Field of a test body in a stream of charged particles

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It is shown that the potential of a field created by a test body in a relativistic stream of charged particles includes not only a component which decreases rapidly because of the Debye screening, but also an oscillatory component decreasing slowly with distance and associated with the magnetic interaction between currents. An exact analytic solution is obtained of the problem of a field formed by a charged current-carrying filament placed inside a plasma with a Bennett profile. The long-range field of the interaction between charges and such a current-carrying filament, and also of the interaction of filaments with one another helps to understand better the nature of filamentation of a plasma in high-current channels. The long-range magnetic interaction should affect not only the structure properties of a plasma but also its correlation properties, as well as its thermodynamic and kinetic characteristics.

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1. INTRODUCTION

An electrostatic field created in a plasma by a single charge decreases rapidly with distance and becomes very weak at distances comparable with the Debye-Hückel radius.[1] On the other hand, a magnetic field penetrates quite freely a plasma in a state of thermodynamic equilibrium. In this case the state of a system is governed by its total energy. The energy of individual charges is conserved in the course of their motion in a static magnetic field. Consequently, a test body carrying a current creates in an equilibrium classical plasma a magnetic field of exactly the same kind as in vacuum.

The situation changes if a plasma is not in a state of thermal equilibrium, for example, if a current flows in the plasma. In the presence of a current the electrons move as a whole (relative to ions) at a drift velocity v_0. If $v_0 \neq 0$, a plasma is not in a thermodynamic equilibrium state. Increase in deviation from equilibrium causes a magnetic field to influence the structure and properties of a plasma because the individual elements of the current interact with one another via the magnetic field they create themselves. The spatial distribution of such a plasma is established in such a way that the pressure gradients are balanced out by the electric and magnetic forces. Introduction of a test charge or a current element in this plasma disturbs the balance of forces. In general, the electrostatic and magnetostatic perturbations are described by a combined system of equations and cannot be separated from one another. The result of the relatively free penetration of a magnetic field into a plasma is a long-range component in the case of perturbations created in the plasma by test bodies; this component is additional to a short-range one describing the Debye screening. The magnetic interaction between charges is usually ignored because it is relativistically weak compared with the electric interaction. However, such neglect of the magnetic interaction is not permissible in problems dealing with dense relativistic beams or with other pinch systems characterized by high values of the current.

We shall consider the direction of the current to be special. The simplest geometry is that of a cylindrically symmetric plasma homogeneous along the current and in the azimuthal direction. The charge density then depends only on the distance r from the axis. A test body which does not disturb the symmetry of the problem is a thin filament located on the axis. In this geometry the coordinates z and φ are cyclic and the problem reduces to a system of ordinary differential equations describing the distribution of a field created in a plasma by a charged current-carrying filament. In the practically important case of the Bennett distribution the problem has an exact analytic solution and a complete investigation can be made for an arbitrary ratio of the drift velocity to the speed of light. In the case of a dense plasma when the scale of spatial changes in the field is small compared with the plasma radius we shall investigate the field of a filament for an arbitrary charge density distribution.

2. INITIAL EQUATIONS

Under steady-state conditions the fields created by plasma charges in a cylindrically symmetric current channel are described by the following electrostatic and magnetostatic equations:

$$\operatorname{div} \mathbf{E} = 4\pi\rho, \quad \operatorname{rot} \mathbf{H} = (4\pi/c)\,\mathbf{j}.$$

The charge ρ and the current $\mathbf{j}$ densities can be expressed in terms of the particle density n_α and in terms of the average velocities of their directional motion $\mathbf{v}_\alpha$:

$$\rho = \sum_\alpha e_\alpha n_\alpha, \quad \mathbf{j} = \sum_\alpha e_\alpha n_\alpha \mathbf{v}_\alpha. \tag{2.1}$$

The summation in Eq. (2.1) is carried out over the types of particle: α assumes the values of e and i for electrons and ions, respectively. The charge densities n_α are expressed in terms of the distribution functions f_α:

$$n_\alpha = \frac{g_\alpha}{(2\pi\hbar)^3} \int f_\alpha \, d^3p, \tag{2.2}$$

where g_α is the g factor. The charge distribution functions themselves depend on the fields $\mathbf{E}$ and $\mathbf{H}$ in the plasma. In

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general, these dependences can be found from the transport equation.

The presence of a current in a plasma means that the electron subsystem moves as a whole relative to the ion subsystem, so that $\mathbf{v}_0 = \mathbf{v}_e - \mathbf{v}_i \neq 0$. We shall assume that the drift velocity $\mathbf{v}_0$ is high compared with the dispersion (scatter) of the charge velocities:

$$v_0 \gg v_{T\alpha}, \quad \alpha = e, i. \tag{2.3}$$

In the case of a Boltzmann plasma the quantity $v_{T\alpha} = (2T_\alpha/m_\alpha)^{1/2}$ is the average velocity of thermal dispersal of charges.

The conditions of Eq. (2.3) mean that the electron and ion subsystems reach independently their states of equilibrium and this happens before the relaxation of the plasma as a whole. The effect is due to the rapid fall of the Coulomb cross sections on increase in the relative velocity of the colliding charges. The inequalities of Eq. (2.3) allow us to ignore the friction of electrons on ions, and to assume both subsystems to be in equilibrium in electric and magnetic fields created by the charges themselves. The equilibrium within each separate subsystem (of electrons and ions) does not mean at all that the plasma as a whole is also in equilibrium. Only in the absence of drift, when $\mathbf{v}_0 = 0$, do the temperatures of the subsystems become rapidly equalized and the plasma as a whole reaches an equilibrium state. Therefore, a convenient measure of the deviation of the state of a plasma from a thermodynamic equilibrium is provided by the ratio of the drift velocity to the speed of light $\beta = v_0/c$.

In this formulation of the problem the charge distribution functions can be found for arbitrary values of the ratio $\beta_\alpha = v_\alpha/c$, essentially without solving the transport equations. In fact, in the case of a subsystem in equilibrium the distribution function depends only on the additive integrals of motion. Under steady-state conditions in the presence of a cylindrical symmetry the time t and the coordinates z and φ are cyclic variables. The integrals of motion, which are canonically conjugate with these variables, are the total energy E_α, the projection of the generalized momentum P_z along the direction of the current, and the projection of the moment M_z. If the subsystem does not rotate as a whole, the distribution function is independent of M_z.

Belyaev and Budker[2] showed that the distribution function is a relativistic invariant. The relativistically invariant form of the dependence on the total energy and generalized momentum is

$$Z_\alpha = C_\alpha + (c/T_\alpha) P_i u_i = C_\alpha + (\gamma_\alpha/T_\alpha)(E_\alpha - \mathbf{P}_\alpha \mathbf{v}_\alpha). \tag{2.4}$$

Here, P_i is the four-vector of the energy-momentum; u_i is the four-vector of the velocity of a reference system linked to particles of type α, relative to the laboratory system; $\gamma_\alpha = (1-\beta_\alpha^2)^{-1/2}$ is the relativistic factor; T_α is the temperature of charges of type α, C_α is a relativistic scalar quantity related to the number of particles N_α per unit length of a discharge channel by the normalization condition

$$N_\alpha = 2\pi \int_0^\infty n_\alpha(r)\, r\, dr. \tag{2.5}$$

In the case of fermions the dependence of the distribution function on Z_α is of the form

$$f_\alpha = (1+\exp Z_\alpha)^{-1}. \tag{2.6}$$

The total energy of a charge of type α moving in an electromagnetic field is

$$E_\alpha = e_\alpha A_0 + \varepsilon_\alpha(\mathbf{P} - (e_\alpha/c)\mathbf{A}), \tag{2.7}$$

where A_0 and $\mathbf{A}$ are the scalar and vector potentials of the field and

$$\varepsilon_\alpha(\mathbf{p}) = (m_\alpha^2 c^4 + \mathbf{p}^2 c^2)^{1/2}$$

is the dependence of the energy of the charge on the kinematic momentum $\mathbf{p} = \mathbf{P} - (e_\alpha/c)\mathbf{A}$. The vector potential of the field for a cylindrically symmetric plasma can be selected conveniently in the following gauge: $\mathbf{A} = (0,0,A_z(r))$, so that $\operatorname{div} \mathbf{A} = 0$. It follows from Eqs. (2.4), (2.6), and (2.7) that the dependence of the distribution function f_α on the kinematic momentum $\mathbf{p}$ of the charges is

$$f_\alpha(\mathbf{p}) = \{1+\exp[C_\alpha + (\gamma_\alpha/T_\alpha)(e_\alpha(A_0 - \beta_\alpha A_z) + \varepsilon_\alpha(\mathbf{p}) - \mathbf{p}\mathbf{v}_\alpha)]\}^{-1}. \tag{2.8}$$

Substituting the distribution function (2.8) into Eq. (2.2) and integrating over the momenta, we obtain the following expression for the charge density as a functional of the field:

$$n_\alpha\{A\} = \frac{g_\alpha \gamma_\alpha}{6\pi^2}\left(\frac{m_\alpha c}{\hbar}\right)^3 B_0\left(\frac{m_\alpha c^2}{T_\alpha}, \frac{\gamma_\alpha U_\alpha}{m_\alpha c^2}\right). \tag{2.9}$$

Here,

$$U_\alpha = e_\alpha(A_0 - \beta_\alpha A_z) + (T_\alpha C_\alpha + m_\alpha c^2)/\gamma_\alpha;$$

U_α can be regarded as the "potential" of the force exerted on charges of type α by an electromagnetic field of the collective interaction. The function $B_0(x,y)$ in Eq. (2.9) can be represented in the form of a single integral:

$$B_0(x,y) = 3\int_1^\infty \frac{z(z^2-1)^{1/2}\,dz}{1+\exp[x(y+z-1)]}. \tag{2.10}$$

Noting that $\mathbf{E} = -\operatorname{grad} A_0$ and $\mathbf{H} = \operatorname{curl} \mathbf{A}$, we can use Eqs. (2.1) and (2.9) to reduce the electrostatic and magnetostatic equations to

$$\frac{1}{r}\frac{d}{dr} r \frac{dA_0}{dr} = -4\pi \sum_\alpha e_\alpha n_\alpha\{A\},$$
$$\frac{1}{r}\frac{d}{dr} r \frac{dA_z}{dr} = -\frac{4\pi}{c} \sum_\alpha e_\alpha v_\alpha n_\alpha\{A\}. \tag{2.11}$$

The expressions in Eq. (2.11) represent a complete system of equations which describe the electric and magnetic fields in a plasma.

3. BENNETT DISTRIBUTION

In the limiting case of the Boltzmann statistics, when unity in the denominator of Eq. (2.10) can be ignored compared with the exponential function, the charge density subjected to the normalization condition (2.5) is given by the explicit expression

$$n_\alpha=\frac{N_\alpha}{2\pi\nu_\alpha}\exp\left\{-\frac{\gamma_\alpha e_\alpha}{T_\alpha}(A_0-\beta_\alpha A_z)\right\},$$

where

$$\nu_\alpha=\int_0^\infty \exp\left[-\frac{\gamma_\alpha e_\alpha}{T_\alpha}(A_0-\beta_\alpha A_z)\right]r\,dr$$

is the normalization integral. In the case of a Boltzmann plasma the equations (2.11) governing the structure of the field are

$$\frac{1}{r}\frac{d}{dr}r\frac{dA_i}{dr}=-2\sum_\alpha\frac{e_\alpha N_\alpha\gamma_\alpha}{\nu_\alpha}u_{\alpha i}\exp\left\{-\frac{e_\alpha}{T_\alpha}A_k u_{\alpha k}\right\},$$

$$i,\ k=0,\ z. \qquad (3.1)$$

The notation in Eq. (3.1) is simplified by introducing relativistic two-component vectors of the field potential $A_i=(A_0,A_z)$ and of the velocity $u_{\alpha i}=(\gamma_\alpha,\gamma_\alpha\beta_\alpha)$. The scalar product of the vectors is $A_iB_i=A_0B_0-A_zB_z$; summation is understood when the Latin indices $i,k,\ldots$ are repeated.

In general, the state of a system is governed by six parameters:

$$T_\alpha,\ N_\alpha,\ \beta_\alpha,\quad \alpha=e,\ i. \qquad (3.2)$$

Since the electron and ion subsystems are in a thermodynamic equilibrium, the values of T_α and β_α are independent of the radius. A suitable selection of the reference system can ensure that one of the quantities β_α vanishes. We shall assume that ions considered as a whole are immobile in the laboratory reference system. Then, $\beta_i=0, \beta_e=\beta,\ldots$, and $\gamma_e=\gamma$.

A detailed investigation[3] has shown that an equilibrium in a Boltzmann plasma subject to just the collective forces of the interaction between charges is possible only if the energy of the magnetic compression $e_e^2N_e^2\beta^2/2$ of the charges, the energy of their electrostatic repulsion $1/2(\Sigma e_\alpha N_\alpha)^2$, and the energy of thermal motion in the radial direction $\Sigma N_\alpha T_\alpha/\gamma_\alpha$ are balanced out exactly:

$$\frac{1}{2}e^2N_e^2\beta^2=\frac{1}{2}\left(\sum_\alpha e_\alpha N_\alpha\right)^2+\sum_\alpha\frac{N_\alpha T_\alpha}{\gamma_\alpha}. \qquad (3.3)$$

Another important property of the system (3.1) is the scale invariance of the equations. In fact, if $A_i(r)$ represents solutions of Eq. (3.1), then the functions $A_i(br)$ are also solutions for any value of b. This means that if the parameters of the system (3.2) satisfy the energy balance equation of Eq. (3.3), a plasma equilibrium is possible for an arbitrary value of the discharge channel radius. In other words, the value of the pinch radius considered in the Boltzmann statistics approximation is governed not only by the conditions of a mechanical equilibrium in a plasma in its own field, but it depends also on finer properties of the system and on its interaction with the ambient medium and external fields.

In general, the system (3.1) can be integrated only partially. In the special case when the parameters of Eq. (3.2) satisfy the relationships

$$\sum_\alpha e_\alpha N_\alpha=\frac{2T_i}{e_i},\qquad e_eN_e\beta^2=2\left(-\frac{T_i}{e_i}+\frac{T_e}{\gamma e_e}\right), \qquad (3.4)$$

a system of two second-order equations in Eq. (3.1) degenerates into one second-order equation which can be integrated analytically. The solution first found by Bennett[4] is

$$n_\alpha^B(r)=(N_\alpha/\pi r_0^2)(1+r^2/r_0^2)^{-2},\quad \nu_\alpha=r_0^2/2,$$

$$A_0^B(r)=\frac{2T_i}{e_i}\ln\left(1+\frac{r^2}{r_0^2}\right), \qquad (3.5)$$

$$A_z^B(r)=\frac{2}{\beta}\left(\frac{T_i}{e_i}-\frac{T_e}{e_e\gamma}\right)\ln\left(1+\frac{r^2}{r_0^2}\right).$$

In the case of the Bennett distribution the dependences of the charge density on the radius are the same. Using the relationships in Eq. (3.4) and the condition $\beta_i=0$, we find that out of the six parameters in Eq. (3.2), three of them

$$\beta,\quad G_e=e_e^2N_e\gamma/T_e,\quad G_i=e_i^2N_i/T_i, \qquad (3.6)$$

remain arbitrary. When the relationships of Eq. (3.4) are obeyed, the energy balance condition (3.3) is satisfied identically.

4. FIELD OF A TEST FILAMENT

We shall assume that a current-carrying thin charged filament is introduced into a plasma whose structure is described by the system (2.11). The perturbation resulting from the presence of this filament does not disturb the symmetry of the system and only the quantitative values of A_0 and A_z are affected. We shall assume that the filament current and charge are sufficiently small and we shall linearize the system (2.11) with respect to perturbations $A_0^{(1)}$ and $A_z^{(1)}$ of the field potentials. The index "(1)" indicates that these are perturbations of the potentials and we shall omit this index in future. It is convenient to represent the linearized electrostatic and magnetostatic equations in the tensor form:

$$\frac{1}{r}\frac{d}{dr}r\frac{dA_i}{dr}=a_{ik}A_k; \qquad (4.1)$$

the tensor a_{ik} is determined by the unperturbed values of the potential of the field $A_i^{(0)}$. Its components are of the form

$$a_{00}=\sum_\alpha\varkappa_\alpha^2,\quad a_{0z}=-a_{z0}=-\sum_\alpha\varkappa_\alpha^2\beta_\alpha,\quad a_{zz}=-\sum_\alpha\varkappa_\alpha^2\beta_\alpha^2,$$

$$\varkappa_\alpha^2=\frac{4\pi e_\alpha^2}{T_\alpha}\frac{g_\alpha\gamma_\alpha}{(2\pi\hbar)^3}\int f_\alpha(1-f_\alpha)\,d^3p$$

$$=\frac{2}{3\pi}\frac{e_\alpha^2\gamma_\alpha^2}{m_\alpha c^2}\left(\frac{m_\alpha c}{\hbar}\right)^3B_1\left(\frac{m_\alpha c^2}{T_\alpha},\ \frac{\gamma_\alpha U_\alpha^{(0)}}{m_\alpha c^2}\right).$$

The repeated Latin indices indicate summation; in Eq. (4.1), we have

$$a_{ik}A_k=a_{i0}A_0+a_{iz}A_z.$$

The function B_1 is the derivative of the function B_0 of Eq. (2.8):

$$B_1(x,y)=\frac{\partial B_0}{\partial y}=\frac{3x}{2}\int_1^\infty\frac{z(z^2-1)^{1/2}\,dz}{\mathrm{ch}[x(y+z-1)/2]}.$$

In the case of a Boltzmann plasma we have $B_1(x,y)=xB_0(x,y)$, so that

$$\varkappa_\alpha^2=\frac{4\pi e_\alpha^2 n_\alpha(r)\gamma_\alpha}{T_\alpha}=\frac{2}{\nu_\alpha}\frac{e_\alpha^2 N_\alpha\gamma_\alpha}{T_\alpha}\exp\left\{-\frac{e_\alpha}{T_\alpha}A_i u_{\alpha i}\right\}.$$

In the special case of the Bennett distribution of Eq. (3.5), we obtain

$$\varkappa_{B\alpha}^2=(4G_\alpha/r_0^2)(1+r^2/r_0^2)^{-2}. \tag{4.2}$$

If a plasma is compressed until it becomes strongly degenerate, the function $x\,\mathrm{ch}^{-1}[x(y+z-1)/2]$ is equivalent to the δ function:

$$\lim_{x\to\infty}\frac{x}{\mathrm{ch}[x(y+z-1)/2]}=\frac{1}{\pi}\delta(y+z-1).$$

In the case of a strong degeneracy the function B_1 is

$$B_1(x,y)=\begin{cases}0, & y>0,\\ (3/2\pi)(1+|y|)[(1+|y|)^2-1]^{1/2}, & y<0.\end{cases}$$

We shall assume, for simplicity, that the test filament is located along the axis of flow of the current $r=0$. Near the filament the field perturbations increase logarithmically. Denoting the charge per unit length of the filament by σ_0 and the current along the filament by $I=c\sigma_z$, we find that the boundary conditions for the potentials in the limit $r\to 0$ can be written in the form

$$r(dA_i/dr)=-2\sigma_i,\quad r\to 0. \tag{4.3}$$

Introduction into a plasma of a charged current-carrying filament alters the total current and charge per unit length of the system as a whole. However, outside the system the field is governed by the total values of the charge and current. Therefore, the boundary condition imposed on perturbations of the potentials outside the plasma are also of the form

$$r(dA_i/dr)=-2\sigma_i,\quad r\to\infty. \tag{4.4}$$

The system (4.1) subject to the boundary conditions (4.3) and (4.4) describes completely the field perturbations induced in a plasma by a charged current-carrying filament. In general, electric and magnetic fields are mutually coupled. Only in the limiting case of $\beta\to 0$, when there is no current in a plasma, can we find a reference system in which only one element of the tensor a_{ik}, namely a_{00}, differs from zero. In this case the system (4.1) splits into two independent equations. One of them describes the Debye screening:

$$\frac{1}{r}\frac{d}{dr}r\frac{dA_0}{dr}=a_{00}A_0,$$

and the other represents free penetration of a magnetic field into a plasma:

$$\frac{1}{r}\frac{d}{dr}r\frac{dA_z}{dr}=0.$$

5. EXACT SOLUTION IN THE CASE OF THE BENNETT DISTRIBUTION

The system (4.1) with the r-dependent components of the tensor a_{ik} cannot be integrated analytically. In the present section we shall consider perturbations of a plasma with the Bennett profile for which we can obtain an exact solution of this system of equations, which we can then investigate fully.

In the case of the Bennett distribution of Eq. (3.5) the system (4.1) becomes

$$\hat{L}A_i=c_{ik}A_k, \tag{5.1}$$

where

$$\hat{L}=(d/d\zeta)(1-\zeta^2)(d/d\zeta),$$

and instead of r, we shall now introduce a new variable

$$\zeta=\mathrm{th}\ln(r/r_0)=-[1-(r/r_0)^2][1+(r/r_0)^2]^{-1}.$$

The components of the tensor c_{ik} are independent of the coordinate:

$$c_{00}=\sum_\alpha G_\alpha,\quad c_{0z}=-c_{z0}=-G_e\beta,\quad c_{zz}=-G_e\beta^2. \tag{5.2}$$

The parameters G_α are given by Eq. (3.6). Their values are not limited by the Bennett equilibrium conditions (3.4) and can be arbitrary.

The eigenfunctions of the operator $\hat{L}$

$$\hat{L}F+\nu(\nu+1)F=0, \tag{5.3}$$

are the Legendre functions

$$F=\alpha P_\nu(\zeta),$$

defined in the interval $(-1,1)$. When r is increased from zero to infinity, the quantity ζ varies from -1 to $+1$. The operator $\hat{L}$ is invariant under the change of the sign of ζ. Therefore, $P_\nu(-\zeta)$ is a solution of (5.3), and $P_\nu(\zeta)$ and $P_\nu(-\zeta)$ are linearly independent.

It is natural to seek a solution of Eq. (5.1) in the form

$$A_i=\alpha_i P_\nu(\zeta). \tag{5.4}$$

Substituting Eq. (5.4) into Eq. (5.1) and allowing for Eq. (5.3), we obtain a system of homogeneous linear algebraic equations for the two-component vector α_i:

$$(c_{ik}+\lambda^2\delta_{ik})\alpha_k=0,\quad \lambda^2=\nu(\nu+1). \tag{5.5}$$

The condition of solubility of the system (5.5) is

$$\det(c_{ik}+\lambda^2\delta_{ik})=0$$

and it defines two eigenvalues λ^2:

$$\lambda_{1,2}^2=-[{}^1/_2S\pm({}^1/_4S^2+|\Delta|)^{1/2}]. \tag{5.6}$$

Here,

$$S=c_{00}+c_{zz}=\mathrm{Sp}\,c_{ik}=G_i+G_e/\gamma^2$$

is the trace of the matrix c_{ik} and Δ is the determinant of the matrix:

$$\Delta=\det c_{ik}=c_{00}c_{zz}-c_{0z}c_{z0}=-G_eG_i\beta^2<0.$$

The determinant Δ is negative because the numbers of particles N_α and their temperatures T_α are positive quantities. It vanishes when electrons cease to drift relative to ions, and the plasma reaches an equilibrium. It follows from Eq. (5.6) that $\lambda_1^2<0$ and $\lambda_2^2>0$. The eigenvalues ν are defined in terms of $\lambda_{1,2}$ in accordance with Eq. (5.5) and are given by

$$\nu_1=-{}^1/_2+({}^1/_4+\lambda_2^2)^{1/2},\quad \nu_2=-{}^1/_2+({}^1/_4-|\lambda_1^2|)^{1/2}. \tag{5.7}$$

In this definition the eigenvalue ν_1 is real, whereas ν_2 is complex for $|\lambda_1|>1/2$.

The general solution of the system (5.1) can be represented conveniently in the form

$$A_i=\alpha_i P_{\nu_1}(\zeta)+\beta_i P_{\nu_1}(-\zeta)+\gamma_i P_{\nu_2}(\zeta)+\delta_i P_{\nu_2}(-\zeta), \tag{5.8}$$

where the vectors α_i and β_i satisfy the expressions in Eq. (5.5) with $\lambda^2=\lambda_2^2$, whereas γ_1 and δ_i satisfy the expressions in Eq. (5.5) with $\lambda^2=\lambda_1^2$. In view of the homogeneity and linearity of the equations in the system (5.5), these vectors are given by the equations only to within constant factors. The four constant factors are found by applying the boundary conditions: two near the filament (4.3) and two outside the plasma (4.4).

The Legendre function $P_\nu(\zeta)$ is regular for $\zeta=1$ and we have $P_\nu(1)=1$, and it also has a logarithmic singularity in the limit $\zeta\to-1$. The absence of a singularity of $P_\nu(\zeta)$ at $\zeta=1$ is the justification for the selection of the general solution in the form of Eq. (5.8). For this general solution the vectors α_i and γ_i are governed by the boundary conditions of Eq. (4.3) near the filament irrespective of the vectors β_i and δ_i, which are obtained using the boundary conditions (4.4) outside the plasma.

In the vicinity of this singularity at $\zeta=-1$ the Legendre function has the following asymptote[5]:

$$P_\nu(\zeta)=\frac{\sin\nu\pi}{\pi}\left[\ln\frac{1+\zeta}{2}+\gamma+2\psi(\nu+1)+\pi\operatorname{ctg}\nu\pi\right],$$

$$\zeta\to-1.$$

In the direct vicinity of the filament, i.e., at $r\to0$, we have

$$r(dA_i/dr)=(2/\pi)\,(\alpha_i\sin\nu_1\pi+\gamma_i\sin\nu_2\pi),\qquad r\to0. \tag{5.9}$$

Outside the plasma in the limit $r\to\infty$ the logarithmic derivative of the field potential is

$$r(dA_i/dr)=-(2/\pi)\,(\beta_i\sin\nu_1\pi+\delta_i\sin\nu_2\pi),\quad r\to\infty. \tag{5.10}$$

Comparing Eqs. (5.9) and (5.10) with Eqs. (4.3) and (4.4), we obtain the following system of equations for the vector coefficients in Eq. (5.8):

$$\begin{gathered}\alpha_i\sin\nu_1\pi+\gamma_i\sin\nu_2\pi=-\pi\sigma_i,\\ (c_{ik}+\lambda_2^2\delta_{ik})\,\alpha_k=0,\quad (c_{ik}+\lambda_1^2\delta_{ik})\,\gamma_k=0.\end{gathered} \tag{5.11}$$

The vectors β_i and δ_i are identical, apart from the sign, with α_i and γ_i:

$$\beta_i=-\alpha_i,\quad \delta_i=-\gamma_i.$$

Multiplying the first equation in the system (5.11) by the matrix c_{ik} on the left and then using the second and third equations of the system (5.11), we find that

$$-\lambda_2^2\alpha_i\sin\nu_1\pi-\lambda_1^2\gamma_i\sin\nu_2\pi=-\pi c_{ik}\sigma_k.$$

Using this relationship and the first equation in the system (5.11), we obtain the vectors α_i and γ_i in the form

$$\begin{gathered}\alpha_i=\frac{\pi}{\sin\nu_1\pi}\,\frac{c_{ik}+\lambda_1^2\delta_{ik}}{\lambda_2^2-\lambda_1^2}\,\sigma_k,\\ \gamma_i=-\frac{\pi}{\sin\nu_2\pi}\,\frac{c_{ik}+\lambda_2^2\delta_{ik}}{\lambda_2^2-\lambda_1^2}\,\sigma_k.\end{gathered} \tag{5.12}$$

It therefore follows that the field of a test filament in a plasma with the Bennett profile is given by the formula

$$A_i(r)=\frac{\pi}{\lambda_2^2-\lambda_1^2}\left\{\frac{P_{\nu_1}(\zeta)-P_{\nu_1}(-\zeta)}{\sin\nu_1\pi}(c_{ik}+\lambda_1^2\delta_{ik})-\frac{P_{\nu_2}(\zeta)-P_{\nu_2}(-\zeta)}{\sin\nu_2\pi}(c_{ik}+\lambda_2^2\delta_{ik})\right\}\sigma_k,\qquad \zeta=\operatorname{th}\ln(r/r_0). \tag{5.13}$$

If $|\lambda_1|>1/2$, the index ν_2 is complex:

$$\nu_2=-{}^1/_2+iu,\quad u=(|\lambda_1^2|-{}^1/_4)^{1/2}. \tag{5.14}$$

The Legendre functions with the complex index of Eq. (5.14), defined in the interval $(-1,1)$, are known as the cone functions. An important property of these functions is that for real u, the functions $P_{-1/2+iu}(\zeta)$ are real and positive.

We shall now consider the changes which occur in Eq. (5.13) in the limit of a small number of particles in a plasma when $G_e\gg1$ and $G_i\ll1$. If the parameters G_α of Eq. (3.6) are small, then the components of the tensor c_{ik} of Eq. (5.2) are also small. The eigenvalues $\lambda_{1,2}^2$ of Eq. (5.6) and, therefore, the Legendre-function indices $\nu_{1,2}$ of Eq. (5.7) are also small. The transition to the limit $\nu\to0$ occurs in accordance with the formula

$$\lim_{\nu\to0}\frac{\pi}{\sin\nu\pi}[P_\nu(\zeta)-P_\nu(-\zeta)]=\ln\frac{1+\zeta}{1-\zeta}. \tag{5.15}$$

Using the relationship (5.15), we can reduce Eq. (5.13) to

$$A_i(r)=-2\sigma_i\ln(r/r_0),\qquad G_e,\ G_i\ll1. \tag{5.16}$$

The field given by Eq. (5.16) does not differ from the field created by a charged current-carrying filament in vacuum. The conditions $G_e, G_i\ll1$ mean that the radius of a plasma channel r_0 is much less than the Debye radius

$$r_d=\Big(\sum_\alpha \varkappa_\alpha^2\Big)^{-1/2}.$$

In fact, using Eq. (4.2), we find that

$$r_0/r_d=2(G_e+G_i)^{1/2}/(1+r^2/r_0^2)\ll1,\qquad G_e,\ G_i\ll1.$$

Under these conditions there is no screening of the field by the plasma.

6. FIELD NEAR A FILAMENT IN A DENSE PLASMA

We shall now consider a high-density plasma, such that the parameters G_α of Eq. (3.6) are large:

$$G_e,\ G_i\gg1. \tag{6.1}$$

We shall assume that β is arbitrary: $\beta\sim1$. Under the conditions of Eq. (6.1) the Debye radius is small compared with the plasma radius:

$$r_0/r_d\sim(G_e+G_i)^{1/2}\gg1.$$

In the range defined by Eq. (6.1) subject to the condition $\beta \sim 1$ the absolute values of the Legendre-function indices are large: $\nu_1 \gg 1, u \gg 1$ [see Eq. (5.14)]. In the case of large indices it is convenient to use the following asymptotic formulas for the Legendre functions:

$$P_\nu(\cos\theta) = \begin{cases} J_0((\nu+{}^1/_2)\theta), & \theta \ll 1, \quad \nu \gg 1, \\ \sin\pi\nu N_0((\nu+{}^1/_2)(\pi-\theta)) + \cos\nu\pi J_0((\nu+{}^1/_2)(\pi-\theta)), & \pi-\theta \ll 1, \quad \nu \gg 1, \end{cases} \tag{6.2}$$

$$P_{-{}^1/_2+iu}(\cos\theta) = \begin{cases} I_0(u\theta), & \theta \ll 1, \quad u \gg 1, \\ \pi^{-1}e^{u\pi}K_0(u(\pi-\theta)), & \pi-\theta \ll 1, \quad u \gg 1. \end{cases} \tag{6.3}$$

Here, $J_0(x)$ and $N_0(x)$ are the Bessel functions of the first and second kind, $I_0(x)$ is a Bessel function with an imaginary argument, $K_0(x)$ is the Macdonald function. At distances from the filament small compared with the plasma radius, $r \ll r_0$, we can use the asymptotes (6.2) and (6.3) to reduce Eq. (5.13) for the field to

$$A_i(r) = \frac{\pi}{\lambda_2^2 - \lambda_1^2} \times \left\{ (c_{ik}+\lambda_1^2\delta_{ik}) \left[N_0\left(\frac{2\nu_1 r}{r_0}\right) - \operatorname{tg}\frac{\pi\nu_1}{2} J_0\left(\frac{2\nu_1 r}{r_0}\right) \right] - \frac{2}{\pi}(c_{ik}+\lambda_2^2\delta_{ik}) K_0\left(\frac{2|\lambda_1| r}{r_0}\right) \right\} \sigma_k, \quad r \ll r_0. \tag{6.4}$$

Near the filament over distances $r \ll r_0$ the filament field consists of two components. The last term in the braces of Eq. (6.4) decreases exponentially on increase in r and describes the Debye screening process. However, in addition to the exponentially decreasing component, the filament field in the plasma also has a slowly decreasing and oscillating component [represented by the first term in the braces of Eq. (6.4)]. If $\beta \sim 1$ and $G_e \sim G_i \gg 1$, the parameters ν_1 and $|\lambda_1|$ are both large and of the same order of magnitude: they are of the order of the ratio of the plasma radius r_0 to the Debye radius r_d in the region where the filament is located. Then, at distances $r \ll r_0$ the characteristic spatial scale of both screening and oscillations is of the order of the Debye radius.

In the vicinity of the filament the field is largely determined by the local properties of the plasma. In fact, at distances from the filament which are small compared with the plasma radius, $r \ll r_0$, the plasma parameters vary only slightly and we can assume in the expressions in Eq. (4.1) that $a_{ik}(r) = a_{ik}(0)$. The eigenfunctions of the Laplace operator

$$\hat{l} = \frac{1}{r}\frac{d}{dr}r\frac{d}{dr}$$

are the cylindrical functions $Z_0(\mu r)$:

$$\hat{l}Z_0(\mu r) + \mu^2 Z_0(\mu r) = 0.$$

Substituting $A_i = \alpha_i Z_0(\mu r)$ in Eq. (4.1), we obtain a system of algebraic equations

$$a_{ik}(0)\alpha_k + \mu^2\alpha_i = 0.$$

These equations are governed by the properties of the plasma in the direct vicinity of the filament and, therefore, they are valid for an arbitrary radial profile of the charge density.

The eigenvalues μ^2 are given by the same formulas (5.6), but now S and Δ are the trace and determinant of the matrix $a_{ik}(0)$. The general solution of the system (4.1) near a filament for a plasma of arbitrary profile can be represented in the form

$$A_i(r) = \alpha_i K_0(|\mu_1| r) + \beta_i I_0(|\mu_1| r) + \gamma_i N_0(\mu_2 r) + \delta_i J_0(\mu_2 r), \quad r \ll r_0. \tag{6.5}$$

The vector coefficients α_i and γ_i in Eq. (6.5) are determined by the boundary conditions (4.3) on the filament:

$$\alpha_i = \pi(\mu_2^2 - \mu_1^2)^{-1}(a_{ik} + \mu_1^2\delta_{ik})\sigma_k,$$

$$\gamma_i = -2(\mu_2^2 - \mu_1^2)^{-1}(a_{ik} + \mu_2^2\delta_{ik})\sigma_k.$$

The coefficients β_i and δ_i can be determined using the boundary conditions outside the plasma (4.4) if we know the nature of the solution not only for $r \ll r_0$ but for any value $r \sim r_0$. As far as the vector β_i is concerned, we find that under the conditions (6.1) it is exponentially small and can be assumed to be zero. In fact, $I_0(|\mu_1| r)$ is a function increasing exponentially when r is increased. Therefore, for $r \sim r_0$ we have $I_0(|\mu_1| r) \sim \exp(|\mu_1| r_0)$. Since the filament charge and current are finite, it follows from Eq. (4.4) that $\beta_i \exp(|\mu_1| r_0) \sim \sigma_i$ and hence $\beta_i \sim \sigma_i \exp(-|\mu_1| r_0)$.

The coefficient δ_i in Eq. (6.5) cannot be determined simply from the local properties of the plasma near the filament even subject to the condition (6.1). This coefficient is found by applying the boundary conditions outside the plasma. It is not exponentially small and it depends on the fine details of the structure of the whole plasma. In the special case of the Bennett distribution, we have

$$|\mu_1| = 2|\lambda_1|/r_0, \quad \mu_2 = 2\lambda_2/r_0.$$

Comparing Eqs. (6.5) and (6.4), we find the vector δ_i for the Bennett profile:

$$\delta_i^B = -\gamma_i \operatorname{tg}({}^1/_4\pi\mu_2 r_0).$$

Under the conditions of Eq. (6.1) the vector δ_i is a rapidly oscillating function of the plasma parameters.

In the limiting case of a dense plasma defined by Eq. (6.1) the vector δ_i should be regarded as a random quantity. The long-range component of the field of a test filament in a dense plasma seems to consist of two parts: a fixed part $\gamma_i N_0(\mu_2 r)$ and a term $\delta_i J_0(\mu_2 r)$ with a random amplitude, whose average value is zero. If the inequalities of Eq. (6.1) are obeyed by such a large margin that fluctuations of the quantities G_α are also large, then fluctuations of the parameter $\mu_2 r_0$ are large compared with the period of the tangent. Then, Eq. (6.5) should be averaged over the plasma fluctuations. The averaged expression for the field in the vicinity of the filament, subject to the conditions of Eq. (6.1) fulfilled by a large margin, no longer contains the term $\delta_i J_0(\mu_2 r)$.

If the inequalities of Eq. (6.1) are not too strong, so that the variance of the parameter $\mu_2 r_0$ is small, there is no justification for the averaging procedure. In this case the filament field depends strongly on its position in the plasma and on the parameters of the plasma itself.

If $\beta \sim 1$, the components of the tensors $c_{ik} + \lambda_1^2\delta_{ik}$ and $c_{ik} + \lambda_2^2\delta_{ik}$ in Eq. (6.4) are of the same order of magnitude. A typical oscillation period corresponding to $\beta \sim 1$ is of the order of the Debye screening radius. It follows that in the case of a relativistic electron beam the field of a charged body inside the beam is not simply Debye-screened, but penetrates to a considerable distance inside the plasma and it also oscillates.

When β decreases, the period of the spatial oscillations of the long-range component of the field increases,

$$|\mu_2|^{-1} \sim r_d/\beta,$$

and the relative amplitude of the oscillating term of the field decreases.

We shall now consider how the filament field changes as β approaches zero and a plasma assumes an equilibrium state. If $G_\alpha \gg 1$ and $\beta \ll 1$, the determinant Δ becomes much smaller than the trace S. We then have

$$\lambda_1^2 \approx -S, \quad \lambda_2^2 \approx |\Delta|/S, \quad \beta \ll 1,$$

where $|\lambda_1^2| \gg 1$ and $\lambda_2^2 \ll 1$. The Legendre-function indices are then $\nu_1 = \lambda_2^2 \ll 1$ and $\nu_2 = -1/2 + i|\lambda_1|$. In the first term of Eq. (5.13) we can go to the limit (5.15), whereas in the second we can use the asymptotic formula (6.3). Bearing in mind that $\sin \nu_2\pi \approx -1/2 \exp(|\lambda_1|\pi)$, we obtain the following expression for the filament field

$$A_i(r) = \frac{2}{\lambda_2^2 - \lambda_1^2} \left\{ (c_{ik} + \lambda_1^2\delta_{ik}) \ln \frac{r}{r_0} + (c_{ik} + \lambda_2^2\delta_{ik}) \left[K_0\left(\frac{2|\lambda_1|r}{r_0}\right) - K_0\left(\frac{2|\lambda_1|r_0}{r}\right) \right] \right\} \sigma_k, \quad G_\alpha \gg 1, \quad \beta \ll 1.$$

Inside an electron beam at distances $r \ll r_0$ we can ignore the second term in the brackets. We then finally have

$$A_i(r) = \frac{2}{\lambda_2^2 - \lambda_1^2} \times \left\{ (c_{ik} + \lambda_1^2\delta_{ik}) \ln \frac{r}{r_0} + (c_{ik} + \lambda_2^2\delta_{ik}) K_0\left(\frac{2|\lambda_1|r}{r_0}\right) \right\} \sigma_k,$$

$$0 < r \leqslant r_0, \quad G_\alpha \gg 1, \quad \beta \ll 1. \tag{6.6}$$

If $\beta \ll 1$, the matrix elements in Eq. (6.6) are

$$c_{ik} + \lambda_1^2\delta_{ik} = \begin{pmatrix} G_e\beta^2 & -G_e\beta \\ G_e\beta & -(G_e + G_i) \end{pmatrix},$$

$$c_{ik} + \lambda_2^2\delta_{ik} = \begin{pmatrix} G_e + G_i + |\Delta|/S & -G_e\beta \\ G_e\beta & -G_e\beta^2 + |\Delta|/S \end{pmatrix}.$$

In the limit $\beta \to 0$, we obtain

$$c_{ik} + \lambda_1^2\delta_{ik} = \begin{pmatrix} 0 & 0 \\ 0 & -(G_e + G_i) \end{pmatrix} \tag{6.7}$$

$$c_{ik} + \lambda_2^2\delta_{ik} = \begin{pmatrix} G_e + G_i & 0 \\ 0 & 0 \end{pmatrix} \quad \beta \to 0.$$

Noting that $\lambda_2^2 - \lambda_1^2 = G_e + G_i$ in the limit $\beta \to 0$, and substituting Eq. (6.7) into Eq. (6.6), we find that

$$A_0(r) = 2\sigma_0 K_0(2|\lambda_1|r/r_0), \quad A_z(r) = -2\sigma_z \ln(r/r_0), \quad \beta \to 0. \tag{6.8}$$

The first formula in Eq. (6.8) describes the Debye screening of the filament charge, whereas the second gives the distribution of the magnetic field created by the filament current. This magnetic field does not differ in its effects from the current in vacuum, which is to be expected because in the limit $\beta = 0$ a plasma assumes an equilibrium state. An equilibrium plasma is penetrated freely by a magnetic field. The distribution function of an equilibrium plasma is determined by the total energy of the charges. However, the energy of a charge moving in a static magnetic field is independent of the field.

In the case of metals the high density of conduction electrons $n \sim 10^{22}$ cm^{-3} usually makes the drift electron velocity very small. We find that $j = env_0 \sim 10^3 v_0$. Even at a relatively high current density $j \sim 10^6$ A/cm^2, the drift velocity is only $v_0 \sim 10^3$ cm/sec. In the case of metals these effects can appear only in the presence of extremely high currents, for example, those encountered in experiments involving exploding wires. Oscillations of the field of a test body observed at large distances are characteristic of relativistic streams of charged particles and also of pinch systems in which the magnetic compression by the magnetic field of the current itself plays the dominant role.

7. CORRELATION PROPERTIES OF CHARGED-PARTICLE STREAMS

Experiments have shown that the plasma in a high-current discharge does not usually represent one structure but consists of a number of thin channels.[6] An investigation of the correlation properties of plasma streams makes it possible to understand the nature of filamentation of a high-current discharge into a system of such channels.

We shall assume that in a high-current channel the action of the excess force of the magnetic self-compression and rapid radiation cooling causes part of the plasma stream to contract into a thin filament. We shall consider the field created in the plasma by this filament. This field determines the correlation properties of the individual plasma charges relative to this filament.

If the charge per unit length of the filament and the current in the filament are small compared with the total charge and current in the system, then the filament field satisfies the system of equations (4.1). In the filament itself at $r \to 0$ we still have the boundary condition (4.3). In contrast to a test body, in the present case the average field of the collective interaction and the average plasma density should be considered for the system as a whole, including the charge and current of the part of the plasma that has contracted into the investigated filament. In this case the filament charge and current are not introduced into the plasma from outside, but are the properties of the plasma itself. The field outside the discharge, governed by the total values of the charge and current, is not affected by the filament contraction. Therefore, in an investigation of the correlation properties of the plasma the boundary condition outside the discharge is of the form

$$r(dA_i/dr) = 0, \quad r \to \infty. \tag{7.1}$$

In the case when the contraction into a filament occurs in a plasma with the Bennett profile, the general solution for the field still has the form (5.8). The vectors α_i and γ_i are given by the formulas in Eq. (5.12) and we also have $\beta_i = \delta_i = 0$ in accordance with the boundary condition (7.1).

Under the conditions of Eq. (6.1), when the Debye radius is small compared with the plasma radius, the filament field is given by Eq. (6.5) and the condition (7.1) yields the following value for the vector δ_i in Eq. (6.5): $\delta_i = \gamma_i \operatorname{ctg} \nu_1 \pi$. The average value of the vector δ_i vanishes and when the conditions of Eq. (6.1) are satisfied by a large margin, the field should be averaged over fluctuations. In the intermediate range $r_d \ll r \ll r_0$ the filament field averaged over the plasma fluctuations is

$$\langle A_i(r)\rangle = \frac{\pi}{\lambda_2^2 - \lambda_1^2} N_0 \left(\frac{2\nu_1 r}{r_0}\right) (c_{ik} + \lambda_1^2 \delta_{ik}) \sigma_k, \quad r_d \ll r \ll r_0.$$

When β or ν_1 decreases, the period of the spatial oscillations of the field increases and in the range $\nu_1 \sim 1$ it becomes of the order of the plasma channel radius r_0. Oscillations then disappear and the division of the field into the local and "random" components becomes meaningless. It is important to note that the filament field corresponding to $\nu_1 \sim 1$ is an oscillatory function of r. The plasma charges moving in this field may be captured by the filament and go over to the region of finite radial motion as a result of energy losses due to emission of radiation. This process may cause splitting of the plasma into concentrated tubular structures which are located at minima of the filament potential. These structures can then split into separate filaments in such a way that the mutual energy of their correlation interaction is minimal.

Filamentation of a strong current into separate channels is manifested by an instability of a system of this kind when radial plasma oscillations of wavelengths $\lambda > \lambda_{cr} = r_d/\beta$ are built up in magnitude.[3] The critical wavelength λ_{cr} is equal to the period of spatial oscillations of the long-range component of the correlation field. An analysis of the stability given in Ref. 3 and an independent analysis of the correlation properties demonstrate that there is a tendency for a high-current channel to split into separate filaments and this tendency is common to pinch systems irrespective of the spatial structure of the discharge-channel plasma.

Our derivation of the distribution function (2.8) is valid in the case of a system which is stationary and homogeneous along the current. It allows us to find a two-dimensional distribution of the field created by a one-dimensional test body, but it is unsuitable for the calculation of microfields generated in a plasma by individual charges. The field of charges moving in a plasma is neither stationary nor homogeneous along the direction of the current. However, it is clear that the presence of a long-range component of the field may alter drastically the structure of a plasma, as well as its thermodynamic and kinetic properties. It would be of great interest to find the field generated in relativistic plasma streams by individual charges. The long-range component of the field may give rise to bound states of electron pairs because of their correlation interaction, i.e., it may give rise to the Cooper effect.[7] The correlation energy of the interaction between two electrons should include an oscillatory function decreasing slowly with distance. Near the positions corresponding to minima of the correlation energy the relative motion of electrons is finite. Cooling of the electron subsystem by the loss of energy as radiation may have the effect that the more favorable (from the energy point of view) state of a system will be that in which electron pairs form bound states. This presents an interesting possibility of a transition of a plasma in a high-current channel to the superconducting state. This transition of a plasma in a relativistic stream to the superconducting state can be investigated only outside the approximation adopted above that the system is stationary and homogeneous along the current axis.

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