Stimulated emission of ultrarelativistic electrons in strong electric and magnetic fields

D. F. Zaretskiĭ and E. A. Nersesov

I. V. Kurchatov Institute of Atomic Energy

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We consider induced emission of ultrarelativistic emissions in strong electric (magnetic) fields that are uniform along the direction of the electron motion and are not uniform in the transverse direction. It is shown that in such a system the gain is increased compared with that of a magnetic undulator by a value of the order of γ($\gamma = \varepsilon/m_e c^2$ is the relativistic factor). Numerical estimates are presented that determine the possibility of amplification of a test wave in the optical and near-infrared bands.

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1. INTRODUCTION

The motion and radiation of relativistic electrons in stationary spatially periodic magnetic fields (undulators) has been investigated in detail recently in connection with the problem of free-electron lasers (FEL).[1–4] An expression was obtained for the amplification coefficient of an electromagnetic wave propagates along[5–7] and at an angle[8] to the direction of motion of the electron beam. It turns out here that in all realistic cases the gain for electrons of energy $\varepsilon > 100$ MeV is so small in the optical band, that the probability of generating the corresponding coherent radiation in undulators becomes quite problematic. It seems vital therefore to seek for other ways of generating coherent radiation in the soft ultraviolet in the FEL regime, without the use of undulators. One such method may be the use of ultrarelativistic electrons that are uniform along the direction of motion but are not uniform in a direction transverse to the stationary electric and magnetic fields. If the field potential in such fields increases in a direction perpendicular to the beam from the center ot the periphery, the electrons can execute harmonic oscillations.

We consider in this paper stimulated emission of relativistic electrons in strong electric and magnetic fields that are uniform along the electron-motion direction and are not uniform in the transverse direction. It is assumed that the potential energy of the interaction of the electrons with the field depends quadratically on the transverse coordinates. We shall assume that a one-dimensional case is realized, when the field varies only along one of the axes in the transverse direction. A generalization of the solution to the two-dimensional case does not lead to results that are fundamentally new.

We calculate in the ultrarelativistic limit the gain for a test wave propagating both along the electron beam and at an angle to it. An expression is obtained for the gain as a function of the length of the interaction region, of the angle of entry of the particles into the field, of the field intensity, of the polarization of the test wave, and others. We note that fields having a configuration close to those considered in the present paper are produced in electric and magnetic quadrupole lenses.

2. BASIC EQUATIONS

The static field in which the relativistic electron move will be specified via the 4-potential $A_1 = (\Phi_1, \mathbf{A}_1)$, whose value is assumed to vary along the Z axis in accord with the quadratic law $A_1(z) = A_{01}(z^2/d^2)$, where A_{01} is the maximum amplitude of the 4-potential and $-d/2 \leqslant z \leqslant d/2$. We assume the field to be uniform in the direction of motion of the electron beam (the X axis) and in the perpendicular direction (the Y axis).

Let $p_\parallel \gg p_\perp$, where $p_\parallel$ and $p_\perp$ are respectively the longitudinal and transverse components of the initial momentum of the electron. In accordance with the considered character of the particle motion, we represent the total initial energy ε by the sum $\varepsilon \approx \varepsilon_\parallel + \varepsilon_\perp$, where $\varepsilon_\parallel = (p_\parallel^2 c^2 + m_e^2 c^4)^{1/2}$ and $\varepsilon_\perp = p_\perp^2/2\gamma m_e$($\gamma = \varepsilon/m_e c^2$ is the relativistic factor and m_e is the electron mass) are the energies corresponding to motion longitudinal and transverse relative to the Z axis. We assume satisfaction of the strong inequality $\varepsilon \approx \varepsilon_\parallel \gg \varepsilon_\perp$.

The field of a traveling electromagnetic field is defined by the 4-potetial ($\hbar = c = 1$)

$$A_2(x) = {}^1/_2 A_{02}[e_2 e^{-ikx} + \text{c.c.}], \tag{1}$$

where $e_2 = (0, \mathbf{e}_2)$ is a unit vector of the wave-field polarization; A_{02} is the amplitude of the field; $k = (\omega, \mathbf{k})$ is the 4-momentum of the field quantum. We have used in (1) the usual notation for the scalar product of 4-vectors: $kx = (kx) = \omega t - \mathbf{k}\cdot\mathbf{r}$.

As the basic equation, neglecting small spin correction, we use the Klein-Gordon equation in the fields $A_{1,2}(x)$. The dimensionless parameter $K = eA_{01}/m_e$, which characterizes the intensity of the interaction of the electron with the electric (magnetic) field can be $\gtrsim 1$, and accordingly we take the field $A_1(x)$ into account in all orders of perturbation theory. The wave field $A_2(x)$ is assumed weak enough and we consider it in first-order perturbation theory. The Klein-Gordon equation takes in a field with potential $\mathbf{A}_1$ the form

$$\left[-\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\mu} - 2ie\left(A_1{}^\mu \frac{\partial}{\partial x^\mu}\right) + (eA_1)^2 - m_e{}^2\right]\Psi = 0. \tag{2}$$

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We seek the solution of (2) in the form[9]

$$\Psi(\mathbf{r},t)=(2\varepsilon V)^{-1/2}e^{-ipx}F(z,t), \tag{3}$$

where $p=(\varepsilon,\mathbf{p})$ is the 4-momentum of the free electron (when the field is turned off). In (3) we use the usual normalization to a single particle in the volume V.

The form of the subsequent formulas depends on the type of the field. Thus, in the case of an electrostatic field ($\mathbf{A}_1=0$) Eq. (2) takes the form

$$\left[-\frac{\partial^2}{\partial t^2}+\Delta-2ie\Phi_1\frac{\partial}{\partial t}+e^2\Phi_1{}^2-m_e{}^2\right]\Psi=0 \tag{4}$$

(Φ_1 is the scalar potential of the field); in the case of a magnetic field ($\Phi_1=0$) Eq. (2) is written in the form

$$\left[-\frac{\partial^2}{\partial t^2}+\Delta-2ie(\mathbf{A}_1\nabla)-e^2A_1{}^2-m_e{}^2\right]\Psi=0 \tag{5}$$

($\mathbf{A}_1$ is the vector potential of the field). Since the general method of solving Eqs. (4) and (5), as well as the approximations used; are analogous in this approximation, we confine ourselves hereafter to the electrostatic field and will present without derivation the results for a stationary magnetic field.

Substitution of (3) in (4) leads to an equation of the Schrödinger type for the transverse motion of the particle:

$$i\frac{\partial F}{\partial t}=\left[-\frac{1}{2\gamma m_e}\frac{\partial^2}{\partial z^2}+e\Phi_1(z)\right]F. \tag{6}$$

The wave function of the particle (3) breaks up into factors that correspond to its longitudinal and transvere motion

$$\Psi(\mathbf{r},t)=(2\varepsilon S)^{-1/2}\exp[-i(\varepsilon_\parallel t-\mathbf{p}_\parallel\boldsymbol{\rho})]F(z,t) \tag{7}$$

(in accord with the representation of the function (7), it is convenient to express the volume V in the form $V=SL$, where S is the normalization area in the XY plane, L is the normalization length in the direction of the Z axis, and ρ is the radius vector of the particle in the XY plane)

In the derivation of (6) we left out the small terms $\partial^2F/\partial t^2$, and $e^2\Phi_1^2F$, this is valid if the inequalities $\varepsilon\gg\varepsilon_\perp$ and $\varepsilon\gg V_{0E}$ are satisfied ($V_{0E}=e\Phi_{01}$ is the depth of the potential well in which the transverse motion of the electrons takes place. The inequality $V_{0E}/\varepsilon\ll1$ can be formulated in the form of the condition on the field intensity and the particle energy, namely $K/\gamma\ll1(K=e\Phi_{01}/m_e)$.

The eigensolutions of the stationary Schrödinger equation

$$-\frac{1}{2\gamma m_e}\frac{\partial^2\varphi_n}{\partial z^2}+e\Phi_{01}\frac{z^2}{d^2}\varphi_n=\varepsilon_n\varphi_n$$

form an orthnormalized basis of oscillator functions with energy eigenvalues $\varepsilon_n=(n+1/2)\Omega_E$, where

$$\Omega_E=\frac{1}{d}\left(\frac{2e\Phi_{01}}{\gamma m_e}\right)^{1/2}=\frac{1}{d}\left(\frac{2V_{0E}}{\varepsilon}\right)^{1/2}. \tag{8}$$

To solve the nonstationary equation (6) we assume that the interaction of the electron with the field is turned on instantaneously at the instant of time $t=0$ (the corresponding condition for the entry of the particle into the field will be formulated below). The basic function that describes the state of the particle at instants $t>0$ is then

$$\Psi_i(\mathbf{r},t)=(2\varepsilon S)^{-1/2}\exp\{-i(\varepsilon_\parallel t-\mathbf{p}_\parallel\boldsymbol{\rho})\}\sum_n a_n\varphi_n(z)\times\exp\{-i\Omega_E(n+1/2)t\}, \tag{9}$$

where the expansion coefficients are given by

$$a_n=\int F(z,0)\varphi_n(z)\,dz \tag{10}$$

and can be interpreted as the probability amplitudes of the transition of the particle from free-motion states $\tilde F(z,t\leqslant0)$ into definte oscillator states $\varphi_n(z)$ when the field is suddenly turned on.

We formulate now the condition for the suddenness of turning on the interaction. This approximation takes place in the case when the characteristic time of entry of the particle in the field $\tau\sim d/v\approx d$ (v is the electron velocity) is much shorter than the period $T\sim(\varepsilon_{n+1}-\varepsilon_n)^{-1}$ of the natural oscillations of the system, i.e., when the inequality $d\ll\pi/\Omega_E$ is satisfied. It is easily seen that this criterion can be formulated in the form of a condition on the value of the parameter K and on the particle energy $(K/\gamma)^{1/2}\ll1$.

We make now a few remarks concerning the form of the function (9) and of the coefficients (10). Since the potential well for the transverse motion has a finite depth, the complete basis of the eigenfunctions contains, strictly speaking not only the oscillator functions $\varphi_n(z)$ but also continuous-spectrum functions. It is therefore legitimate to retain the discrete states in the expansion (9) only when the energy of the transverse motion of the particle on entering the field does not exceed the height of the potential barrier: $\varepsilon_\perp\lesssim V_{0E}$. When this condition is satisfied the well is filled with states that are not too close in energy to the height of the barrier, the summation in (9) extends to the value $n=n_0$ corresponding to the upper level ($n_0=V_{0E}/\Omega_E$). The restriction on the value of $\varepsilon_\perp$ can be formulated in the form of a condition on the angle $\Theta_0=p_\perp/\varepsilon$ of entry of the particle into the field:

$$\Theta_0\leqslant(2V_{0E}/\varepsilon)^{1/2}. \tag{11}$$

We turn now to the coefficients a_n, which give the populations of the levels in the well. In accord with the meaning of the expansion (9), these coefficient should satisfy the normalization condition $\Sigma_n|a_n|^2=1$. To satisfy this condition we use for the function F $(z,t\lesssim0)$ normalization in a volume with side $L\gg1/p_\perp$, so that in accordance with the definition (10) the coefficients are given by the formula

$$a_n=L^{-1/2}\int_{-L/2}^{L/2}\exp(ip_\perp z)\varphi_n(z)\,dz. \tag{12}$$

The wave funtion of the final state of the particle is obtained from (9) and (10) by formal replacement of all the initial parameters by the final ones:

$$\Psi_f(\mathbf{r},t)=(2\varepsilon'S)^{-1/2}\exp\{-i(\varepsilon_\parallel' t-\mathbf{p}_\parallel'\boldsymbol{\rho})\}\sum_m a_m'\varphi_m(z)\times\exp\{-i\Omega_E(m+1/2)t\},$$

$$a_m'=L^{-1/2}\int_{-L/2}^{L/2}\exp(ip_\perp'z)\varphi_m(z)\,dz. \tag{13}$$

In the case of a magnetostatic field the Schrödinger equation that describes the transverse motion of the particle is obtained from (6) by making the obvious replacement of the potential energy $e\Phi_1(z)$ by $-eA_1(z)(\mathbf{p}\mathbf{e}_1)/\gamma m_e$, where $\mathbf{e}_1$ is a unit vector of the magnetic-field polarization and is perpendicular to the Z axis. The natural frequency of the system is then equal to

$$\Omega_M = \frac{(2eA_{01}|\mathbf{p}\mathbf{e}_1|)^{1/2}}{d\gamma m_e} = \frac{1}{d}\left(\frac{2V_{0M}}{\varepsilon}\right)^{1/2} \tag{14}$$

and depends parametrically not only on the magentic-field amplitude but also on the energy and direction of motion of the particle. Owing to this dependence it is necessary to formulate an additional criterion, upon satisfaction of which we obtain for the magnetic field a Schrödinger equation of the type (6): $|\mathbf{p}\cdot\mathbf{e}_1|\gg eA_{01}$. Obviously, this condition restricts the possible directions of the longitudinal component $\mathbf{p}_\parallel$ of the particle momentum relative to the vector $\mathbf{A}_{01}$.

As noted above, the field $A_2(x)$ of a traveling electromagnetic wave field is taken into account in first-order perturbation theory. The perturbation operator linear in the wave field takes for the case of an electrostatic field the form

$$\hat{V}_E = -2ie(\mathbf{A}_2\nabla), \tag{15}$$

and for a magnetostatic field

$$\hat{V}_M = -2ie(\mathbf{A}_2\nabla) - 2e^2(\mathbf{A}_1\mathbf{A}_2). \tag{16}$$

Again, as above, we confine ourselves to the electrostatic field. The results for the magnetic field will be given in final form without proof.

The processes considered in this paper are characterized by an S-matrix element the expression for which in first-order perturbation theory in terms of the field A_2 is of the form

$$S_{fi} = -\frac{i}{2(\varepsilon\varepsilon')^{1/2}S}\sum_{m,n} a_m'^* a_n \int \exp\{i(\varepsilon_\parallel' + \Omega m \pm \omega - \varepsilon_\parallel - \Omega n)t\}$$
$$\times \exp\{-i(\mathbf{p}_\parallel' \pm \mathbf{k}_\parallel - \mathbf{p}_\parallel)\boldsymbol{\rho}\}\,d\boldsymbol{\rho}\,dt\Big\{e(\mathbf{A}_{02}\mathbf{p}_\parallel)\int \varphi_m^*(z)$$
$$\times \exp(\mp ik_\perp z)\varphi_n(z)\,dz$$
$$-ie(A_{02})_z\int \varphi_m^*(z)\exp(\mp ik_\perp z)\frac{d\varphi_n(z)}{dz}\,dz\Big\} \tag{17}$$

(in this and following formulas we leave out the subscript E, which is of no importance in the derivation; the upper and lower signs correspond respectively to emission and absorption).

We assume an infinite region of interaction between the electrons in the field (more accurately, we assume satisfaction of the condition $\delta\varepsilon/\varepsilon > 2\pi/l\Omega$, where $\delta\varepsilon/\varepsilon$ is the relative energy scatter in the initial electron beam and l is the linear dimension of the interaction region. The parameter $l\Omega/2\pi$ determines the effective number of the electron oscillations in the transverse direction over a length l. In this case the integrations in (17) are between infinite limits, and as a result we obtain a product of three δ-functions connected with the system energy and momentum conservation laws for longitudinal motion. In the upshot we obtain from (17) the following expression for the S-matrix elements that describe processes with emission (S_{fi}^e) and absorption (S_{fi}^a) of a wave quantum with energy ω:

$$S_{fi}^{e(a)} = -\frac{i}{2(\varepsilon\varepsilon')^{1/2}S}(2\pi)^3\delta^{(2)}(\mathbf{p}_\parallel' \pm \mathbf{k}_\parallel - \mathbf{p}_\parallel)$$
$$\times \sum_{m,n} a_m'^* a_n \delta[\varepsilon_\parallel' \pm \omega - \varepsilon_\parallel - (n-m)\Omega]$$
$$\times \{e(\mathbf{A}_{02}\mathbf{p}_\parallel)I_{m,n}^{(1)} - ie(A_{02})_z I_{m,n}^{(2)}\};$$
$$I_{m,n}^{(1)} = \int \varphi_{m'}^*(z)\exp(\mp ik_\perp z)\varphi_n(z)\,dz,$$
$$I_{m,n}^{(2)} = \int \varphi_m^*(z)\exp(\mp ik_\perp z)\frac{d\varphi_n(z)}{dz}\,dz. \tag{18}$$

From the conservation laws contained in the δ functions of (18) follows a relation for the frequency ω of the wave for which amplification is possible

$$\omega_s = \frac{s\Omega}{1-(p_\parallel/\varepsilon_\parallel)\cos\theta} \approx \frac{2\gamma^2 s\Omega}{1+\gamma^2\theta^2}, \tag{19}$$

where $s = |n - m|$; θ is a small angle between the vectors $\mathbf{k}$ and $\mathbf{p}_\parallel$.

When transforming to probabilities of the processes, the singularity that remains after integration with respect to dp' in (18), as is customary in problems of induced emission in a given field, is eliminated by subsequent integration that takes into account, for example, the finite character of the interaction region, the anharmonicity of the field, etc. In our formulation, the total probabilities of the processes should be averaged over the initial energy distribution of the electrons in the beam, given by the distribution function $f(\varepsilon)$. We assume that this function is normalized to unity by the condition $\int f(\varepsilon)d\varepsilon = 1$ and that the width of the function $\delta\varepsilon \ll \varepsilon$.

Taking all the foregoing into account, the probabilities per unit time of the processes with emission (dw_e) and absorption (dw_a) of a quantum ω are given by the expressions

$$dw_{e,a} = \frac{\pi}{2}\sum_{m,n}|a_n|^2\frac{\delta[\varepsilon_\parallel' \pm \omega - \varepsilon_\parallel - (n-m)\Omega]}{\varepsilon\varepsilon'_{e,a}}$$
$$\times |e(\mathbf{A}_{02}\mathbf{p}_\parallel)I_{m,n}^{(1)} - ie(A_{02})_z I_{m,n}^{(2)}|^2 f(\varepsilon)\,d\varepsilon. \tag{20}$$

Equation (20) was obtained as a result of integration over the phase space of the scattered particle, using the sum rule for the coefficients a_n and the condition that the functions $\varphi_n(z)$ are orthonormalized. Using the known expressions for the integrals $I_{m,n}^{(1)}$ and $I_{m,n}^{(2)}$ in terms of Laguerre polynomials[10] we obtain from (20)

$$dw_e^{(s)} = \frac{\pi}{2}\frac{\delta(\varepsilon_{\parallel e}' + \omega_s - \varepsilon_\parallel - s\Omega)}{\varepsilon\varepsilon_e'}$$
$$\times \sum_{n=s}|a_n|^2\frac{(n-s)!}{n!}e^{-\alpha}\Big\{e(\mathbf{A}_{02}\mathbf{p}_\parallel)\alpha^{s/2}L_n^s(\alpha)$$
$$+\frac{1}{\sqrt{2}}\frac{e(A_{02})_z}{z_0}[n\alpha^{(s-1)/2}L_{n-1}^{s-1}(\alpha) + \alpha^{(s+1)/2}L_{n+1}^{s+1}(\alpha)]\Big\}^2 f(\varepsilon)\,d\varepsilon;$$

$$dw_a^{(s)}=\frac{\pi}{2}\frac{\delta(\varepsilon_{\|a}'-\omega_s-\varepsilon_\|+s\Omega)}{\varepsilon\varepsilon_a'}$$

$$\times\sum_{n=0}|a_n|^2\frac{n!}{(n+s)!}e^{-\alpha}\left\{e(\mathbf{A}_{02}\mathbf{p}_\|)\alpha^{s/2}L_{n+s}^s(\alpha)\right.$$

$$\left.+\frac{1}{\sqrt{2}}\frac{e(A_{02})_z}{z_0}[(n+1)\alpha^{(s-1)/2}L_{n+s}^{s-1}(\alpha)+\alpha^{(s+1)/2}L_{n+s}^{s+1}(\alpha)]\right\}^2$$

$$\times f(\varepsilon)d\varepsilon \tag{21}$$

(the superscript s of the probabilities indicates that the processes considered are accompanied by transitions through $s=|n-m|$ levels in the oscillator potential for the transverse motion of the particle motion; $\alpha\equiv(k_\perp z_0)^2/2$, where $(n\alpha)^{1/2}\sim(\varepsilon_\perp/\varepsilon)^{1/2}s\gamma^2\theta<1$ is the characteristic oscillator length).

In the case of large quantum numbers n, we use the known asymptotic representation of Laguerre polynomials[11]:

$$L_n^s(\alpha)\approx\frac{\Gamma(n+s+1)}{n!}e^{\alpha/2}(n\alpha)^{-s/2}J_s(2\sqrt{n\alpha})\qquad(n\gg1), \tag{22}$$

where $J_s(2\sqrt{n\alpha})$ is a Bessel function.

We shall distinguish hereafter between two limiting cases, when $(n\alpha)^{1/2}<1$ and $(n\alpha)^{1/2}\gg1$. The inequality $(n\alpha)^{1/2}\sim(\varepsilon_\perp/\varepsilon)^{1/2}s\gamma^2\theta<1$ corresponds to the dipole approximation and is realized in a collinear geometry, when the traveling wave propagates in the direction of the longitudinal motion of the particle. In the dipole approximation, the asymptotic behavior of the function (22) is given by[11]

$$L_n^s(\alpha)\approx n^s e^{\alpha/2}/s!\qquad((n\alpha)^{1/2}<1,\ n\gg s),$$

with the aid of which we obtain from (21)

$$dw_e^{(s)}=\frac{\pi}{2}(eA_{02})^2\frac{\delta(\varepsilon_{\|e}'+\omega_s-\varepsilon_\|-s\Omega)}{\varepsilon\varepsilon_e'}\sum_{n=s}|a_n|^2n\frac{(n\alpha)^{s-1}}{[(s-1)!]^2}$$

$$\times\left\{|\mathbf{e}_2\mathbf{e}_p|\varepsilon\frac{\alpha^{1/2}}{s}+\frac{|\mathbf{e}_2\mathbf{e}_z|}{2^{1/2}z_0}\left[1+\frac{n\alpha}{s(s+1)}\right]\right\}^2f(\varepsilon)d\varepsilon;$$

$$dw_a^{(s)}=\frac{\pi}{2}(eA_{02})^2\frac{\delta(\varepsilon_{\|a}'-\omega_s-\varepsilon_\|+s\Omega)}{\varepsilon\varepsilon_a'} \tag{23}$$

$$\times\sum_{n=0}|a_n|^2(n+s)\frac{(n\alpha)^{s-1}}{[(s-1)!]^2}$$

$$\times\left\{|\mathbf{e}_2\mathbf{e}_p|\varepsilon\frac{\alpha^{1/2}}{s}+\frac{|\mathbf{e}_2\mathbf{e}_z|}{2^{1/2}z_0}\left[1+\frac{n\alpha}{s(s+1)}\right]\right\}^2f(\varepsilon)d\varepsilon,$$

where $\mathbf{e}_p=\mathbf{p}_\|/p_\|$; $\mathbf{e}_z$ is the unit vector of the Cartesian axis Z. From (23) it follows, particular, that for a wave with vector $\mathbf{A}_{02}$ perpendicular to the Z axis the probabilities of emission and absorption of a quantum vanish in the limit as $\alpha\to0$.

In the case of arbitrary polarization of the wave, the first term in the expressions in curly brackets in (23) is small compared with the second to the extent that $((n\alpha)^{1/2}s)(\varepsilon/\varepsilon_\perp)^{1/2}\theta\sim\gamma^2\theta^2\ll1$. Thus (we confine ourselves hereafter to the case of the lowest value $s=1$).

$$dw_e^{(s=1)}=\frac{\pi}{4}\left(\frac{eA_{02}}{z_0}\right)^2|\mathbf{e}_2\mathbf{e}_z|^2\frac{\delta(\varepsilon_{\|e}'+\omega_1-\varepsilon_\|-\Omega)}{\varepsilon\varepsilon_e'}$$

$$\times\sum_{n=0}|a_n|^2nf(\varepsilon)d\varepsilon, \tag{24}$$

$$dw_a^{(s=1)}=\frac{\pi}{4}\left(\frac{eA_{02}}{z_0}\right)^2|\mathbf{e}_2\mathbf{e}_z|^2\frac{\delta(\varepsilon_{\|a}'-\omega_1-\varepsilon_\|+\Omega)}{\varepsilon\varepsilon_a'}$$

$$\times\sum_{n=0}|a_n|^2(n+1)f(\varepsilon)d\varepsilon.$$

The averaging over ε in Eqs. (23) and (24) is easily carried out using the conditions $\Omega\ll\omega\ll\varepsilon_\|\approx\varepsilon\approx|\mathbf{p}|$. In this approximation the δ functions can be represented in the form

$$\delta(\varepsilon_{\|e,a}'\pm\omega_s-\varepsilon_\|\mp s\Omega)=\frac{\delta(\varepsilon-\varepsilon_{e,a})}{|\partial\varepsilon_{\|e,a}'/\partial\varepsilon_\|-1|}=\frac{\gamma^2\varepsilon\delta(\varepsilon-\varepsilon_{e,a})}{\omega_s}, \tag{25}$$

where $\varepsilon_{e,a}=\varepsilon_0\pm\Delta\varepsilon$ are the energies of the electrons that emit or absorb a quantum of energy ω_s at a given oscillation frequency Ω;

$$\varepsilon_0=m_e(\omega_s/2s\Omega)^{1/2},\quad\Delta\varepsilon=\omega_s/2. \tag{26}$$

The rate of amplification of the wave is determined by the difference between the induced-emission and absorption probabilities. Using expressions (24) and integrating with respect with the aid of (25) we obtain for the difference between the total probabilities of emitting and absorbing per unit time a quantum of frequency ω_1

$$\Delta w^{(s=1)}=\frac{\pi}{4}\left(\frac{eA_{02}}{z_0}\right)^2|\mathbf{e}_2\mathbf{e}_z|^2\frac{\gamma^2}{\varepsilon\omega_1}\left[-f(\varepsilon_0)+\omega_1\frac{df}{d\varepsilon}\sum_n n|a_n|^2\right]. \tag{27}$$

In the derivation of (27) for the difference between the values of the function $f(\varepsilon)$ at the points ε_e and ε_a we have used the approximate equation

$$f(\varepsilon_e)-f(\varepsilon_a)\approx2\Delta\varepsilon\frac{df}{d\varepsilon}\approx\frac{\omega_s}{(\delta\varepsilon)^2},\quad\delta\varepsilon\gg\omega_s, \tag{28}$$

in which the derivative is calculated at the point $\varepsilon=\varepsilon_0$.

In accord with the definition (12), the level populations at $n\gg1$ are

$$|a_n|^2\approx\frac{1}{\pi\eta_0}\frac{1}{(2n)^{1/2}}\left[\frac{\sin(\beta+(2n)^{1/2})\eta_0}{\beta+(2n)^{1/2}}\right.$$

$$\left.+(-1)^n\frac{\sin(\beta-(2n)^{1/2})\eta_0}{\beta-(2n)^{1/2}}\right]^2$$

$$\beta\equiv p_\perp z_0=\pm(2\varepsilon_\perp/\Omega)^{1/2},\quad\eta_0\equiv L/2z_0.$$

It suffices to retain in this equation one of the terms (its choice is determined by the sign of the transverse momentum of the particle on entering the field). Let, for the sake of argument, $p_\perp>0$ and

$$|a_n|^2=\frac{(\eta_0/\pi)}{(2n)^{1/2}}\left(\frac{\sin\chi}{\chi}\right)^2,\quad \chi=(\beta-(2n)^{1/2})\eta_0. \tag{29}$$

An analysis of (29) shows that the level populations have an extremely strongly pronounced maximum near the value $n\approx\beta^2/2\approx\varepsilon_\perp/\Omega$, with a width $\delta n\sim(\pi/\eta_0)(\varepsilon_\perp/\Omega)^{1/2}$. Using for $|a_n|^2$ a representation with the aid of a δ-function

$$|a_n|^2\approx(\beta/(2n)^{1/2})\delta(n-\beta^2/2)$$

and changing from summation over n to integration, we find that the expression in the square brackets of (27) is equal to

$$[\;]=-f(\varepsilon_0)+\omega_1\frac{df}{d\varepsilon}\frac{\varepsilon_\perp}{\Omega}. \tag{30}$$

The ratio of the absolute value of the first term to the second in (30) is defined by the parameter

$$\zeta=\left|f(\varepsilon_0)\Big/\frac{\omega_1\varepsilon_\perp}{\Omega}\frac{df}{d\varepsilon}\right|\sim\frac{\delta\varepsilon\Omega}{\omega_1\varepsilon_\perp}. \tag{31}$$

At $\zeta>1$, in accordance with the sign of the first term, the particles are accelerated on account of a predominant absorption of the wave quanta. On the contrary, at $\zeta<1$ the wave is amplified and the gain is determined in this case, as in the case of a magnetic undulator, by the sign and magnitude of the derivative $df/d\varepsilon$ (the recoil effect). The amplification condition imposes a lower limit on the transverse energy of the particles:

$$\varepsilon_\perp>\varepsilon_{\perp\,\mathrm{amp}}=\frac{\Omega\delta\varepsilon}{\omega_1}\sim\frac{m_e}{\gamma}\frac{\delta\varepsilon}{\varepsilon}.$$

Assuming that the condition (31) is satisfied, we obtain from (27)

$$\Delta w^{(s=1)}=\frac{\pi}{4}\left(\frac{eA_{02}}{z_0}\right)^2|\mathbf{e}_2\mathbf{e}_z|^2\frac{\gamma^2\varepsilon_\perp}{\varepsilon\Omega}\frac{df}{d\varepsilon},\quad (\varepsilon_\perp/\varepsilon)^{1/2}\gamma^2\theta<1. \tag{32}$$

We turn now to the limiting case $(n\alpha)^{1/2}\sim(\varepsilon_\perp/\varepsilon)^{1/2}\cdot s\gamma^2\theta\gg1$, when the dipole approximation cannot be used. Bearing in mind the known asymptotic representation of the Bessel function,[11] we find that the number s should satisfy the condition $|2(n\alpha)^{1/2}-s|\lesssim s^{1/2}$, and Eq. (22) takes then the form

$$L_n^s(\alpha)\approx0{,}53n^se^{\alpha/2}(n\alpha)^{-s/2-1/6},\qquad (n\alpha)^{1/2}\gg1,\quad n\gg s.$$

again, assuming the amplification condition (31) to be satisfied (the parameter ζ is in the general case independent of s), we obtain from (21) the following expression for the difference between the total emission and absorption probabilities per unit time:

$$\Delta w^{(s)}=\frac{\pi}{2}(eA_{02})^2\frac{\gamma^2(df/d\varepsilon)}{\varepsilon(s/2)^{2/3}}\sum_n|a_n|^2(0.53)^2\cdot\times\left\{|\mathbf{e}_2\mathbf{e}_p|\varepsilon+(2n)^{1/2}\frac{|\mathbf{e}_2\mathbf{e}_z|}{z_0}\right\}^2. \tag{33}$$

For a wave with arbitrary polarization, as in the dipole-approximation case, the first term of the expression in the curly brackets of (33) is small if the condition $\theta<(\varepsilon_\perp/\varepsilon)^{1/2}\lesssim\Theta_0$ is satisfied. Omitting this term, we obtain from (33)

$$\Delta w^{(s)}\approx\frac{\pi}{4}\left(\frac{eA_{02}}{z_0}\right)^2|\mathbf{e}_2\mathbf{e}_z|^2\frac{\gamma^2(df/d\varepsilon)}{\varepsilon(s/2)^{2/3}}\sum_n n|a_n|^2, \tag{34}$$

$$\left(\frac{\varepsilon_\perp}{\varepsilon}\right)^{1/2}\gamma^2\theta\gg1.$$

The differences Δw of the total probabilities of emitting and absorbing per unit time a quantum of frequency ω determine the gain G at this frequency:

$$G=\frac{\Delta w\omega N_e}{E_2^2/8\pi}, \tag{35}$$

where N_e is the electron density in the beam and E_2 is the amplitude of the electric-field intensity in the amplified wave.

3. THE GAINS

Substituting Eqs. (32) and (34) in the definition (35) of G, and using the approximate equality (28), we obtain for the corresponding gains per pass the following expressions:

$$G^{(s=1)}=\frac{2\pi^2e^2|\mathbf{e}_2\mathbf{e}_z|^2N_el\varepsilon_\perp}{m_e^2\omega_1}\left(\frac{\varepsilon}{\delta\varepsilon}\right)^2,\quad \left(\frac{\varepsilon_\perp}{\varepsilon}\right)^{1/2}\gamma^2\theta<1; \tag{36}$$

$$G^{(s)}=\frac{2\pi^2e^2|\mathbf{e}_2\mathbf{e}_z|^2N_el\varepsilon_\perp}{m_e^2\omega_s(s/2)^{2/3}}\left(\frac{\varepsilon}{\delta\varepsilon}\right)^2,\quad \left(\frac{\varepsilon_\perp}{\varepsilon}\right)^{1/2}\gamma^2\theta\gg1 \tag{37}$$

(the number s in (37) is determined by the value of the frequency ω_s of the amplified wave). We emphasize once more that Eqs. (36) and (37) are valid under the assumption that the transverse energy of the particle as it enters the field does not exceed the height of the potential barrier: $\varepsilon_\perp\lesssim V_{0\perp}$. We note that the presented expressions (36) and (37) are universal and are equally applicable to an electrostatic and to a magnetostatic field (the contribution of the term $\sim(\mathbf{A}_1\cdot\mathbf{A}_2)$ which enters in the expression for the operator of the perturbation for the constant magnetic field is small in the ratio

$$\Omega|\mathbf{e}_1\mathbf{e}_2|/(\varepsilon\varepsilon_\perp)^{1/2}|\mathbf{e}_2\mathbf{e}_z|).$$

The expressions (36) and (37) for the gains are valid in the limit $\delta\varepsilon/\varepsilon>2\pi/\Omega$. A more realistic situation is one in which the inverse condition holds: $\delta\varepsilon/\varepsilon<2\pi/\Omega$. In this case the spontaneous-emission line width is determined not by the energy scatter of the particles, but by the diffraction width connected with the fact that the region of interaction of the electrons with the field is finite. From the formal point of view the derivative $df/d\varepsilon\sim1/(\delta\varepsilon)^2$ in (36) and (37) should be replaced[8,12] by

$$s^2\frac{(l\Omega)^2}{\pi(\delta\varepsilon)^2}\left(\frac{\delta\varepsilon}{\varepsilon}\right)^2\frac{d}{du}\frac{\sin^2u}{u^2},\quad s\leqslant s_0=\frac{\varepsilon}{\delta\varepsilon}\frac{\pi}{l\Omega},$$

where $u=l\Omega(\varepsilon-\varepsilon_0)/\varepsilon_0$; the maxima of the gains correspond to $u\approx-1.5$. The gains should be calculated from the formulas

$$G^{(s=1)}=\frac{\pi e^2|\mathbf{e}_2\mathbf{e}_z|^2N_el^3\varepsilon_\perp}{\varepsilon^2d}\left(\frac{2V_{0\perp}}{\varepsilon}\right)^{1/2}\frac{d}{du}\frac{\sin^2u}{u^2}, \tag{38}$$

$$G^{(s)}=\frac{\pi e^2|\mathbf{e}_2\mathbf{e}_z|^2N_el^3\varepsilon_\perp}{\varepsilon^2d}\left(\frac{2V_{0\perp}}{\varepsilon}\right)^{1/2}\frac{d}{du}\frac{\sin^2u}{u^2}s^{1/3},\quad s\leqslant s_0. \tag{39}$$

We note that formula (38) coincides with the expression obtained in Ref. 3 for the electrostatic field by another method.

4. DISCUSSION OF RESULTS

We turn to an analysis of basic expressions obtained in the paper. As follows from (19), amplification of the wave at the frequency $\omega_1 = 2\gamma^2\Omega$ is possible in the collinear scheme. When a wave is launched at an angle θ to the direction $\mathbf{p}_\parallel$, if $(\varepsilon_\perp/\varepsilon)^{1/2}\gamma^2\theta \gtrsim 1$, amplification is possible at the frequency $\omega_s = 2\gamma^2 s\Omega\ (\gamma^2\theta^2 \ll 1)$. Amplification at higher frequencies (on account of the increase of s) is accompanied, however, other conditions being equal, by a decrease of the gain: $G^{(s)} \sim G^{(s=1)}/s(s/2)^{2/3}(s > s_0)$.

From the point of view of the effectiveness of the scheme considered in the paper, it is of interest to compare the gains obtained [Eqs. (38) and (39)] with the gains in a magnetic undulator. In this case, of course the gains must be compared at the same frequency. From the condition that the frequencies be equal we obtain the following relation between the parameters of the fields in the considered scheme and of the undulator (at equal electron energies):

$$\left(\frac{2K}{\gamma}\right)^{1/2} = \frac{2\pi d}{\lambda_u(1+K_u^2)}, \quad \left(\frac{K}{\gamma}\right)^{1/2} \ll 1, \tag{40}$$

where $K_u = eA_u/m_e$ is a parameter that characterizes the intensity of the electron interaction with the undulator field (A_u is the amplitude of the vector potential of this field); λ_u is the undulator period.

We use an equation given in Ref. 14 for the gain in a magnetic undulator in the collinear scheme:

$$G_o = \frac{4\pi^3 e^2 N_e l_u^3 K_u^2}{\gamma m_e \omega (1+K_u^2)\lambda_u^2} \frac{d}{du} \frac{\sin^2 u}{u^2}. \tag{41}$$

for convenience in the comparison we rewrite the expression for $G^{(s=1)}$ [Eq. (38)] in comparable terms:

$$G^{(s=1)} = \frac{4\pi e^2 N_e l^3 \varepsilon_\perp K}{d^2 \omega m_e^2 \gamma} \frac{d}{du} \frac{\sin^2 u}{u^2}. \tag{42}$$

The ratio of the gains, as follows from (41) and (42) is equal to (we assume equal dimensions of the regions of electron-field interaction, $l_u = l$)

$$G^{(s=1)}/G_u = 2\gamma\varepsilon_\perp/m_e K^2(1+K_u^2). \tag{43}$$

It follows hence that in the case $K \sim K_u \sim 1$ and $\varepsilon_\perp \sim m_e$ the gain in the scheme considered by us exceeds that of a magnetic undulator by a factor of the order of γ. The result can be understood by comparing the matrix elements of the perturbation operators for the undulator, $\hat{V}_u \sim e^2(A_1 \cdot A_2)$ and for a field with transverse gradient $\hat{V}_\perp \sim e(A_{02})_z(\partial/\partial z)$ in the collinear scheme. The ratio of the squares of these matrix elements, which determine the gains, is of the order of

$$(V_u/V_\perp)^2 \sim \left(\frac{e^2 A_u A_{02} |\mathbf{e}_1\mathbf{e}_2|}{(eA_{02}/z_0)\,|\mathbf{e}_1\mathbf{e}_z|\, n^{1/2}}\right)^2 \sim \frac{(eA_u)^2}{\varepsilon_\perp \varepsilon}.$$

If $\varepsilon_\perp \sim V_u = eA_u \sim m_e$, this ratio turns out to be of the order of γ^{-1}.

In conclusion, we present by way of example numerical estimates of the amplified-wave frequencies and of the gain $G(s=1)$ for a storage ring with the following beam and field characteristics: $\varepsilon = 150$ MeV ($\gamma = 300$); $\delta\varepsilon/\varepsilon = 0.5\times10^{-3}$; $N_e = 2\times10^{11}$ cm^{-3}; $d = 0.2$ cm; $l = 1$ m (typical length of the linear section in a storage ring); $V_{01} = 1\times10^6$ eV; $V_{01}/d^2 \approx 60$ kG/cm; $\varepsilon_\perp = V_{01}/2$. At the parameters chosen by us the inequality $\delta\varepsilon/\varepsilon < 2\pi/l\Omega$ is satisfied, therefore the gain is calculated using Eq. (38). The value of the amplified frequency for these parameters is $\omega_1 \approx 1.8$ eV, and the gain at this frequency is $G^{(s=1)} \approx 0.7$.

The indicated field characteristics can be realized by a set of quadrupole (multipole) magnetic lenses made, e.g., of magnets based on a samarium-cobalt alloy. In the case of an electrostatic field the field configuration considered in the paper can be obtained between a system of parallel plates that carry in vacuum charges of like sign (negative for electrons). The use of magnetic lenses, however, is preferable since the electrostatic field intensities are limited by the autoionization effect.

Stimulated emission of ultrarelativistic electrons in electric (magnetic) fields with large transverse field gradients makes it thus possible, under certain conditions, to obtain noticeably larger gains than in ordinary magnetic undulators. In this case we are dealing with optical and near-ultraviolet frequencies. The possibility of generating harder coherent radiation calls for a separate analysis.

The calculation in the present paper was performed for a harmonic dependence of the potential on the transverse coordinate of the particle. A separate analysis is necessary to cast light on the role of anharmonicity.

In the derivation of the equations we used a quantum treatment of the electron motion in the field. However, the gains (38) and (39) do not depend on the Planck constant $\hbar$ and can therefore be obtained within the framework of a classical description.

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