar\omega_0/\tau = mc^2/\omega_0\tau^2. \quad (8)$$

The maximal measurement error is independent of the number of photons in the pulse, though it may be attained at a lower frequency ω_0 the greater N .

Let us make a number of estimates. At $m = 10^{-3}$ kg, $\omega_0 = 10^{16}$ sec $^{-1}$, and $\tau = 1$ sec we have an optimal power $P_{\text{opt}} \approx 10^{-2}$ W. Even if $\tau \sim 10^{-2}$ sec, P_{opt} will have a reasonable value from the experimental standpoint. Consequently, effects associated with the acceleration of the body in Doppler momentum measurements may appear even at contemporary experimental techniques.

The spectrum broadening described above could be eliminated were it possible to compensate for the recoil produced by reflecting the photon, for example, by a similar photon impinging on the body from the opposite direction. But it makes sense to speak of simultaneity only to within the coherence time τ . Even in the case of biphoton fields,² the probability of simultaneous photoelectric readings is close to unity only if the resolution time is less than τ . But if the radiated field is in a coherent state with the mean number N of photons the compensation will be affected also by the indeterminacy of the number of photons per pulse.

It can be stated that (6) determines the theoretical sensitivity limit of the Doppler method of measuring the momentum of a free-falling body. This result is of particular importance in searching for supersensitive measurement methods in experiments designed to detect gravitational radiation.

¹J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, 1955. (Russ. transl., Nauka, Moscow, 1964).

²V. B. Braginskii and Yu. I. Vorontsov, *Usp. Fiz. Nauk*, **114**, 41 (1974) [*Sov. Phys. Usp.* **17**, 644 (1975)].

³Yu. I. Vorontsov, *Usp. Fiz. Nauk*, **133**, 351 (1981) [*Sov. Phys. Usp.* **24**, 150 (1981)].

⁴I. S. Gonorovskii, *Electronic Circuits and Signals*, [in Russian] Sov. Radio, Moscow, 1967, p. 162.

⁵D. N. Klyshko, *Photons and Nonlinear Optics* [in Russian], Nauka, Moscow, 1980, p. 200.

Translated by Robert H. Silverman