Hidden supersymmetry of stochastic dissipative dynamics

M. V. Feĭgel'man and A. M. Tsvelik

L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR, Moscow
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It is shown that the generating functional of dissipative dynamics in a potential force field can be reduced to the form of the functional integral of Euclidian supersymmetric field theory. The existence of a nonequilibrium (current) steady state is related to spontaneous supersymmetry breaking in the corresponding field theory. A variational principle of minimum entropy production is formulated. A supersymmetric diagrammatic technique is developed which is a compact invariant formulation of the well-known diagrammatic technique for dynamical problems. The nonlinear-dynamics problem for the fluctuating pendulum is considered in detail. The coefficient of angular diffusion under essentially nonequilibrium conditions, which determines, in particular, the line width of a synchronized self-excited oscillator, is computed.

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1. INTRODUCTION

The purpose of the present paper is to study the hidden symmetry inherent in the well-known equations of stochastic dissipative dynamcis:

$$\dot{\varphi}_x = -\Gamma\frac{\delta V}{\delta\varphi_x} + \xi(x,t), \tag{1}$$

where $\xi(x,t)$ is white noise:

$$\langle \xi(x,t)\,\xi(x',t')\rangle = 2T\Gamma\delta(x-x')\,\delta(t-t'), \tag{2}$$

$V\{\varphi_x\}$ is a functional of the variables φ_x, and Γ is the kinetic coefficient. Equation (1) can describe both the relaxation to thermodynamic equilibrium of a classical statistical system with energy $V\{\varphi_x\}$ and temperature T and a broad class of relaxation processes that occur in physical and nonphysical systems under conditions very different from thermodynamic equilibrium conditions[1,2] (in this case the parameter T in (2) is not temperature). It is well known (see, for example, Refs. 1 and 2) that the stochastic equivalent of Eqs. (1) and (2) is the Fokker-Planck equation (FPE) for the distribution functional $\mathscr{P}\{\varphi_x\}$ of the quantities φ_x:

$$\frac{1}{\Gamma}\frac{\partial\mathscr{P}}{\partial t} = \int\limits_{(x)} \frac{\delta}{\delta\varphi_x}\left(T\frac{\delta\mathscr{P}}{\delta\varphi_x} + \frac{\delta V}{\delta\varphi_x}\mathscr{P}\right). \tag{3}$$

Equation (3) has an obvious time-independent solution

$$\mathscr{P}\{\varphi_x\} = A^{-1}\exp\left(-\frac{1}{T}V\{\varphi_x\}\right), \tag{4}$$

which gives the "equilibrium" (see below) distribution function in the case in which the normalization integral

$$A = \int \exp\left(-\frac{1}{T}V\{\varphi_x\}\right)D\varphi_x < \infty \tag{5}$$

converges. Equation (3) is the simplest example of the Fokker-Planck equations that satisfy the so-called potential conditions.[3,4] These conditions, supplemented by the normalization condition (5), are equivalent[3] to the existence of detailed balance in the system. More general potential conditions, which contain the conditions for detailed balance as a particular case, are given in Ref. 4. From the standpoint of the solution to Eq. (1) or (3) thermodynamically nonequilibrium systems in a steady state with detailed balance are formally indistinguishable from equilibrium systems (a fact which has been used to advantage in investigations by, for example, Haken's group[2]). Therefore, below we shall call such states equilibrium states no matter how far the system is from the true state of thermodynamic equilibrium.

If the integral (5) diverges, then the equilibrium solution (4) is physically absurd, and we should seek that steady-state solution to (3) which satisfies the normalization condition. It is very important that, regardless of the convergence of the integral (5), the FPE is reducible to the form of the functional Schrödinger equation with imaginary time with the aid of the substitution[1)] $\mathscr{P}\{\varphi_x\} = \exp\left(-\frac{1}{2T}V\{\varphi_x\}\right)\Psi\{\varphi_x\}$:

$$\frac{T}{\Gamma}\frac{\partial}{\partial t}\Psi = \int\limits_x\left(T^2\frac{\delta^2\Psi}{\delta\varphi_x^2} - U\{\varphi_x\}\Psi\right),$$
$$U\{\varphi_x\} = \int\limits_{(x)}\left[\frac{1}{4}\left(\frac{\delta V}{\delta\varphi_x}\right)^2 - \frac{T}{2}\frac{\delta^2 V}{\delta\varphi_x^2}\right]. \tag{6}$$

The Schrödinger equation (6) has a remarkable property: for any potential $V\{\varphi_x\}$ satisfying (5), the ground-state wave function $\Psi_0\{\varphi_x\}$ has the form $A^{-1/2}\exp(-V\{\varphi_x\}/2T)$, and the ground-state energy $E_0 = 0$. This property is a trivial consequence of the existence of a time-independent solution of the FPE (3), but it indicates a hidden symmetry connected with the Schrödinger equation (6). As is well known, the functional Schrödinger equation with imaginary time corresponds to Euclidean quantum field theory. We shall show that the field theory corresponding to (6) possesses the property of supersymmetry. Supersymmetric field theories have been intensively investigated in recent years.[9–15] It is known, in particular, that the energy of vacuum is equal to zero in any theory with unbroken supersymmetry.

In the second section of the paper we shall obtain the Lagrangian formulation of the supersymmetric field theory corresponding to the equation (1) of dissipative dynamics, derive a variational principle that generalizes the theorem of minimum entropy production, and construct a superconvar-

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iant diagrammatic technique suitable for application. In the third section we construct a Hamiltonian formalism, and discuss the connection between the spontaneous breaking of the supersymmetry and the appearance of a nonequilibrium steady-state solution to the FPE (3). In the third section we illustrate the proposed method by using it to solve the nonlinear-dynamics problem for a pendulum in a heat bath. A specific new result is the computation of the coefficient of diffusion in the steady-state background in the essentially nonlinear region.

2. THE LAGRANGIAN FORMULATION OF THE THEORY

To derive an explicitly supersymmetric representation, it is convenient to use the dynamical-generating-functional method.[16–19] The dynamical generating functional has the form

$$Z\{l_{x,t}\}=\Big\langle\int e^{i\int_{(x,t)} l\varphi}\prod_{(x,t)}\delta\left(\frac{\dot\varphi_x}{\Gamma}+\frac{\delta V}{\delta\varphi_x}-\frac{1}{\Gamma}\xi(x,t)\right)\times\det\left(\frac{1}{\Gamma}\frac{\partial}{\partial t}+\frac{\delta^2V}{\delta\varphi_x\delta\varphi_{x'}}\right)D\varphi(x,t)\Big\rangle_\xi. \quad (7)$$

The ξ-averaged quantity in (7) is computed with the aid of Gaussian integration:

$$\langle\Phi\rangle_\xi=\int\Phi\exp\Big\{-\int_{(x,t)}\xi^2/4T\Gamma\Big\}D\xi(x,t). \quad (8)$$

On account of the definition (7), $Z\{l_{x,t}=0\}=1$. The $\varphi(x,t)$-field correlation functions are obtained by differentiating $Z\{l_{x,t}\}$, e.g.,

$$\langle\varphi(x,t)\varphi(x',t')\rangle=\frac{\delta^2Z}{\delta l_{xt}\,\delta l_{x't'}}\Big|_{l_{x,t}=0}. \quad (9)$$

Rewriting the δ function and the determinant in (7) by integrating them over the secondary boson $\hat\varphi$ and fermion $\psi,\bar\psi$ fields, and performing the averaging in accordance to the formula (8), we obtain

$$Z\{l\}=\int\exp\Big(i\int_{(x,t)}l\varphi\Big)\exp\Big\{\int_{(x,t)}\Big[\hat\varphi_x\Big(\frac{\dot\varphi_x}{\Gamma}+\frac{\delta V}{\delta\varphi_x}\Big)+\frac{T}{\Gamma}\hat\varphi_x^2+\bar\psi_x\Big(\frac{1}{\Gamma}\frac{\partial}{\partial t}+\frac{\delta^2V}{\delta\varphi_x\delta\varphi_{x'}}\Big)\psi_{x'}\Big]\Big\}D\varphi\,D\hat\varphi\,D\bar\psi D\psi \quad (10)$$

(the integration over $\hat\varphi$ in (10) is performed along the imaginary axis).

Thus far the potential character of the forces in Eq. (1) have not been important: the representation (10) remains valid when $\delta V/\delta\varphi_x$ is replaced by an arbitrary K_x. Let us now use the potential character in order to represent $Z\{l(x,t)\}$ in the form of the functional integral of Euclidian field theory. Setting $\hat\varphi=F-\dot\varphi/2T$, we reduce (10) to the form

$$Z\{l\}=\int\exp\Big(i\int_{(x,t)}l\varphi\Big)\exp\Big[-\frac{1}{2T}(V\{\varphi_f\}-V\{\varphi_{in}\})\Big]D\varphi_f\times D\varphi_{in}\int_{\{\varphi_{in}\}}^{\{\varphi_f\}}e^{-S}D\varphi\,DF\,D\bar\psi\,D\psi, \quad (11)$$

where the action S has the form

$$S=\int dx\,dt\mathscr{L}=\frac{1}{T}\int dx\,dt\Big[-\frac{1}{\Gamma}F^2+F\frac{\delta V}{\delta\varphi}+\frac{\dot\varphi^2}{4\Gamma}-\bar\psi\Big(\frac{\partial}{\partial t}\frac{1}{\Gamma}+\frac{\delta^2V}{\delta\varphi_x\,\delta\varphi_{x'}}\Big)\psi\Big] \quad (12)$$

[in the course of the derivation of (12) we made the substitutions $TF\to F$; $T^{1/2}\psi\to\psi$, $T^{1/2}\bar\psi\to\bar\psi$]. In the expression (11) $\{\varphi_{in}\}$ and $\{\varphi_f\}$ denote the initial and final configurations of the φ field. Notice that the expression (12) has the form of the action of Euclidian field theory, and does not include any temporal irreversibility (under the transformation $t\to it$ S it transform into the action of pseudo-Euclidian unitary field theory). The entire irreversibility inherent in dissipative dynamics is now contained in the factor $\exp[-(V(\varphi_f)-V(\varphi_i))/2T]$ outside the integral. This separating out of the irreversible terms is possible in view of the potentiality of the forces in Eq. (1) (this circumstance is also noted in Refs. 20–23).

As can be seen from (11), the path integration splits up into two successive operations: the evaluation of an Euclidian-field-theory path integral with prescribed initial $\{\varphi_{in.}\}$ and final $\{\varphi_f\}$ conditions and then the averaging over $\{\varphi_{in.}\}$ and integration over $\{\varphi_f\}$. The $\{\varphi_f,\varphi_{in.}\}$ functional that arises after the first stage is (for $\{l_{x,t}\}=0$) the associated distribution functional $\mathscr{P}\{\varphi_f,\varphi_{in}\}$:

$$\mathscr{P}\{\varphi_f,\varphi_{in}\}=\exp\Big(-\frac{1}{2T}(V\{\varphi_f\}-V\{\varphi_{in}\})\Big)\times\int_{\{\varphi_{in}\}}^{\{\varphi_f\}}\exp(-S)D\varphi\,DF\,D\psi D\bar\psi. \quad (13)$$

The expression (13) allows us to formulate for the nonequilibrium steady states a variational principle generalizing the principle, valid in the linear region, of minimum entropy production.

Let the integral in (13) be evaluated between states $\{\varphi_{in}\}$ and $\{\varphi_f\}$ separated in time t by a very long interval (longer than all the characteristic times of the system). Then this integral can be represented in the form $\exp(-\mathscr{F}\{j_x\}t)$, where $\mathscr{F}\{j_x\}$ is a t-independent functional of the extensive quantities j_x, the fluxes determining the steady state (the analog of the boundary-conditions-dependent free energy). The index of the exponential function in the factor outside the integral is also proportional to t:

$$(V\{\varphi_f\}-V\{\varphi_{in}\})/2T=tR\{j_x\}/2,$$

where $R\{j_x\}$ is the flux-dependent rate of entropy production (for the problem with a single variable the set $\{j\}$ reduces to the single flux $j=(\varphi_f-\varphi_{in})/t$, whose magnitude in the steady state is to be determined.[2)] Since the subsequent integration of (13) over $D\varphi_{in}$ should yield a t-independent steady-state distribution function $\mathscr{P}\{\varphi_f\}$, and, because of the presence of the very large t factor in the exponents, only the neighborhood of the minimum of the functional

$$\Omega\{j\}=\mathscr{F}\{j\}-{}^1/_2R\{j\},$$

makes a contribution to the Dj integral, which is equivalent to the $D\varphi_{\rm in}$ integral, we have two conditions determining the production of entropy in the realized (r) steady state:

$$2\delta\mathscr{F}/\delta j_x = X_x \quad (\delta\Omega/\delta j_x|_{j_x^r}=0), \tag{14}$$
$$R = R\{j_x^r\} = 2\mathscr{F}\{j_x^r\} \quad (\Omega(j_x^r)=0).$$

The quantities $X_x = \delta R/\delta j_x$ are the generalized forces conjugate to the fluxes j_x. Thus, the rate R of entropy production in the steady state is determined by the conventional extremum of t he "thermodynamic potential" Ω in the case when the forces X_x are prescribed. In the case of linear kinetics the path integral (13) is Gaussian, and, instead of the extremum of $\mathscr{F}$, we can directly use the extremum of the action S; the contribution of the fluctuations can be substantial in the nonlinear region. It should be noted that, in the case of the free energy $\mathscr{F}$ of Euclidian field theory, the two conditions (14) are, generally speaking, inconsistent; their consistency in our theory is due to the presence of supersymmetry in it.

In order to demonstrate the manifest supersymmetry of the action (12), let us introduce the anticommuting (Grassman) coordinates θ and $\bar\theta$ and the superfield

$$\Phi = \varphi + \bar\theta\psi + \bar\psi\theta + \bar\theta\theta F. \tag{15}$$

Using the rules governing integration over the Grassman variables,[24,11] i.e., the relations

$$\int d\theta = \int d\bar\theta = 0; \qquad \int \theta\, d\theta = \int \bar\theta\, d\bar\theta = 1; \qquad \theta^2 = \bar\theta^2 = 0\,,$$

we can easily verify that the action (12) can be written in the form

$$S = \frac{1}{T}\int d\bar\theta\, d\theta\, dt \left[\int_{(x)} \frac{1}{\Gamma}\Phi\bar D D\Phi + V(\Phi)\right], \tag{16}$$

where D, and $\bar D$ are covariant superderivatives:

$$D = \frac{\partial}{\partial\bar\theta} + \frac{\theta}{2}\frac{\partial}{\partial t}, \qquad \bar D = \frac{\partial}{\partial\theta} + \frac{\bar\theta}{2}\frac{\partial}{\partial t}. \tag{17}$$

The operators D and $\bar D$ generate a simple superalgebra:

$$\{\bar D, D\} = \partial/\partial t$$

and are the generators of a supersymmetry group consisting of simultaneous translations in time and in one of the Grassman coordinates:

$$e^{\bar\varepsilon D}:\ \bar\theta \to \bar\theta + \bar\varepsilon,\ \ t \to t + {}^1/{}_2\bar\varepsilon\theta; \tag{18}$$
$$e^{\varepsilon\bar D}:\ \theta \to \theta + \varepsilon,\ \ t \to t + {}^1/{}_2\bar\theta\varepsilon.$$

The φ, ψ, and F fields behave under infinitesimal transformations in the following manner:

$$e^{\bar\varepsilon D}:\ \delta\varphi = \bar\varepsilon\psi,\ \delta\psi = 0,\ \delta\bar\psi = \bar\varepsilon(\dot\varphi/2 + F),\ \delta F = -\bar\varepsilon\dot\psi/2; \tag{19}$$
$$e^{\varepsilon\bar D}:\ \delta\varphi = -\bar\psi\varepsilon,\ \delta\psi = (\dot\varphi/2 - F)\varepsilon,\ \delta\bar\psi = 0,\ \delta F = -\dot{\bar\psi}\varepsilon/2.$$

By substituting (19) into (16) or (12), we can easily verify the invariance of the action under the operations of the supersymmetry group [this, by the way, follows already from the fact that we can represent (12) in the covariant form (16)]. Thus, our problem reduces to the investigation of Euclidian supersymmetric field theory. The central question is then the question whether spontaneous supersymmetry breaking is possible in the theory in question. As will be shown below with the aid of a Hamiltonian formalism, the supersymmetry is spontaneously broken in those and only in those cases in which the normalization integral (5) diverges; in other words, spontaneous supersymmetry breaking in the theory with the action (16) is unambiguously tied with the existence of a nonequilibrium steady-state solution of the FPE (3).

In those cases in which the supersymmetry is not broken, and an equilibrium distribution function $\mathscr{P}_0 = A^{-1}\exp(-V\{\varphi\}/T)$ exists, the problem consists in the computation of the correlation functions and, in particular, the spectrum of the characteristic times of the relaxation of the system to the equilibrium distribution $\mathscr{P}_0$, which, in the language of supersymmetric field theory, implies the computation of the spectrum of the excitations above the ground state. In problems that can be investigated with the aid of perturbation theory, the representation of the supertheory allows us to construct in terms of the superfields Φ a diagrammatic technique that is a convenient compact formulation of the well-known diagrammatic technique for dynamical problems.[17,19]

To construct the diagrammatic technique, let us separate the interaction terms $V_{\rm int}$ from the potential $V(\Phi)$ entering into (16):

$$V(\Phi) = \frac{1}{2}(\nabla\Phi)^2 + \frac{m^2}{2}\Phi^2 - \mathscr{E}\Phi + V_{int}(\Phi) \tag{20}$$

and fine the bare correlator

$$G_0(z, z') = \langle\Phi(z)\Phi(z')\rangle,$$

(z denotes the set x, t, and θ) corresponding to the "unperturbed" Lagrangian

$$\mathscr{L}_0 = \int\left[\frac{1}{\Gamma}\Phi\bar D D\Phi + \frac{1}{2}(\nabla\Phi)^2 + \frac{m^2}{2}\Phi^2 - \mathscr{E}\Phi\right] dx\, d\bar\theta\, d\theta. \tag{21}$$

We shall now consider the case of the flux-free state, i.e., the state in which spontaneous supersymmetry breaking does not occur, and the "supersymmetric mean value" of Φ is equal to zero:

$$\langle\Phi\rangle_{ss} = \int\langle\Phi\rangle d\bar\theta\, d\theta = \langle F\rangle = 0. \tag{22}$$

At the level of the bare Lagrangian (21), this implies that $m^2/\mathscr{E} \neq 0$. The formal expression for the bare correlator has the form

$$G_0(z, z') = \frac{\delta^2}{\delta l_z\,\delta l_{z'}}\int D\Phi \times \exp\left[-\frac{1}{T}\int\mathscr{L}_0\, d\bar\theta\, d\theta\, dt + \int l_z\Phi\, dz\right]\Bigg|_{l_z=0} \tag{23}$$

[the factor $\exp(-V_f - V_i)/2T$] outside the integral is unimportant when the perturbation theory is constructed with $\langle\Phi\rangle_{ss} = 0$). The correlator (23) is given by the solution to the linear equation (with a source on the right-hand side) for the extremum of $\mathscr{L}_0$:

$$\left(\frac{1}{\Gamma}(\bar D D - D\bar D) - \nabla^2 + m^2\right) G_0 = T\delta^s(z - z'). \tag{24}$$

The supersymmetric delta function $\delta^s(z-z')$ is given by the relation

$$\int \delta^s(x-x', t-t', \theta-\theta')\Phi(x', t', \theta')dx'\, dt'\, d\theta' = \Phi(x, t, \theta). \tag{25}$$

Substituting Φ in the form (15) into (25), we obtain

$$\delta^s(x, t, \theta) = \delta(x)\delta(t)\bar\theta\theta \tag{26}$$

and, using the identity

$$(D\bar D - \bar D D)^2 = \partial^2/\partial t^2,$$

which follows from the definitions (17), we have

$$G_0(z, z') = T\frac{-\nabla^2+m^2-\Gamma^{-1}(D\bar D-\bar D D)}{-\Gamma^{-2}\partial^2/\partial t^2+(-\nabla^2+m^2)^2}$$

$$\times(\bar\theta-\bar\theta')(\theta-\theta')\delta(x-x')\delta(t-t'). \tag{27}$$

In the Fourier representation we have

$$G_0(\omega, k, \theta, \theta') = T\frac{(k^2+m^2)(\bar\theta-\bar\theta')(\theta-\theta')+(2i\omega/\Gamma)(\bar\theta'\theta-\bar\theta\theta') - (\omega^2/2\Gamma)\bar\theta\theta\bar\theta'\theta'}{\omega^2/\Gamma^2+(k^2+m^2)^2}. \tag{28}$$

The expansion of $V_{\rm int}(\Phi)$ in a series in powers of Φ gives the interaction vertices. The resulting diagrammatic technique differs from the usual technique only in that it contains integrations over the supercoordinates θ and $\bar\theta$. This leads, in particular, to the vanishing of all the single-tail diagrams, since, as can be seen from (28), the integral over the "point of attachment of the tail" vanishes:

$$\int G(\omega, k, \theta, \theta')d\bar\theta\, d\theta = 0$$

(on account of the supercovariance of the theory, the exact correlator $G(\omega,k,\theta,\theta')$ possesses the same structure with respect to the supercoordinates θ and θ' as the bare G_0 from (28).

To conclude this section, let us prove the fluctuation-dissipation theorem in our technique. As can be seen from the representation (10) of the generating functional, the function $\mathscr{E}\left(V_{\mathscr{E}}\{\varphi\} = V\{\varphi\} - \int \mathscr{E}_x\varphi_x\, dx\right)$ characterizing the response to the field is given by the expression

$$\mathscr{G}(x, t; x', t') = \frac{\delta\varphi(x, t)}{\delta\mathscr{E}(x', t')}\bigg|_{\mathscr{E}=0} = -\langle\varphi(x, t)\hat\varphi(x', t')\rangle. \tag{29}$$

The correlation function is

$$K(x, t; x', t') = \langle\varphi(x, t)\varphi(x', t')\rangle. \tag{30}$$

By definition [see the transition to the expression (11)]

$$\hat\varphi = F - \dot\varphi/2T,$$

and therefore

$$\mathscr{G}(x, t; x', t') = \frac{1}{2T}\frac{\partial}{\partial t'}K(x, t; x', t') - \langle\varphi(x, t)F(x', t')\rangle. \tag{31}$$

Going over to the frequency representation, and taking into account the realness of the correlator $\langle\varphi F\rangle$ in Euclidian field theory, we obtain

$$K_\omega(x, x') = \frac{2T}{\omega}\,\mathrm{Im}\,\mathscr{G}_\omega(x, x'). \tag{32}$$

Let us emphasize that the fluctuation-dissipation theorem (32) is not valid for nonequilibrium steady states. The point is that the physical correlation functions $K_\omega(x,x')$ and $\mathscr{G}_\omega(x,x')$ are obtained in this case through the analytic continuation of the formulas of Euclidian field theory (see the example in Sec. 4), as a result of which the correlator $\langle\varphi F\rangle_\omega$ acquires an imaginary part and, instead of (32), we have

$$K_\omega(x, x') = \frac{2T}{\omega}[\mathrm{Im}\,\mathscr{G}_\omega(x, x') + \mathrm{Im}\langle\varphi(x)F(x')\rangle_\omega]. \tag{33}$$

3. THE HAMILTONIAN FORMALISM

To derive the equations of the theory in the Hamiltonian formalism, it is convenient to proceed from the path integral (11) with the action (12). Integrating $F(x,t)$ over the field, we obtain the φ-, ψ-, and $\bar\psi$-field path integral with the Lagrangian

$$\mathscr{L}' = \frac{1}{T}\left[\frac{\dot\varphi_x^2}{4\Gamma} + \frac{\Gamma}{4}\left(\frac{\delta V}{\delta\varphi_x}\right)^2 - \bar\psi_x\left(\frac{1}{\Gamma}\frac{\partial}{\partial t} + \frac{\delta^2 V}{\delta\varphi_x\,\delta\varphi_{x'}}\right)\psi_{x'}\right]. \tag{34}$$

The Hamiltonian operator corresponding to $\mathscr{L}'$ can be written in the usual manner with the aid of the momentum operator $\delta/\delta\varphi_x$ and the Fermi creation and annihilation operators a_x^+ and a_x:

$$\hat H = \int_{(x)}\left[-T^2\frac{\delta^2}{\delta\varphi_x^2} + \frac{1}{4}\left(\frac{\delta V}{\delta\varphi_x}\right)^2\right] + \frac{T}{2}\iint_{(x,x')}\frac{\delta^2 V}{\delta\varphi_x\delta\varphi_{x'}}[a_x^+ a_{x'}]. \tag{35}$$

In going over to (35), we dropped the factor Γ, which determines the general time scale. It should be noted that the writing down of the last term in (35) in precisely the commutator form is dictated by the theory's supersymmetry, which does not allow the energy reference level to be shifted arbitrarily.

The total fermion number $N = \int_{(x)} a_x^+ a_x$ is an integral of the motion, so that the Schrödinger equation with the Hamiltonian $\hat H$ splits up into independent equations for the various N numbers. Of special interest are the equations corresponding to the cases of zero and maximum fermion number, i.e., to the cases in which $N=0$ and $N=N_{\rm max}$. The first case $a_x|\Psi_-\rangle = 0$ for all x, and therefore the Hamiltonian reduces to

$$\hat H_- = \int_{(x)}\left[-T^2\frac{\delta^2}{\delta\varphi_x^2} + \frac{1}{4}\left(\frac{\delta V}{\delta\varphi_x}\right)^2 - \frac{T}{2}\frac{\delta^2 V}{\delta\varphi_x^2}\right] \tag{36}$$

and the corresponding Schrödinger equation coincides with Eq. (6), which was derived from the Fokker-Planck equation (3). In the case of highest fermion population the wave function satisfies the condition $a_x^+ |\Psi_+\rangle = 0$, and the Hamiltonian reduces to

$$\hat{H}_+ = \int_{(x)} \left[-T^2 \frac{\delta^2}{\delta\varphi_x^2} + \frac{1}{4}\left(\frac{\delta V}{\delta\varphi_x}\right)^2 + \frac{T}{2}\frac{\delta^2 V}{\delta\varphi_x^2} \right]. \tag{37}$$

The Schrödinger equation with the Hamiltonian $\hat{H}_+$ is obtained from the backward FPE (it is now the Kolmogorov equation), just as (6) is obtained from (3). If, instead of the field φ_x, we have only one variable φ, then the states $|\Psi_+\rangle$ and $|\Psi_-\rangle$ constitute the entire basis, and we do not obtain any new equations; such a theory (supersymmetric quantum mechanics) is investigated in Refs. 13 and 14 (without any ties to the FPE). In the remaining cases there are a number of states with intermediate fermion numbers ($0 < N < N_{\max}$). The equations for the wave functions of these states apparently do not have simple analogs in the FPE theory.

As is well known (see, for example, Refs. 10, 11, and 13), the Hamiltonian of any supersymmetric theory can be represented in the form of the square of the Hermitian supercharge operator (or in the form of a sum of the squares of such operators if there are several of them). In the present case, as can easily be verified through a direct calculation, the Hamiltonian H from (35) can be represented in the form $\hat{H} = \hat{Q}^2$, where

$$\hat{Q} = \int_{(x)} \left[\left(-T\frac{\delta}{\delta\varphi_x} + \frac{1}{2}\frac{\delta V}{\delta\varphi_x} \right) a_x + \left(T\frac{\delta}{\delta\varphi_x} + \frac{1}{2}\frac{\delta V}{\delta\varphi_x} \right) a_x^+ \right]. \tag{38}$$

Therefore, we can use the equation

$$\hat{Q}\Psi = E^{1/2}\Psi \tag{39}$$

to compute the spectrum and wave functions of the theory. Equation (39) is especially convenient for the elucidation of the question of spontaneous supersymmetry breaking in the theory in question.[13,14] In the supertheory in which spontaneous supersymmetry breaking does not occur, the energy of vacuum $E_0 = 0$. Equation (39) with $E = 0$ possesses two obvious formal solutions:

$$\Psi_- = e^{-V\{\varphi\}/2T} |-\rangle, \tag{40}$$

where $|-\rangle$ is the fermion vacuum ($a_x|-\rangle = 0$), and

$$\Psi_+ = e^{V\{\varphi\}/2T} |+\rangle \qquad (a_x^+ |+\rangle = 0). \tag{41}$$

If any of the solutions (40) and (41) is normalizable, then it is the true wave function of the ground state. If, on the other hand, the potential $V\{\varphi\}$ is such that both solutions, (40) and (41), cannot be normalized [e.g., if $V(\varphi) = \cos\varphi + \mathscr{E}\varphi$], then the equation $\hat{Q}\Psi = 0$ does not possess suitable solutions, and, consequently, $E_0 > 0$; in this case we have spontaneous supersymmetry breaking to deal with, and, to determine the ground state, we must solve the equation $\hat{Q}\Psi = (E_0)^{1/2}\Psi$.

The normalizability condition for the wave function Ψ_- coincides with the condition (5) for the existence of an equilibrium distribution function $\mathscr{P} \propto \exp(-V\{\varphi\}/T)$ that is a solution of Eq. (3). Therefore, the phenomenon of spontaneous supersymmetry breaking in the theory with the action (16) is unequivocally tied with the existence of nonequilibrium (current) steady-state solutions of the Fokker-Planck equation (3).[3)] This circumstance allows us to use the methods of field theory to compute the fluxes and fluctuations in such steady states. In the next section we shall, as an example, carry out such a calculation for the problem with one degree of freedom.

4. NONLINEAR DYNAMICS OF A FLUCTUATING PENDULUM

Let us consider the equation for the highly dissipative motion of a pendulum in a heat bath under the action of the moment of gravity and a torque:

$$\dot{\varphi} = -\Gamma(m\sin\varphi - \mathscr{E}) + \xi(t) \qquad (\langle \xi(t)\xi(t')\rangle = 2\Gamma T\delta(t-t')). \tag{42}$$

This equation occurs in many physical problems; in particular, it describes the phase drift of a synchronized self-excited oscillator[1] or a laser[2] and the current through a Josephson junction under conditions of strong dissipation.[25,26] The Fokker-Planck equation that is equivalent to (42) has the form

$$\frac{\partial\mathscr{P}}{\partial\tau} = \frac{\partial}{\partial\varphi}\left(T\frac{\partial\mathscr{P}}{\partial\varphi} + (m\sin\varphi - \mathscr{E})\mathscr{P} \right) \tag{43}$$

(we have denoted Γt by τ). The corresponding potential has the form

$$V(\varphi) = -m\cos\varphi - \mathscr{E}(\varphi - \pi) \tag{44}$$

(the term $\mathscr{E}\pi$ has been added for computational convenience in what follows). For $\mathscr{E} \neq 0$ the equilibrium distribution function $\exp[-V(\varphi)/T]$ is normalizable, and a steady state with a nonzero flux $j = \langle\dot{\varphi}\rangle$ is realized. The computation of the flux and the steady-state distribution function is simple, and was carried out long ago.[1,25,26] The investigation of the fluctuations in the background of such a steady state is significantly less trivial, since it requires the solution of the complete (nonstationary) Eq. (43) without the use of the fluctuation-dissipation theorems (which are valid only for the equilibrium solution with $\mathscr{E} = 0$). In particular, it is of interest to compute the phase-diffusion coefficient D determining the asymptotic behavior of the fluctuations at large t:

$$\langle (\varphi(t) - \varphi(0) - jt)^2 \rangle = 2D|t| \tag{45}$$

(this quantity determines, in particular, the line width of the synchronized self-excited oscillator[1]). We shall compute the quantity D for $\mathscr{E} < m$ and sufficiently low temperatures T, when the flux is produced by relatively infrequent (the precise criterion will be given below) fluctuation-induced turnovers of the pendulum ($\mathscr{E}$ can then be of the order of m and much higher than T, so that we are investigating the fluctuations in the essentially nonlinear region). For this purpose, it is convenient to express the associated distribution function $\mathscr{P}(\varphi, \varphi_0, t)$ in terms of the eigenfunctions of the Schrödinger equation corresponding to (43) [see Eq. (6)]:

$$T\frac{\partial\Psi}{\partial\tau} = +T^2\frac{\partial^2\Psi}{\partial\varphi^2} - \left[\frac{1}{4}(m\sin\varphi - \mathscr{E})^2 - \frac{T}{2}m\cos\varphi \right]\Psi. \tag{46}$$

The potential energy $U(\varphi)$ in (46) is periodic in φ; therefore, the wave functions Ψ and the energy levels E are characterized by a quasimomentum P and the band number α. The distribution function $\mathscr{P}(\varphi, \varphi_0, t)$ can be represented in terms of $\Psi_{P,\alpha}(\varphi)$ as follows:

$$\mathscr{P}(\varphi,\varphi_0,t)=\exp\left\{-\frac{V(\varphi)-V(\varphi_0)}{2T}\right\}\sum_\alpha\int_{-1/2}^{1/2}dp$$

$$\times\exp\left\{-\frac{E_\alpha(P)}{T}\tau\right\}\Psi_{P,\alpha}(\varphi)\Psi^*_{P,\alpha}(\varphi_0). \tag{47}$$

Below we shall be interested in the asymptotic behavior of the fluctuations at large values of the time τ; therefore, we shall need only the states in the lowest band (we shall omit the band number $\alpha=0$). Let us write down the expression for the mean $\langle\exp iq(\varphi-\varphi_0)\rangle$ with allowance for the foregoing:

$$\langle e^{iq(\varphi-\varphi_0)}\rangle=\int_{-\infty}^{+\infty}d\varphi\int_0^{2\pi}\frac{d\varphi_0}{2\pi}e^{iq(\varphi-\varphi_0)}\mathscr{P}(\varphi,\varphi_0,t)$$

$$=\exp\left\{-\frac{\tau}{T}E(i\mathscr{E}/2T-q)\right\}S_1(q)S_2(q), \tag{48}$$

where

$$S_1(q)=\int_0^{2\pi}\Psi_{i\mathscr{E}/2T-q}(\varphi)e^{-V(\varphi)/2T}d\varphi,$$

$$S_2(q)=\int_0^{2\pi}\Psi^*_{i\mathscr{E}/2T-q}(\varphi_0)e^{V(\varphi_0)/2T}d\varphi_0. \tag{49}$$

Knowing $\langle\exp[iq(\varphi-\varphi_0)]\rangle$, we can, by differentiating it with respect to q at $q=0$, determine the quantity D of interest to us (as well as higher-order correlators). We shall find the spectrum and the wave functions for real values of P in the quasiclassical approximation and then continue them analytically to $P\approx i\mathscr{E}/2T$. We shall construct the quasiclassical approximation for Eq. (46) in a manner that will not destroy the hidden supersymmetry of the problem. For this purpose, it is convenient to proceed from a first-order equation of the type (39). Let us write the operator $\hat{Q}$ in the matrix representation:

$$\hat{Q}=T\frac{\partial}{\partial\varphi}\sigma_2+\frac{V'}{2}\sigma_1,\quad \sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-1\\1&0\end{pmatrix}. \tag{50}$$

We shall seek the wave function in the form

$$\Psi(\varphi)=\begin{pmatrix}\exp(-S(\varphi)/T)\\ p(\varphi)\exp(-S(\varphi)/T)\end{pmatrix}. \tag{51}$$

The upper component of the column in (51) gives that solution to Eq. (46) which corresponds to the fermion vacuum and is necessary to us here. In terms of the functions S and p, the equation $\hat{Q}\Psi=E^{1/2}\Psi$ has the form ($Q_0=E^{1/2}$)

$$\frac{dS}{d\varphi}=\frac{1}{2}\frac{dV}{d\varphi}-Q_0p, \tag{52}$$

$$T\frac{dp}{d\varphi}=\frac{dV}{d\varphi}p-Q_0(1+p^2), \tag{53}$$

where $V(\varphi)$ is defined in (44).

The equation for $\Psi(\varphi)$ should be solved with the boundary condition

$$\Psi_P(\varphi+2\pi)=e^{2\pi iP}\Psi_P(\varphi),$$

which leads, when (51) and (52) are taken into account, to equations for $p(\varphi)$:

$$p(\varphi+2\pi)=p(\varphi), \tag{54}$$

$$-\pi\mathscr{E}/T+2\pi iP=Q_0\int_\varphi^{2\pi+\varphi}p(\alpha)d\alpha. \tag{55}$$

The parameter of our quasiclassical approximation is the ratio Q_0/T, since Eqs. (52) and (53) with $Q_0=0$ possess the obvious solutions

$$S(\varphi)={}^1/_2V(\varphi),\quad p(\varphi)\propto\exp(V(\varphi)/T).$$

For $Q_0/T\ll1$, we can seek the solution to (53) approximately in the regions with $p\ll1$ and $p\gg1$. In the first case, neglecting the term Q_0p^2, we have

$$p=e^{V(\varphi)/T}\left(C_1-Q_0/T\int_{\varphi_1}^{\varphi}e^{-V(\alpha)/T}d\alpha\right), \tag{56}$$

where $\varphi_1=\arcsin(\mathscr{E}/m)$ is the minimum point of the potential $V(\varphi)$ and C_1 is a constant. A similar expression is valid for $\varphi\approx\varphi_1+2\pi$:

$$p=e^{V(\varphi)/T}\left(C_3+Q_0/T\int_\varphi^{\varphi_1+2\pi}e^{-V(\alpha)/T}d\alpha\right). \tag{57}$$

For φ close to $\pi-\varphi_1$ [the maximum of $V(\varphi)$], $p(\varphi)\gg1$, and therefore we should drop the Q_0 term in (53) and solve the linear equation for $1/p$. As a result, we have

$$p=e^{V(\varphi)/T}\left(C_2+Q_0/T\int_{\varphi_1}^{\varphi}e^{V(\alpha)/T}d\alpha\right)^{-1}. \tag{58}$$

Matching the solutions (56), (58) and (58), (57) approximately, and taking the boundary conditions (54) and (55) (with φ replaced by φ_1) into account, we obtain a system of equations for Q_0 and C_1, C_2, C_3:

$$C_1-J_+=C_2^{-1},\qquad (C_2+J_0)^{-1}=C_3+J_-,$$
$$C_3\exp(-2\pi\mathscr{E}/T)=C_1,\qquad \exp(-\pi\mathscr{E}/T+2\pi ip)=1+C_2^{-1}J_0, \tag{59}$$

where we have introduced the notation

$$J_+=\frac{Q_0}{T}\int_{\varphi_1}^{\pi-\varphi_1}e^{-V(\alpha)/T}d\alpha,\qquad J_-=\frac{Q_0}{T}\int_{\pi-\varphi_1}^{2\pi+\varphi_1}e^{-V(\alpha)/T}d\alpha,$$

$$J_0=\frac{Q_0}{T}\int_{\varphi_1}^{\varphi_1+2\pi}e^{V(\alpha)/T}d\alpha.$$

It is not difficult to show that $J_+e^{\mathscr{E}\pi/T}+J_-e^{-\mathscr{E}\pi/T}=J_0$. After this, the system (59) is easy to solve, and we obtain for the spectrum $E(p)$ the expression

$$E(p)=Q_0^2=2T^2I^{-2}(\mathscr{E})(\operatorname{ch}\pi\mathscr{E}/T-\cos2\pi P), \tag{60}$$

where

$$I(\mathscr{E})=\int_{\varphi_1}^{2\pi+\varphi_1}e^{V(\varphi)/T}d\varphi,\qquad \varphi_1=\arcsin(\mathscr{E}/m). \tag{61}$$

The solution of (59) also yields the relation

$$\frac{Q_0}{TC_2}=I^{-1}(\mathscr{E})(\exp\{-\mathscr{E}\pi/T+2\pi iP\}-1) \tag{62}$$

which is useful for what follows. Let us emphasize that the small parameter of our quasiclassical approximation is precisely the ratio Q_0/T, and not, for example, T/m, as would have been the case in the usual quasiclassical approach. The corrections to the expressions (60) and (61) are exponentially small together with $Q_0/T \sim I^{-1}(\mathscr{E}) e^{\pi\mathscr{E}/2T}$; therefore, the use of the complete (and not the saddle-point) expression (61) for $I(\mathscr{E})$ is entirely legitimate. Let us also note that the applicability of (60) and (61) is not limited by the specific potential $V(\varphi)$ given in (44).

Let us now consider the "form factors" S_1 and S_2 defined in (49). The dominant contribution to $S_1(q)$ is made by the region $\varphi \gtrsim \varphi_1$, where $p(\varphi)$ is exponentially small. The wave function in this region has the form

$$\Psi(\varphi) = A^{-1} e^{-V(\varphi)/2T}, \tag{63}$$

where A is a normalization constant, given by the equation

$$A^2 = \int_0^{2\pi} e^{-V(\varphi)/T}\, d\varphi. \tag{64}$$

Taking (63) and (64) into account, we find that $S_1(q) = A$. The dominant contribution to the integral $S_2(q)$ is made by the region $\varphi \gtrsim \pi - \varphi_1$. The wave function in this region has the form

$$\Psi(\varphi) = A^{-1} \exp\left(-\frac{V(\varphi)}{2T}\right) \exp\left[\frac{Q_0}{T} \int_0^{\varphi} p(\varphi')\, d\varphi'\right]. \tag{65}$$

The integral of the function $p(\varphi)$ is evaluated with the aid of the formula (58), after which, taking (62) into consideration, we obtain

$$\Psi_P^*(\varphi) = A^{-1} e^{-V/2T} \left(1 + I^{-1}(\mathscr{E})\, e^{-\mathscr{E}\pi/2T - 2\pi i P} \int_0^{\varphi} e^{V(\alpha)/T}\, d\alpha\right). \tag{66}$$

The integral $S_2(q)$ is computed by parts, and, as a result, we have

$$S_1(q) S_2(q) = \frac{2\sin\pi q}{q} \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{i(\pi-\varphi)q} I^{-1}(\mathscr{E})\, e^{V(\varphi)/T}. \tag{67}$$

Now the expression (48) for $\langle \exp[iq(\varphi - \varphi_0)] \rangle$ is completely determined. Substituting the spectrum (60) and the form factors (67) into (48), and differentiating with respect to q at $q = 0$, we obtain

$$j = \Gamma \frac{4\pi T}{I^2(\mathscr{E})} \operatorname{sh} \frac{\pi\mathscr{E}}{T}, \tag{68}$$

$$D = \Gamma \frac{4\pi^2 T}{I^2(\mathscr{E})} \operatorname{ch} \frac{\pi\mathscr{E}}{T} + \frac{T}{2} j \frac{\partial}{\partial\mathscr{E}} \ln I(\mathscr{E}), \tag{69}$$

where $I(\mathscr{E})$ is defined in (61). The corrections to (68) and (69) are of the order of $I^{-2}(\mathscr{E}) \operatorname{ch}(\pi\mathscr{E}/T)$, and are small if

$$\frac{T}{m}\left(1 - \frac{\mathscr{E}^2}{m^2}\right)^{-1/2} e^{-\pi(m-\mathscr{E})/T} \ll 1 \quad (\mathscr{E} < m). \tag{70}$$

The formula (68) for the current is equivalent [in the region (70) of applicability] to the exact formula obtained from the solution to the steady-state FPE.[1,25,26] The expression (69) for the diffusion coefficient D is a new result; estimative considerations[1] gave only the first term in (69). Notice that, although the linear approximation is limited by the condition $\mathscr{E} \ll T$, the fluctuation-dissipation relation

$$D = T\partial j/\partial\mathscr{E}$$

is valid in the much broader (at low T) region $\mathscr{E} \ll m$; the second term in (69) is of the same order of magnitude as the first term when $\mathscr{E} \sim m$. By computing $I(\mathscr{E})$ by the method of steepest descent, we can represent the expression (69) in a simpler, approximate form:

$$D = j\left[\pi \operatorname{cth} \frac{\pi\mathscr{E}}{T} + \frac{1}{2} \arcsin \frac{\mathscr{E}}{m} + \frac{I\mathscr{E}}{2(m^2 - \mathscr{E}^2)}\right]. \tag{71}$$

5. CONCLUSION

In this paper we have shown that the investigation of the equations (1) of stochastic dissipative dynamics can be reduced to the study of the supersymmetric Euclidean field theory with the Lagrangian (16) [or the Hamiltonian (35)], the existence of a nonequilibrium ($\langle \varphi \rangle \neq 0$) steady state being unequivocally tied with the spontaneous breaking of the supersymmetry in the corresponding field theory. In doing this, we made essential use of the potential condition for the forces in Eq. (1). The potential condition is, generally speaking, more general than the condition for detailed balance, and coincides with it only when the additional normalizability condition (5) is fulfilled. We have in the paper considered a specific example of a potential system without detailed balance: a pendulum under the action of a torque. Another example of this sort is given in Miguel and Chaturverdi's recent paper[4] (we give the formulas in our notation):

$$\dot\psi(x) = -(\Gamma + i\Omega)\frac{\delta V}{\delta \psi_x^*} + \xi(x, t),$$

$$\langle \xi\xi \rangle = \langle \xi^* \xi^* \rangle = 0, \qquad \langle \xi(x, t)\, \xi^*(x', t') \rangle = \delta(x - x')\,\delta(t - t')\, 2T$$

and can also be investigated with the aid of our method (in this case we obtain a theory with a complex superfield).

In the paper we have constructed for the superfield correlation functions $\langle \Phi(x,t,\theta)\Phi(x',t',\theta') \rangle$ a supersymmetric diagrammatic technique that is a convenient invariant formulation of the diagrammatic technique constructed earlier[16–19] for the correlators and response functions of dissipative dynamics. A simple proof is given for the fluctuation-dissipation theorem for systems with detailed balance.

It should be noted that the specific mechanism underlying the dissipative equations (1) or (3) is quite unimportant for the construction of the theory, as a result of which any system for which the condition for detailed balance is fulfilled and the distribution function has the form (4) is formally indistinguishable from a system in a thermodynamic-equilibrium state with the free-energy functional $V\{\varphi\}$. In those cases in which the equilibrium—in this respect—distribution (4) is not normalizable, there is realized a current steady state whose distribution function has to be determined. We may find it useful to employ for this purpose the above-formulated variational principle [the formula (14)], which generalizes the principle, valid in the linear region, of least entropy production, entropy here being used (as in Ref. 21) in the generalized sense connected with the formal free enrgy $V\{\varphi\}$.

As an illustration of the direct computation of the flux and the fluctuations in the steady state, we have considered the problem of the nonlinear dynamics of a pendulum in a heat bath. We have obtained for the coefficient D of angular diffusion a formula which is valid in the essentially nonlinear region as well. The quantity D determines, in particular, the line width of the synchronized self-excited oscillator (e.g., the single-mode laser) under conditions of a finite frequency detuning.

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1)A tranformation of this type has been used in many papers (see, for example, Refs. 5–8).

2)Let us, to avoid any misunderstanding, emphasize that the terms "free energy," "entropy," etc. are used in a generalized, and not in a thermodynamic, sense, as is the concept of equilibrium distribution function (see the Introduction, as well as Ref. 21).

3)This fact was pointed out to the authors by A. M. Polyakov.

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