# Kinetic effects in electron cyclotron resonance in a nonuniform magnetic field

A. V. Zvonkov and G. N. Chulkov

*I. V. Kurchatov Institute of Atomic Energy*
(Submitted 25 May 1982)
Zh. Eksp. Teor. Fiz. **84**, 60–70 (January 1983)

The mechanism of electron cyclotron resonance in a nonuniform magnetic field is considered. It is shown that effects such as the relativistic dependence of the cyclotron frequency on the electron velocity, the accelerated electron motion in a nonuniform magnetic field, and the nonlinear dependence of the magnetic field on the coordinates decrease the reflection coefficient of a wave propagating away from the weaker magnetic-field region. The limits of applicability of the adiabatic wave equations are indicated.

PACS numbers: 52.35.Mw

## INTRODUCTION

Starting with Budden's paper,[1] the so-called adiabatic wave equation has been widely used for the theoretical investigation of cyclotron resonance in a nonuniform magnetic field. The coordinate dependence of the magnetic field enters in the adiabatic equation parametrically, and resonance effects are taken into account in simplified form with the aid of the Landau contour rule for circuiting around the resonance point.[2–4] The adiabatic equation can therefore lead in some cases to incorrect results. Thus, it is shown in Ref. 5, where cyclotron resonance in a transversely nonuniform magnetic field was investigated, that the reflection and absorption coefficients of a wave incident from the direction of the weaker magnetic field cannot generally speaking be determined from the adiabatic equation.

In addition, interaction-coefficient features that follow from the adiabatic equation, such as the absence of reflection when the electromagnetic wave is incident from the direction of a stronger magnetic field, or the large reflection for incidence from the opposite direction, call in our opinion for a palpable physical explanation.

To cope with these problems it is necessary to investigate in detail the cyclotron resonance interaction in a nonuniform magnetic field with allowance for the effects of the thermal motion of the electrons, going outside the framework of the adiabatic equation.

We consider in this paper electron cyclotron resonance of an extraordinary wave propagating along a magnetic field that varies in the propagation direction in accord with a linear (or nearly linear) law. We propose for the cyclotron resonance in an inhomogeneous magnetic field a mechanism that explains lucidly both the anisotropy of the reflection and absorption coefficients relative to the sign of the wave vector, as well as expressions for them. We show that effects such as the relativistic dependence of the cyclotron frequency on the electron velocity, the change of the electron longitudinal velocity due to nonuniformity of the magnetic field, and the nonlinear coordinate dependence of the magnetic field cause the reflection to decrease. The qualitative arguments are confirmed by direct successive-approximation calculations of the transmission, reflection, and absorption coefficients with allowance for these effects. The results indicate the limits of applicability of the adiabatic equation.

## 1. INTERACTION OF AN ELECTROMAGNETIC WAVE WITH AN ELECTRON BEAM

It is convenient to start the investigation of cyclotron resonance by considering the interaction of an electromagnetic wave with a beam of electrons all having the same velocity $v_\parallel$ along the magnetic field. The simple form of the electron distribution function makes it easiest to examine in this case the physical processes that occur at resonance.

If the beam is cold (the electron transverse velocity is zero) and the cyclotron frequency depends linearly on the longitudinal coordinate, $\omega_e(z) = \omega(1 - z/L)$, the equation for the extraordinary wave, which is integral in this case, can be solved exactly. The case of a wave and a beam that move away from the region of the stronger magnetic field $v_\parallel > 0$) was investigated in Ref. 6, and expressions were obtained for the oscillation transmission ($\tau$), reflection ($\rho$), and absorption ($\eta$) coefficients on passing through the resonance region. Resonant interaction at other signs of $k$ and $v_\parallel$ ($k$ is the wave vector of the incident wave) is considered in detail in Appendix 1 of the present paper. As a result we obtain for a wave propagating away from the stronger magnetic field

$$\tau=e^{-\pi\gamma_-},\quad \rho=0,\quad \eta=1-e^{-\pi\gamma_-}, \tag{1}$$

and for a wave coming from a weaker magnetic field

$$\tau=e^{-\pi\gamma_+},\quad \rho=\frac{\gamma_+}{\gamma_-}(1-e^{-\pi\gamma_+})(1-e^{-\pi\gamma_-}),$$
$$\eta=(1-e^{-\pi\gamma_+})\left(1-\frac{\gamma_+}{\gamma_-}(1-e^{-\pi\gamma_-})\right), \tag{2}$$

where $\gamma_\pm = A(1 \pm v_\parallel/c)$, $A = \omega_p{}^2L/\omega c$, $\omega$ is the wave frequency, $\omega_p$ is the electron plasma frequency, and $c$ is the speed of light; $v_\parallel$ enters with the appropriate sign.

It can be seen from (1) and (2) that the coefficients $\rho$ and $\eta$ depend substantially on the wave propagation direction. A wave incident from the left from a strong magnetic field region propagates without reflection, while a wave coming from the right is partially reflected.

It must be noted that when the beam moves from the stronger magnetic field the reflection coefficient can be larger than unity—the disequilibrium of the electron distribution function leads to generation of a reflected wave having an amplitude larger than the incident one.

 0038-5646/83/010035-06$04.00  

The mechanism of cyclotron resonance in a nonuniform magnetic field has remained unclear to this day. We present a lucid picture of the resonant-interaction process and indicate the cause of the anisotropy of $\rho$ and $\eta$. To this end we consider the interaction between a wave and a beam of electrons with velocity $v_\parallel$. When account is taken of the Doppler effect, the resonance condition takes the form $\omega = \omega_e + kv_\parallel$ and is satisfied at the point $z_s = kv_\parallel L/\omega$. The nonuniformity of the magnetic field restricts the resonant interaction to a region of size $\delta z_s \sim (Lv_\parallel/\omega)^{1/2}$ about the point $z_s$. (We assume validity of the inequality $\delta z_s < kv_\parallel L/\omega$, which is usually satisfied.) In this region part of the wave-field energy is converted into the energy of the electron transverse motion. This gives rise to microscopic currents that induce in turn secondary electric fields.[3] After passage through the resonance zone, the electric field of the oscillations is a superposition of large-scale and small scale components. The large-scale part is identified with the field of the transmitted wave. The small-scale oscillations, which describe the secondary fields, are similar to Van Kampen waves. We shall call them hereafter the modulated beam. The phase of the small-scale oscillations is of the form (see Appendix 1)

$$\Phi(t, z) = -\omega t + \frac{\omega}{2v_\parallel L}(z^2 + z_s^2). \quad (3)$$

In the quasiclassical approximation, the wave vector of these oscillations is

$$k_b = \partial\Phi/\partial z = \omega z/v_\parallel L.$$

It can be easily seen that at the point $z = -z_s$ there is satisfied the condition of linear conversion of the oscillations of the modulated beam into a reflected wave $k_b = -k$. A reflected wave can appear, however, only if the beam passes initially through the resonance point $z_s$ and then through the reflection point $-z_s$. This condition is satisfied for a wave that approaches resonance from a weaker magnetic field, regardless of the direction of the beam motion. If the wave propagates from the stronger magnetic field, this condition is not satisfied for any beam direction.

In the second case the beam passes first through the reflection point and then lands at the resonance point. For the modulated beam the reflection point turns out to be "in the past."

Note the analogy between reflection in cyclotron resonance in a nonuniform magnetic field and the phenomena of nonlocal reflection[7] and transparentization of barriers.[8,9] These effects are also based on modulation of a particle beam by a wave with formation of small-scale oscillations at the resonance point, followed by reradiation of the reflected or transmitted wave at the transformation point.

## 2. EFFECT OF THERMAL SCATTER IN THE DISTRIBUTION FUNCTION

We now show how the simple reasoning of Sec. 1 can be used to consider cyclotron resonance in a plasma when the electron longitudinal-velocity distribution function has a thermal scatter, and obtain expressions for $\tau, \rho$, and $\eta$. Such expressions were first obtained by Budden[1] by solving the adiabatic equation.

A distribution function having a thermal scatter can be regarded as a set of beams with continuous velocity distributions, the density of each beam being

$$dn(v_\parallel) = n_0 f(v_\parallel)\, dv_\parallel. \quad (4)$$

We assume that the beams interact with the wave independently of one another and that the influence of one beam on the others is negligibly small.

Depending on the velocity, each beam will interact with the wave in a small vicinity of the corresonding resonance point $z_s(v_\parallel) = kv_\parallel L/\omega$. The total transmission coefficient is equal to the product of the transmission coefficients of the individual beams

$$\tau = \prod_i \tau_i(v_\parallel), \quad (5)$$

where the product is taken over all the beam. Taking into acount the explicit form of the expresion for $\tau_i$ and substituting the density in the form (4), we obtain

$$\tau = e^{-\Gamma}, \quad \Gamma = n_0 \int_{-\infty}^{\infty} dv_\parallel f(v_\parallel)\gamma = \pi A, \quad (6)$$

where $\gamma = \gamma_\pm$ and $A$ enters in the density $n_0$. The distribution function $f(v_\parallel)$ is assumed to be symmetric, so that

$$\int_{-\infty}^{\infty} d_v v_\parallel f(v_\parallel) = 0\,.$$

From this we find that the fraction of energy transferred from the wave to the beam is

$$w = 1 - e^{-\Gamma}. \quad (7)$$

If the wave propagates from the stronger magnetic field, this entire energy is absorbed. As a result we get

$$\eta = 1 - e^{-\Gamma}, \quad \rho = 0, \quad (8)$$

where $\eta$ and $\rho$ are the absorption and reflection coefficients.

A wave approaching resonance from a weaker magnetic field is partially reflected. Since the emission of a reflected wave is analogous to absorption of an incident wave, it is natural to assume that the energy fraction reradiated by a modulated beam in the form of a reflected wave is determined by Eq. (7). We then obtain for the reflection coefficient

$$\rho = (1 - e^{-\Gamma})^2. \quad (9)$$

From the energy conservation law we determine the absorption coefficient

$$\eta = w - \rho = e^{-\Gamma}(1 - e^{-\Gamma}). \quad (10)$$

We shall show in the next section that these expressions are valid for an isotropic Maxwellian distribution function with only the Doppler effect taken into account.

In the derivation of (9) and (10) it was implicitly assumed that the reflected waves emited by the different modulated beams are in phase. Actually, as follows from (3), the phase of a modulated beam at the reflection point is

$$\Phi(t, -z_s) = -\omega(t - t_s) - \omega t_s + kz_s.$$

The difference $t - t_s = -2kL/\omega$, equal to the time interval in which a modulated beam from the resonant point reaches the reflection point, is the same for all the beams. The two last terms in this expression, $-\omega t_s + kz_s$, are the phase of the wave at the instant of resonance. Thus, at the reflection points the phases of the modulated beams differ by one and the same amount from the phase of the incident wave at the instant $t_s$. The reflected waves corresonding to different beams are therefore in phase, i.e., an "echo" effect is produced.[10] This phenomenon differs from the nonlinear "echo" effect in that there is no "time reversal" here. What is radiated therefore is the reflected rather than the initial wave.

If the phase of the oscillations of the modulated beam has at the reflection point an increment that depends on the beam velocity, the reflected waves corresponding to different beams will be partially canceled by interference, so that the reflection coefficient is decreased. This dephasing can be due, for example, to the relativistic dependence of the cyclotron frequency on the electron velocity, to the nonuniform motion of the particle in the nonuniform magnetic field under the action of the force $-\mu\nabla H$ (where $\mu = v_\perp^2 m/2H$), and to the nonlinear dependence of the magnetic field on the coordinates. Thus, in the case of weak relativism we have

$$\omega_e(z,\ v) \approx \omega(1 - z/L - v^2/2c^2). \tag{11}$$

The phase of the modulated beam at the reflection point acquires an additional increment $\Delta\Phi = -kLv^2/c^2$. The oscillations of the beams whose velocities are shifted by more than

$$\Delta v \approx \frac{\pi}{2}\frac{c}{\omega L}\frac{c^2}{v},$$

will become mutually canceled by interference. In this case only a fraction of the beam participates and is proportional to

$$\frac{\Delta v}{v} = \frac{\pi}{2}\frac{c}{\omega L}\frac{c^2}{v^2}$$

[cf. $\kappa$ in (19)].

It is easily seen that the longitudinally accelerated motion of the electron and the nonlinear coordinate dependence of the magnetic field also lead to dephasing of the modulated beams and to a change in the reflection coefficients.[1]

This analysis shows that the nonlocality of the reflection manifests itself when account is taken of effects that cause a phase mismatch of the modulated-beam oscillations, and does not appear at all when only the Doppler effect is taken into account. The adiabatic equation, in which it is assumed that all the electrons interact with the wave at one point, yields therefore the same expressions as the kinetic approach if only the Doppler effect is taken into consideration.

The conclusions drawn here are confirmed in the next section by direct calculation of the interaction coefficients.

## 3. CALCULATION OF THE INTERACTION COEFFICIENTS

A consistent allowance for the effects of thermal motion of the electrons in a nonuniform magnetic field leads to an integrodifferential wave equation

$$E'' + E - iA\int d^3p\,\frac{p_\perp^2}{\varepsilon}\frac{\partial f}{\partial p^2}\int_{-\infty}^{0} d\zeta\, E\left(\xi + \frac{\omega}{c}\int_0^{\zeta L/c} dt\, v_\parallel(t)\right) \times e^{i\Phi(\zeta)} = 0, \tag{12}$$

where $E = E_x - iE_y$ corresponds to the extraordinary wave, and $\xi = \omega z/c$, and the unperturbed electron energy distribution is[11]

$$f(p) = \exp\left(-\frac{mc^2}{T}\varepsilon\right)\left\{4\pi(mc)^3\left[2\left(\frac{T}{mc^2}\right)^2 K_1\left(\frac{mc^2}{T}\right) + \frac{T}{mc^2}K_0\left(\frac{mc^2}{T}\right)\right]\right\}^{-1},$$

$$\varepsilon = \left(1 + \frac{p^2}{m^2c^2}\right)^{1/2}, \quad \Phi(\zeta) = -\frac{\omega L}{c}\zeta + \int_0^{\zeta L/c} dt\,\omega_e(\varepsilon, z(t)).$$

We shall solve Eq. (12) by successive approximations, assuming the plasma density to be small ($A < 1$) and treating the integral part as a correction. We can then write

$$E_{n+1}(\xi) = \frac{A}{2}\left\{e^{i\xi}\int_{-\infty}^{\xi} d\xi_1\, e^{-i\xi_1}\int d^3p\,\frac{p_\perp^2}{\varepsilon}\frac{\partial f}{\partial p^2}\int_{-\infty}^{0} d\zeta\, E_n\left(\xi_1 + \frac{\omega}{c}\int_0^{\zeta L/c} dt\, v_\parallel(t)\right)e^{i\Phi(\zeta)} + e^{-i\xi}\int_{\xi} d\xi_1\, e^{i\xi_1}\cdot \right.$$
$$\left.\times\int d^3p\,\frac{p_\perp^2}{\varepsilon}\frac{\partial f}{\partial p^2}\int_{-\infty}^{0} d\zeta\, E_n\left(\xi_1 + \frac{\omega}{c}\int_0^{\zeta L/c} dt\, v_\parallel(t)\right)e^{i\Phi(\zeta)}\right\}; \tag{13}$$

$n$ is the order of the expansion of the solution of (12) in $A$.

We examine now the effects of the Doppler effect, or relativism, and of the electron acceleration on the resonant interaction. In the calculation of $\Phi(\zeta)$ it must be taken into account that the electron velocity is not constant: $v_\parallel(t) = v_\parallel(0) + at$, where $a = v_\perp^2/2L$. If the wave progagates from the region of the stronger magnetic field, we have in the zeroth approximation $E_0 = e^{i\xi}$. Substituting $E_0(\xi)$ in (13) and using for the cyclotron frequency the expression $\omega_e(\varepsilon, z) = (\omega/\varepsilon)(1 - z/L)$, we find that the term of order $A^1$ in the expansion of the transmitted-wave amplitude is of the form

$$E_1^{tr} = \frac{A}{2}\int_{-\infty}^{\infty} d\xi_1\int d^3p\,\frac{p_\perp^2}{\varepsilon}\frac{\partial f}{\partial p^2}\int_{-\infty}^{0} d\zeta\,\exp\left\{-\frac{i\xi_1\zeta}{\varepsilon} + i\Psi(\zeta)\right\},$$

$$\Psi(\zeta) = \frac{\omega L}{c}\left(\frac{v_\parallel}{c} + \frac{1}{\varepsilon} - 1\right)\zeta + \frac{L\omega}{2c^2}\left(\frac{aL}{c} - \frac{v_\parallel}{\varepsilon}\right)\zeta^2 - \frac{aL^2\omega}{6c^3\varepsilon}\zeta^3. \tag{14}$$

It is convenient to begin the calculation of (14) with integration with respect to $d\xi_1$:

$$\int_{-\infty}^{\infty} d\xi_1\exp\left(-\frac{i\xi_1\zeta}{\epsilon}\right) = 2\pi\epsilon\delta(\zeta).$$

The integral with respect to $d\xi$ is then likewise easily calculated:

$$2\pi\int_{-\infty}^{0} d\zeta\, e^{i\Psi(\zeta)}\,\delta(\zeta) = \pi.$$

Finally, recognizing that

$$\int d^3p\, p_\perp^2 \frac{\partial f}{\partial p^2} = -1,$$

we obtain

$$E_1^{\rm tr} = -A\pi/2.$$

By successive iterations we find that the term of order $A^n$ in the expansion of the amplitude of the transmitted wave is (see Appeidix 2)

$$E_n^{\rm tr} = \frac{1}{n!}\left(-\frac{A\pi}{2}\right)^n.$$

Summing the terms of the infinite series, we obtain

$$E^{\rm tr} = e^{-\pi A/2}. \qquad (15)$$

For the first term of the $A$ expansion of the reflected-wave amplitude it follows from (13) that

$$E_1^{\rm ref} = \frac{A}{2}\int_{-\infty}^{\infty} d\xi_1\, e^{2i\xi_1} \int d^3p \frac{p_\perp^2}{\varepsilon}\frac{\partial f}{\partial p^2} \int_{-\infty}^{0} d\zeta \exp\left\{-\frac{i\xi_1\zeta}{\varepsilon} + i\Psi(\zeta)\right\}. \qquad (16)$$

Calculating, just as in (14), the integral with respect to $d\zeta_1$, we obtain

$$\int_{-\infty}^{\infty} d\xi_1 \exp\left\{\frac{i}{\varepsilon}(2\varepsilon-\zeta)\xi_1\right\} = 2\pi\varepsilon\delta(2\varepsilon-\zeta).$$

Since $\zeta<0$, the argument of the $\delta$ function does not vanish, so that $E_1^{\rm ref} = 0$. It can be similarly shown (Appendix 2) that $E_n^{\rm ref} = 0$ for any $n$. There is no reflected wave. In this case the coefficients $\tau, \rho$, and $\eta$ are determined by expressions (6) and (8).

For a wave incident on the resonance from the weaker magnetic field we have $E_0 = e^{-i\xi}$. The amplitude of the transmitted wave is calculated just as for the wave from the left, and is given by expression (15). In first-order approximation in $A$ we obtain from (13) for the amplitude of the reflected wave

$$E_1^{\rm ref} = \frac{A}{2}\int_{-\infty}^{\infty} d\xi_1\, e^{-2i\xi_1} \int d^3p \frac{p_\perp^2}{\varepsilon}\frac{\partial f}{\partial p^2} \times \int_{-\infty}^{0} d\zeta \exp\left\{-i\xi_1\frac{\zeta}{\varepsilon} + i\Psi_1(\zeta)\right\}, \qquad (17)$$

$$\Psi_1(\zeta) = -\frac{\omega L}{c}\left(\frac{v_\parallel}{c} - \frac{1}{\varepsilon} + 1\right)\zeta - \frac{L\omega}{2c^2}\left(\frac{aL}{c} + \frac{v_\parallel}{\varepsilon}\right)\zeta^2 - \frac{aL^2\omega}{6c^3\varepsilon}\zeta^3.$$

The integral with respect to $d\xi_1$ in (17) is equal to $2\pi\varepsilon\delta(2\varepsilon + \xi)$. Consequently the integral with respect to $d\xi$ is equal to $\exp[i\Psi_1(-2\varepsilon)]$. In the limit of weak relativism ($T/mc^2 \ll 1$) the integral over the momentum space is then determined by the region of values $|p/mc| \ll 1$ irrespectively of the value of $L\omega/2c$. In this case $\varepsilon \approx 1 + p^2/2m^2c^2$ and the remaining integral with respect to $d^3p$ can be easily calculated. As a result we obtain

$$E_1^{\rm ref} = \pi A\left(1 - i\frac{2T}{mc^2}\frac{\omega L}{c}\right)^{-1/2}\left(1 - i\frac{4T}{3mc^2}\frac{\omega L}{c}\right)^{-2}. \qquad (18)$$

Calculations of higher order in $A$ were unsuccessful. From (18) we obtain the reflection coefficient

$$\rho = \pi^2A^2(1+4\varkappa^2)^{-1/2}\left(1 + \frac{16}{9}\varkappa^2\right)^{-2}, \quad \varkappa = \frac{T}{mc^2}\frac{\omega L}{c}. \qquad (19)$$

We now take into account the deviation of $H(z)$ from linearity, i.e., we assume that

$$\omega_e(z) = \omega_e(0) + \omega_e'(0)z + \frac{1}{2}\omega_e''(0)z^2 \quad (\omega_e'(0) < 0).$$

Since the calculations are quite straightforward but unwieldy, we present only the final result:

$$\rho = \pi^2A^2\left[1 + 4\varkappa^2\left(1 + \frac{2}{3}\frac{\omega_e''(0)\omega}{(\omega_e'(0))^2}\right)^2\right]^{-1/2}\left(1 + \frac{16}{9}\varkappa^2\right)^{-2}. \qquad (20)$$

At $\kappa \ll 1$, Eq. (20) agrees with (9) accurate to terms of order $\Gamma^2$. At $\kappa \gtrsim 1$ the reflection coefficient turns out to be less than the value obtained from the adiabatic equation.

## CONCLUSIONS

Our analysis shows that cyclotron resonance in a nonuniform magnetic field is a nonlocal phenomenon, and to describe it completely it is necessary to solve the integral wave equation (12). In some cases, however, the interaction coefficients can be determined by using the approximate adiabatic equation. Thus, the adiabatic equation describes correctly a wave incident from the direction of the stronger magnetic field. For a wave incident from the weaker magnetic field it gives the correct transmission coefficient. At the same time the reflection coefficient determined from the adiabatic equation is correct (at least accurate to terms $\sim A^2$) only if $\omega LT/mc^3 \ll 1$. At $\omega LT/mc^3 \gtrsim 1$ the true reflection coefficient turns out to be smaller (and the absorption coefficient is correspondingly larger) than that obtained from the adiabatic equation, and in the limit $\omega^2_pL/\omega c \ll 1$ it is determined by Eq. (20).

The authors thank A. V.Timofeev for helpful discussions and for constant interest in the work.

## APPENDIX

If the distribution function of the electron longitudinal velocities is of the form

$$f(v_\parallel) = \delta(v_\parallel - v_0),$$

The wave equation for the extraordinary wave can be written in the form

$$E'' + E + 2iAp_0^2\int_{-\infty}^{\infty} d\xi_1\left[E(\xi_1) + i\frac{v_0}{c}E'(\xi_1)\right]K(\xi,\xi_1) = 0, \qquad (A.1.1)$$

where

$$K(\xi,\xi_1) = \theta\left(\frac{\xi-\xi_1}{v_0}\right)\exp(-i\,{\rm sign}(v_0)p_0^2(\xi_1^2-\xi^2)),$$

$2p_0^2 = c^2/\omega L\,|v_0|$, the prime denotes differentiation with respect to the argument, and $\theta(x)$ is the unit step function

$$\theta(x)=1, \quad x\geqslant 0,$$
$$\theta(x)=0, \quad x<0.$$

We consider the solution of Eq. (A.1.1), using as an example a beam moving from the right with velocity $-v_0$ (where $v_0>0$), and of a wave propagating in the same direction. The solution can be represented in the form of a contour integral

$$E(\xi)=\int\limits_C \frac{dt}{t^2+1}\exp\left[\xi t-\frac{it^2}{4p_0^2}+i\gamma_1\ln(t-i)-i\gamma_2\ln(t+i)\right]. \tag{A.1.2}$$

Here $\gamma_1=\frac{1}{4}A(1+v_0/c)$ and $\gamma_2=\frac{1}{4}A(1-v_0/c)$. The integrand should vanish at the ends of the integration contour. The choice of the contour is determined by the boundary conditions. For the considered case, the boundary condition is that a transmitted wave and a modulated beam can exist as $\xi\to-\infty$. This condition is satisfied by the contour shown in Fig. 1, case a.

At large values of $|\xi|$ the main contribution to the integral along the contour is made by the vicinity of the points $t_2=-i$, and $t_3=-2ip_0^2\xi$. The point $t_2$ corresponds to the transmitted wave, and the point $t_3$ to a modulated beam. The integral in the vicinity of the point $t_2$ reduces to a $\Gamma$ function. We take the contribution of the point $t_3$ into account by the saddle-point method:

$$E(\xi)\approx C_2+(1-e^{-2\pi\gamma_2})C_3, \tag{A.1.3}$$

where

$$C_2=i\exp\left[\frac{i}{4p_0^2}+i\gamma_1\ln 2-\pi(\gamma_2-{}^1/{}_2\gamma_1)\right]\operatorname{sh}(\pi\gamma_2)\Gamma(-i\gamma_2)|\xi|^{i\gamma_2}e^{-i\xi},$$

$$C_3=\frac{2p\pi^{1/2}\exp(-ip_0^2\xi^2-i\pi/4)}{(2p_0^2|\xi|)^2}.$$

On going to positive values of $\xi$ the integration of contour $C$ is deformed in accord with Fig. 1, case b. We now obtain for (A.1.2)

$$E(\xi)\approx\tilde{C}_2+(1-e^{-2\pi\gamma_2})C_1, \tag{A.1.4}$$

where

$$C_1\approx i\exp\left(\frac{i}{4p_0^2}-i\gamma_2\ln 2+\frac{\pi\gamma_2}{2}\right)\operatorname{sh}(\pi\gamma_1)\Gamma(i\gamma_1)\xi^{-i\gamma_1}e^{i\xi},$$

$$\tilde{C}_2\approx i\exp\left(\frac{i}{4p_0^2}+i\gamma_1\ln 2+\frac{\pi\gamma_1}{2}\right)\operatorname{sh}(\pi\gamma_2)\Gamma(-i\gamma_2)\xi^{i\gamma_2}e^{-i\xi},$$

$C_2$ and $C_1$ describe respectively the incident and reflected waves. From (A.1.3) and (A.1.4) we obtain for the coefficients $\tau$, $\rho$, and $\eta$

$$\tau=e^{-2\pi\gamma_2}, \quad \rho=\frac{\gamma_2}{\gamma_1}(1-e^{-2\pi\gamma_2})(1-e^{-2\pi\gamma_1}),$$

$$\eta=(1-e^{-2\pi\gamma_2})\left(1-\frac{\gamma_2}{\gamma_1}(1-e^{-2\pi\gamma_1})\right).$$

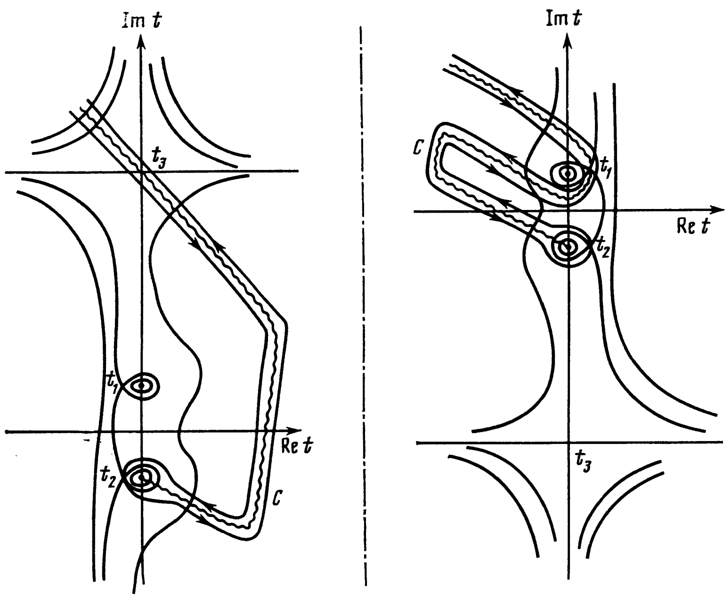
The other variants of the wave propagation and beam motion are considered similarly.

## APPENDIX 2

It follows from (13) that in integrals of mulitplicity $3n$ are encountered in the calculation of the $n$th iteration for the electric field of the transmitted or reflected wave. We shall show how to calculate such integrals. By way of example we consider incidence of the wave on the resonance from the direction of the stronger magnetic field. The transmitted-wave field is determined by integrals of the type

$$I=\int\limits_{-\infty}^{\infty}d\xi_1\,e^{-i\xi_1}\int d^3p_1\frac{p_{\perp 1}^2}{\varepsilon_1}\frac{\partial f(p_1)}{\partial p_1^2}\int\limits_{-\infty}^{0}d\zeta_1\exp\left\{-i\frac{\zeta_1\xi_1}{\varepsilon_1}-i\Omega_1\pm iq_1\right\}$$

$$\times\int\limits_{-\infty,(q_1)}^{(\infty),q_1}d\xi_2\,e^{\mp i\xi_2}\int d^3p_2\frac{p_{\perp 2}^2}{\varepsilon_2}\frac{\partial f(p_2)}{\partial p_2^2}\int\limits_{-\infty}^{0}d\zeta_2\exp\left(-i\frac{\zeta_2\xi_2}{\varepsilon_2}-i\Omega_2\right)$$




a    b

FIG. 1. Integration contour $C$ at $\xi<0$ (a) and $\xi>0$ (b). Thin lines—real- phase lines; wavy line—cut emerging from the branch point $t_2$.

$$\dots\exp\{\pm iq_{n-1}\}\int_{-\infty,(q_{n-1})}^{(\infty),q_{n-1}} d\xi_n\, e^{\mp i\xi_n}\int d^3p\frac{p_{\perp n}^2}{\varepsilon_n}\frac{\partial f(p_n)}{\partial p_n^2}$$

$$\times\int_{-\infty}^{0} d\zeta_n \exp\left\{-i\frac{\xi_n\zeta_n}{\varepsilon_n}-i\Omega_n+iq_n\right\},$$

where

$$\Omega_j=\left(1-\frac{1}{\varepsilon_j}\right)\frac{\omega L}{c}\zeta_j+\frac{a_jL^2\omega}{6c^3\varepsilon_j}\zeta_j^3+\frac{L\omega}{2c^2}\frac{v_\parallel}{\varepsilon_j}\zeta_j^2,$$

$$r_j=\frac{\omega}{c}\int_0^{L\zeta_j/c} dt\, v_{\parallel j}(t)=\frac{\omega L v_{\parallel j}}{c^2}\zeta_j+\frac{a_j\omega L^2}{2c^3}\zeta_j^2,$$

$$q_n=\xi_n+r_n.$$

Only one of these integrals differs from zero. It corresponds to a choice of the upper sign in the exponentials and of the integration limits without the parentheses.

The integrals with other combinations of signs in the exponentials and with other integration limits can be respresented in the form

$$I\sim\int_0^\infty dR\, R\delta(R)F(R), \tag{A.2.2}$$

with $F(R)$ finite as $R\to 0$. Obviously, in this case $I=0$.

The nonzero integral of the type (A.2.1) is calculated in the following manner:

$$I_0=\int_{-\infty}^{\infty} d\xi_1 H(\xi_1)\int_{-\infty}^{\xi_1} d\xi_2 H(\xi_2)\dots\int_{-\infty}^{\xi_{n-1}} d\xi_n H(\xi_n)$$

$$-\int_{-\infty}^{\infty} d\xi_1 H(\xi_1)\int_{q_1}^{\xi_1} d\xi_2 H(\xi_2)\int_{-\infty}^{q_2} d\xi_3 H(\xi_3)\dots\int_{-\infty}^{q_{n-1}} d\xi_n H(\xi_n)\dots$$

$$-\dots\int_{-\infty} d\xi_1 H(\xi_1)\int_{-\infty}^{\xi_1} d\xi_2 H(\xi_2)\dots\int_{-\infty}^{\xi_{n-2}} d\xi_{n-1} H(\xi_{n-1})\int_{q_{n-1}}^{\xi_{n-1}} d\xi_n H(\xi_n), \tag{A.2.3}$$

where

$$H(\xi)=\int d^3p\frac{p_\perp^2}{\varepsilon}\frac{\partial f(p)}{\partial p^2}\int_{-\infty}^{0} d\zeta \exp\left(-i\frac{\zeta\xi}{\varepsilon}-i\Omega+ir\right).$$

Starting with the second, all the integrals in (A.2.3) are zero [they can be reduced to the form (A.2.2)].

The first term in (A.2.3) can be easily calculated (see, e.g., Ref. 12) and is equal to

$$I_0=\frac{1}{n!}\left(\int_{-\infty}^{\infty} d\xi\, H(\xi)\right)^n. \tag{A.2.4}$$

Using the result of the calculation (14) we obtain

$$I_0=\pi^n/n! \tag{A.2.5}$$

The reflected-wave field is determined by integrals that differ from (A.2.1) in the presence of exp $(2i\zeta_1)$ in the integrand.This makes the integral with respect to $d\zeta_1$ equal to

$$\int_{-\infty}^{\infty} d\xi_1\exp\{i(2-\zeta_1/\varepsilon_1-\dots-\zeta_n/\varepsilon_n)\xi_1\}$$
$$=2\pi\delta(2-\zeta_1/\varepsilon_1-\dots-\zeta_n/\varepsilon_n). \tag{a}$$

The argument of the $\delta$ function is larger than zero, so that the subsequent integration yields zero for such integrals. There is no reflected wave.

A wave incident from the weaker magnetic field is treated similarly. In this case we find that the $n$th iteration for the field of the transmitted wave is also given by (A.2.5).


[1]K. G. Budden, Radiowaves in the Ionosphere, Cambridge U. Press, 1961.
[2]A. V. Timofeev and a. K. Nekrasov, Nucl. Fusion **10**, 377 (1970).
[3]A. V. Timofeev, Usp. Fiz. Nauk **110**, 329 (1973) [Sov. Phys. Usp. **16**, 445 (1974)].
[4]A. V.Timoveef and G. N. Chulkov, Fizika plazmy **7**, 129 (1981) (Sov. J. Plasma Phys. **7**, 73 (1981)].
[5]A. V. Zvonkov and a. V.Timoveev, *ibid.* 6, 1219 (1980) [**6**, 668 (1980)].
[6]M. Brambilla, Nucl. Fusion **9**, 343 (1969).
[7]H. L.Berk, C. W. Horton, M. N. Rosenbluth, D. E. Baldwin, and R. N.Sudan, Phys. Fluids **11**, 365 (1968).
[8]A. A. Vodyanitskiĭ, N. S. Erokhin, and S. S. Moiseev, Pis'ma Zh. Eksp. Teor.Fiz. **12**, 529 (1970) [JETP Lett. **12**, 372 (1970)].
[9]B. B. Kadomtsev, Kollektivnye yavleniya v plazme (Collective Phenomena in Plasma), Nauka, 1976, p. 150.
[10]
[11]L. D. Landau and E. M. Lifshitz, Statisticheskaya fizika (Statistical Physics), Nauka, 1964, p. 139 [Pergamon, 1967].
[12]A. A. Abrikosov, A. P. Gor'kov, and I. E. Dzyaloshinskiĭ, Metody kvantovoĭ teorii polya v statisticheskoĭ fizike (Quantum Theoretical Methods in Statistical Physics), Fizmatgiz, 1961, p. 70 [Pergamon, 1975].


Translated by J.G. Adashko