# XY model with random anisotropy

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An XY model with weak random anisotropy is considered. In the two-dimensional case at $n = 1$ and 2 there is no phase transition. At $n \geq 3$ there exists a temperature range $T^* < T < T_c$ in which the low-temperature phase of a "pure" XY model is realized. At $T < T^*$ the parameters $T$ and $h$ are substantially renormalized, namely $T \to T^*$ and $\bar{h}^2 \to 0$, so that low temperatures are effectively unattainable. There is therefore no state of the spin-glass type, although the susceptibility has a quasicusp relative to $h(\bar{x}) \equiv h \neq 0$ at $T = T^*$. A quasi-two-dimensional layered system with weak coupling of special type $g_\perp \cos n(\theta_{i+1} - \theta_i)$ exhibits at $T^* < T < T_c$ the same properties as in the two-dimensional case. At $T < T^*$, however, for any arbitrarily weak coupling $g_\perp$, the system becomes essentially three-dimensional. The renormalization-group equations have "standing pole" solutions, thus indicating a phase transition into a spatially fully disordered state of the spin-glass type.

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## I. INTRODUCTION

The possibility of the existence of a phase of the spin-glass type[1] in disordered magnets having a low spatial dimensionality ($d = 2$ or 3) continues to attract increasing interest. We consider this problem in this paper using as an example a planar ($XY$) classical ferromagnet in the presence of random anisotropy fields. We consider separately a two-dimensional ($2D$) and a quasi-two-dimensional (layered) system.

The "pure" (without random fields) $2D$ $XY$ model has by now been fully investigated (see, e.g., Refs. 2–5). Just as in any $2D$ magnet with continuous symmetry group, thermodynamic fluctuations make a long range order in this system impossible. Nonetheless, a phase transition does take place in the $2D$ $XY$ model and is due to formation of a low-temperature phase characterized by the so-called quasi-long-range order with a slow power-law decrease of the correlations. The destruction of the low-temperature phase and the transition to the paramagnetic phase is due to formation of vortical excitations.[2,3] In the dual representation, the $XY$ model is described by the well-known sine-Gordon model,[5] which is equivalent in turn to the Thirring model of interacting massive fermions.[6-8] This equivalence is preserved also in the case of a disordered model, so that from the technical point of view it is convenient to study the properties of the initial disordered magnet in the language of fermion theory which is known to be renormalizable in the two-dimensional case.

Two types of disordered models are usually considered. These are either models in which the random quantity is the interaction between neighboring spins (models with random bonds), or models in which each spin is in a random external field. In the case of the $XY$ model with random bonds (see, e.g., Refs. 9 and 10) the question of the transition into the spin-glass phase can be raised only in the case of strong disorder, when the system contains a large amount of frustration[11] and strong degeneracy of the ground state sets in.

In the case of the $XY$ model with random field, the situation is somewhat different. The system energy can in this case be written in the form

$$H = -\sum_{i,a} \cos(\theta_{i+a} - \theta_i) - \sum_i h_i \cos(n\theta_i);$$

here $\theta$ is the angle variable that specifies the direction of the planar spin ($0 \leq \theta \leq 2\pi$), $n$ is the anisotropy order, and $h_i$ is the random field: $\overline{h_1 h_k} = \bar{g}\delta_{ik}$. All the variables are assumed specified at the lattice sites, and the summation in the first term is over pairs of nearest neighbors. It is well known (see, e.g., Refs. 12 and 13) that even at arbitrarily small value of the random field ($h_i \ll 1$) there is no long-range order in such a system if the space dimensionality $d \leq 4$. This can be seen already from the following qualitative considerations. Let the characteristic length over which the long-range order is destroyed by $R_c$. In a volume with linear dimension of the order $R_c$ the exchange energy connected with the deformation of the structure is then of the order of $R_c^{d-2}$ ($d$ is the dimensionality of space), whereas the energy gain in the same volume, due to the interaction with the random field, is of the order of the mean squared value $(\bar{g}R_c^d)^{1/2}$. At $d < 4$ the energy gain due to the alignment with the random field is larger (at sufficiently large $R_c$) than the loss of exchange energy, so that formation of a randomly inhomogeneous structure with a characteristic correlation radius $R_c \sim \bar{g}^{1/(d-4)}$ is favored (at this value of $R_c$ both terms in the energy are of the same order). The foregoing estimates show that at the temperature $T = 0$ the system should be in a spin-glass state with "quenched" spins and with a finite correlation radius (which is large in the case of weak disorder). The question of the stability of this state at $T \neq 0$ relative to thermal fluctuations becomes one of the basic ones for the theory of spin glasses and other essentially disordered systems. A consistent approach to the solution of this problem calls for understanding the properties of the ground state of the spin glass and of the excitations near it. For lack of this understanding at the present time, we must use the "rule of contraries" and seek the high-temperature (paramagnetic) phase instability due to spin-glass formation. The available attempts to describe analytically this instability (see, e.g., Refs. 14–16) use as the starting point the self-consistent-field theory (the $1/z$ expansion, where $z$ is the number of nearest neighbors) and, as shown in Ref. 15, encounter serious dif-

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ficulties already at $d \leq 6$.

We propose here an approach that does not use the $1/z$ expansion and is based on the well known properties of the $2D$ $XY$ model. We consider the properties of a purely two-dimensional and of a quasi-two-dimensional (layered) medium. Since we are considering weak random fields, and the characteristic spatial scale of the problem is large compared with the lattice constant, we can use a continual approximation and consider continuous random fields $h(x)$ with a correlator $\overline{h(\mathbf{x})h(\mathbf{x}')} = \tilde{g}\delta(\mathbf{x}-\mathbf{x}')$. We show that in the $2D$ case states of the spin-glass type with "quenched" spins are unstable at all $T \neq 0$, and there is no phase transition at all at $n=1$ or 2. At $n \geq 3$ there exists a temperature region $T^* < T < T_c$ in which the disorder is inessential and the low-temperature phase of the pure $2D$ $XY$ model is realized. At $T < T^*$ there is a substantial renormalization of the parameters $T$ and $g$, leading to $T_{eff} = T^*$ and $\tilde{g}_{eff} = 0$. A short variant of this part of the work was published in Ref. 17. In Sec. III we consider a three-dimensional layered system made up of the two-dimensional systems described above (with $n \geq 3$) and with a weak interlayer interaction of special form $g_\perp \cos n(\theta_{i+1} - \theta_i)$. At $T > T^*$ this interaction is insignificant and the results reduce to those obtained with $g_\perp = 0$. At $T < T^*$ and at an arbitrary relation between $g_\perp$ and $\tilde{g}$ the "two-dimensional" solution is unstable and the system becomes essentially three-dimensional. Analysis within the framework of the "fast parquet" method[18,19] shows that the "moving-pole" solution characteristic of the pure ($\tilde{g}=0$) system is unstable at all $\tilde{g} \neq 0$ (thus again indicating instability of the ferromagnetic state) and is realized by a solution of the "standing pole" type. It seems to us that this last instability is evidence of a phase transition into the spin-glass state at $T < T^*$. It is curious to note that a formal consideration of the fast-parquet equations in the case of more than two "transverse" dimensions (corresponding to $d > 4$) demonstrates the stability of the "moving-pole" solution relative to $g \neq 0$ and agrees with the presence of ferromagnetic order in the considered model at $d > 4$ (Refs. 12 and 13).

## II. 2D XY MODEL WITH RANDOM ANISOTROPY

We consider the $XY$ model in a random-anisotropy field $h(\mathbf{x})\cos n\theta(\mathbf{x})$. Using the known transition to dual variables,[5] the effective Hamiltonian of our model can be written in the form

$$H = \int d^2x[{}^1/_2(\partial_\mu\varphi)^2 + u\cos\tilde{\beta}\tilde{\varphi} + h(\mathbf{x})\cos\beta\varphi], \tag{1}$$

where $\varphi = \theta/T^{1/2}$, $\tilde{\varphi}$ is the variable dual to $\varphi$: $\varepsilon_{\mu\nu}\partial_\nu\tilde{\varphi} = \partial_\mu\varphi$, $\tilde{\beta} = 2\pi/T^{1/2}$, $\beta = nT^{1/2}(\beta\tilde{\beta} = 2\pi n)$. The second term in (1) describes vortical excitations of the $XY$ model, and the parameter $u$ assumes the role of the chemical potential of the vortices. We shall be interested hereafter in the region near the point $\beta^2 = 4\pi$ (or $\tilde{\beta}^2 = \pi n^2$). Since the low-temperature phase of the $XY$ model corresponds to the region $\tilde{\beta}^2 > 8\pi$ (Refs. 4 and 5), it is necessary, if the value of interest to is to be in this region, that $n$ be larger than or equal to 3. It is known[4,5] that in the low-temperature phase the term

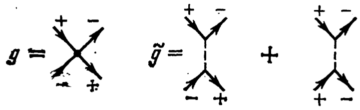

FIG. 1. Diagram representation of the vertices $g$ and $\tilde{g}$.

$u\cos\tilde{\beta}\tilde{\varphi}$ is inessential in the Hamiltonian (1), so that our Hamiltonian becomes the Hamiltonian of the sine-Gordon model with random mass $h(\mathbf{x})$. Using a known transformation, this model reduces to the Thirring model of interacting massive fermions.[6-8] These two models remain equivalent also in the case of a coordinate-dependent mass $h(\mathbf{x})$. (To verify this it is convenient to use the boson–fermion transformation in Euclidean form, introduced by Zamolodchikov.[8]) As a result we arrive at the Thirring model with random mass:

$$H = \int d^2x[\bar{\psi}\hat{\partial}\psi + h(\mathbf{x})(\bar{\psi}\psi) + g_0(\bar{\psi}\psi)^2], \tag{2}$$

where $(1+g/\pi)^{-1} = \beta^2/4\pi$, $\hat{\partial} + \hat{\gamma}_\mu\partial_\mu$, $\hat{\gamma}_\mu$ are Dirac $2D$ matrices, and $(\bar{\psi}\psi) = \psi_+^*\psi_- + \psi_-^+\psi_+$. To average over the random field $h(x)$ it is convenient to use the method of replicas.[20] As a result we have

$$H = \int d^2x\left\{\sum_{a=1}^{N}[\bar{\psi}^a\hat{\partial}\psi^a + g_0(\bar{\psi}^a\psi^a)(\bar{\psi}^a\psi^a)] + \tilde{g}_0\sum_{a,b=1}^{N}(\bar{\psi}^a\psi^a)(\bar{\psi}^b\psi^b)\right\}, \tag{3}$$

where $a, b = 1, 2, \ldots, N$ are the replica indices, and it is necessary to put $N=0$ in the result. Both charges $g_0$ and $\tilde{g}_0$ are assumed small. This means, first, that the analysis is carried out near the point $\beta^2 = 4\pi$ (which corresponds to $g_0 = 0$ and to a temperature $T = 4\pi/n^2 \equiv T^*$), and second that the random field is weak.

In the zeroth approximation the only nonzero Green's function components are

$$G_+^{(0)}(\mathbf{r}) = \langle\psi_+^+(0)\psi_+(\mathbf{r})\rangle = \frac{1}{2\pi}\frac{x+iy}{|\mathbf{r}|^2},$$
$$G_-^{(0)}(\mathbf{r}) = \langle\psi_-^+(0)\psi_-(\mathbf{r})\rangle = \frac{1}{2\pi}\frac{x-iy}{|\mathbf{r}|^2}. \tag{4}$$

The vertices $g$ and $\tilde{g}$ are shown in the diagram representation in Fig. 1. In the parquet approximation, the renormalization of the vertex $g$ is given by the diagram of Fig. 2, and that of the vertex $\tilde{g}$ by the diagrams of Fig. 3. The corresponding parquet equations of the renormalization group are

$$\frac{dg}{d\xi} = -\frac{2}{\pi}g\tilde{g}, \quad \frac{d\tilde{g}}{d\xi} = -\frac{2}{\pi}\tilde{g}^2 + \frac{2}{\pi}g\tilde{g}. \tag{5}$$

Here $\xi = \ln(\Lambda'/\Lambda)$, and $\Lambda$ and $\Lambda'$ are the old and new cutoff scales at small distances. The asymptotic solutions of the system (5) depend essentially on the sign of $g|_{\xi=0} \equiv g_0$. At $g_0 < 0$ $(T > T^*)$ we have

$$g(\xi\to\infty) = g_0\exp(-\tilde{g}_0/|g_0|),$$
$$\tilde{g}(\xi\to\infty) = 0. \tag{6}$$

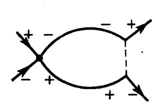

FIG. 2. Diagram representing the vertex $g$.

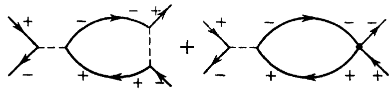

FIG. 3. Diagrams that renormalize the vertex $\tilde{g}$.

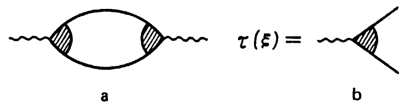


FIG. 5. a) Diagram representation of the four-fermion mean value $\langle (\bar\psi\psi)(0)(\bar\psi\psi)(\mathbf{x})\rangle$; b) vertex part of $\tau(\xi)$.

At $g_0 > 0$ ($T < T^*$) there is realized the "zero charge" asymptotic

$$g(\xi)\approx\frac{\pi}{2\xi\ln\xi},\quad \tilde{g}(\xi)\approx\frac{\pi}{2\xi}. \tag{7}$$

We note that at $\tilde{g}_0 \ll g_0$ the solution reach the asymptotic forms (7) at

$$\xi>\xi^*\simeq\frac{1}{g_0}\ln\frac{g_0}{\tilde{g}_0}\ll\frac{1}{\tilde{g}_0}.$$

Solutions (6) and (7) are shown in Fig. 4 in terms of the effective $XY$-model temperature $T_{eff}(\xi)=4\pi n^{-2}[1+g(\xi)/\pi]^{-1}$. The region of the initial temperatures $T^* < T < T_c = \pi/2$ where $T_c$ is Berezinskii-Kosterlitz-Thouless temperature) corresponds to a line of fixed points in which the correlators are scale-invariant, and the degree $\alpha$ of the spin-spin correlator $\langle\overline{[s(0)s(r)]}\rangle \simeq r^{-\alpha}$ changes from $2/n^2$ to $\frac{1}{4}$. The random fields are insignificant in this region. The effective temperature of the model turns out to be lower than the bare temperature (Fig. 4).

An interesting situation is observed in the region $T < T^*$. The presence of random fields leads, at large scales, to an increase of the effective temperature, which tends to the universal value $T_{eff}(\xi\to\infty)=T^*=4\pi/n^2$. This means that the system cannot be cooled below $T^*$, meaning also that a spin-glass state with "quenched" spins cannot be reached. Nonetheless the temperature dependence of the system susceptibility $\chi\approx\partial\overline{(\langle\cos\beta\varphi\rangle)}/\partial h$ relative to a nonzero mean field $\overline{h(x)}=0$ has a "quasi-cusp" at $T=T^*$. The width of the quasi-cusp $\delta\sim g_0$ is small if $h\ll 1, \tilde{g}_0\ll 1$ and $\tilde{g}_0\ln(1/h)\gg 1$. Indeed, since $\cos\beta\varphi$ corresponds to $(\bar\psi\psi)$, the calculation of $\chi$ reduces to calculation of the four-fermion mean value

$$\int d^2x\langle\overline{(\bar\psi\psi)(0)(\bar\psi\psi)(\mathbf{x})}\rangle,$$

which is the usual polarization operator (Fig. 5a). Therefore

$$\chi\approx\int_0^{\ln(1/h)} d\xi\,\tau^2(\xi),$$

where the angular part of $\tau(\xi)$ (Fig. 5b) is calculated with the aid of Eqs. (5) and the supplementary equation

$$\frac{d(\ln\tau)}{d\xi}=-\frac{1}{\pi}\tilde{g}(\xi)+\frac{1}{\pi}g(\xi). \tag{8}$$

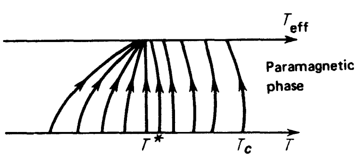


FIG. 4. Qualitative picture of the temperature renormalization at $n\geq 3\,[T_{eff}=T(\xi=\infty)]$.

The result is

$$\frac{1}{\chi}\frac{\partial\chi}{\partial g_0}\Big|_{g_0=-\delta}\approx\frac{1}{\tilde{g}_0}, \tag{9}$$

$$\frac{1}{\chi}\frac{\partial\chi}{\partial g_0}\Big|_{g_0=+\delta}\approx\frac{1}{\tilde{g}_0\ln(\tilde{g}_0\ln(1/h))}.$$

Therefore the relative slope of the $\chi(g_0)$ curve changes substantially near $g_0=0$ in a narrow region having a width of the order of $\tilde{g}_0$. At $g_0>0$ it follows from (8) that $\tau(\xi)\sim[(\ln\xi)/\xi]^{1/2}$, therefore in this region ($T<T^*$) the asymptotic form of the correlator is

$$K(\mathbf{x})=\overline{\langle\cos\beta\varphi(0)\cos\beta\varphi(x)\rangle}=\langle\overline{(\bar\psi\psi)(0)(\bar\psi\psi)(\mathbf{x})}\rangle\sim\frac{\ln\ln x}{x^2\ln x}. \tag{10}$$

As $T\to T^*+0$ we have $K(x)\sim x^{-2}$, and at $T=T^*$ we have $K(x)\sim x^{-2}\ln^{-1}x$. It must be noted that Eqs. (5) are valid only when $T-T^*\sim g_0\ll 1$, and not in the entire region $0<T<T^*$. Nonetheless, the parquet equations in the next order in $g$

$$\frac{dg}{d\xi}=-\frac{2}{\pi}g\tilde{g},\quad \frac{d\tilde{g}}{d\xi}=-\frac{2}{\pi}\tilde{g}^2(1+g/\pi)+\frac{2}{\pi}\tilde{g}\frac{g}{1+g/\pi}, \tag{11}$$

show now new tendencies whatever in the behavior of the solutions $g(\xi)$ and $\tilde{g}(\xi)$. We hope therefore that our results are qualitatively valid in the entire range $0<T<T^*$. (We note that the last term in the second equation of (11) is exact, as can be easily verified by comparing the renormalization of the vertex $g$ with the mass renormalization in the Thirring model.)

All the foregoing is valid only at $n\geq 3$, when $T^*<T_c$. In the cases $n=1, 2$ ($T^*>T_c$) the vortical excitations near $T^*$ cannot be neglected. The qualitative picture is here the following. It is obvious that weak random fields do not change qualitatively the properties of the paramagnetic phase of the pure system at $T>T_c$. The renormalization trajectories for $T<T_c$ are shown in Fig. 6. According to (5) the effective temperature increases to $T_c$, after which vortices appear and the paramagnetic phase sets in. There is therefore no phase transition in the system at $n=1$ and 2.

We note that our results are directly applicable to the $XY$ model with isotropically distributed random fields $h\cos n[\theta+\alpha(\mathbf{x})]$. The Hamiltonian has in this case a different symmetry, so that another behavior of the system is expected at $T<T^*$. A similar system was considered recently in Refs. 21 and 22, where, just as

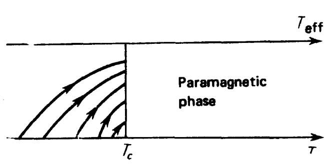


FIG. 6. Renormalization trajectories at $n=1$ and 2.

in our case, a universal temperature $T^* = 4\pi/n^2$, which limits at $n \geq 3$ the low-temperature phase of the $XY$ model, was obtained.

## III. QUASI-TWO-DIMENSIONAL $XY$ MODEL WITH RANDOM ANISOTROPY

In this section we generalize the analysis of Sec. II to include the case of a layered system with a weak coupling between layers. We shall use here the "fast parquet" method developed by Gor'kov and Dzyaloshinskiĭ[18] and by Obukhov[19] in the theory of quasi-one-dimensional metals.

Each layer is a $2D$ $XY$ model with random anisotropy $h(\mathbf{x})\cos n\theta$ (considered in Sec. II), and the coupling between the layers is specified by an interaction of special form $g_\perp \cos n(\theta_{i+1} - \theta_i)$, where $i$ is the number of the layer and $g_\perp$ is a parameter assumed to be small. The Hamiltonian of our problem is then

$$H = \sum_i \int d^2x \left[\frac{1}{2}(\partial_\mu\varphi_i)^2 + u\cos\beta\tilde{\varphi}_i + h_i(\mathbf{x})\cos\beta\varphi_i + g_\perp\cos\beta(\varphi_{i+1}-\varphi_i)\right]. \tag{12}$$

In the fermion representation this Hamiltonian is given by

$$H = \sum_i \int d^2x[(\bar{\psi}\hat{\partial}\psi)_i + h_i(x)(\bar{\psi}\psi)_i + g_0(\bar{\psi}\psi)_i^2 + g_\perp(\psi_+{}^+\psi_-)_i(\psi_-{}^+\psi_+)_{i+1}]. \tag{13}$$

After averaging over the random fields $h_i(x)$ we obtain

$$H_0 = \sum_{i,k} \int d^2x \left[\sum_{a=1}^{N} (\bar{\psi}^a\hat{\partial}\psi^a)\delta_{ik} + g_1{}^{ik}\sum_{a=1}^{N}(\psi_+{}^{a+}\psi_-{}^a)_i(\psi_-{}^{a+}\psi_+{}^a)_k + g_2{}^{ik}\sum_{a=1}^{N}(\psi_-{}^{a+}\psi_-{}^a)_i(\psi_+{}^{a+}\psi_+{}^a)_k + \tilde{g}^{ik}\sum_{a,b=1}^{N}(\bar{\psi}^a\psi^a)_i(\bar{\psi}^b\psi^b)_k\right]. \tag{14}$$

In this bare Hamiltonian the interaction constants $g_{1,2}$ and $\tilde{g}$ differ from zero only inside one layer $i = k$ and for the neighboring layers $i = k \pm 1$:

$$g_{10}{}^{ik} = g_0'\delta_{ik} + g_\perp\delta_{i,\,i\pm1}, \qquad g_{20}{}^{ik} = g_0''\delta_{ik}, \quad \tilde{g}_0{}^{ik} = \tilde{g}_0\delta_{ik} \tag{15}$$

(here $g_0' + g_0'' = g_0$), but the renormalization procedure can initiate also interaction with more remote layers.

The zeroth-approximation Green's functions in each layer are given by Eqs. (4). The vertices $g_1$, $g_2$, and $\tilde{g}$ are shown in the diagram representation in Fig. 7. The renormalization of the vertex $g_1^{ik}$ is given by the diagrams of Fig. 8, that of $g_2^{ik}$ by the diagrams of Fig. 9, and of the vertex $\tilde{g}^{ik}$ by the diagrams of Fig. 10.

The corresponding parquet equations of the renormalization group take the form (there is no summation over repeated indices)

$$\frac{dg_1{}^{ik}}{d\xi} = \frac{1}{\pi}\sum_l g_1{}^{il}g_1{}^{lk} + \frac{2}{\pi}g_1{}^{ik}g_2{}^{kk} - \frac{2}{\pi}g_1{}^{ik}g_2{}^{ik} - \frac{2}{\pi}g_2{}^{ik}\tilde{g}^{ik},$$
$$\frac{dg_2{}^{ik}}{d\xi} = -\frac{1}{\pi}g_1{}^{ik}g_1{}^{ik} - \frac{2}{\pi}g_1{}^{ik}\tilde{g}^{ik},$$
$$\frac{d\tilde{g}^{ik}}{d\xi} = \frac{2}{\pi}\sum_l \tilde{g}^{il}g_1{}^{lk} - \frac{2}{\pi}\tilde{g}^{ik}\tilde{g}^{kk} + \frac{2}{\pi}\tilde{g}^{ik}g_2{}^{kk}. \tag{16}$$

In the Fourier representation we obtain the system

$$\frac{dg_1(q)}{d\xi} = \frac{1}{\pi}g_1{}^2(q) + \frac{2}{\pi}g_1(q)\int\frac{dq'}{2\pi}g_2(q') - \frac{2}{\pi}\int\frac{dq'}{2\pi}g_1(q')g_2(q-q') - \frac{2}{\pi}\int\frac{dq'}{2\pi}g_2(q')\tilde{g}(q-q'),$$
$$\frac{dg_2(q)}{d\xi} = -\frac{1}{\pi}\int\frac{dq'}{2\pi}g_1(q')g_1(q-q') - \frac{2}{\pi}\int\frac{dq'}{2\pi}g_1(q')\tilde{g}(q-q'), \tag{17}$$
$$\frac{d\tilde{g}(q)}{d\xi} = \frac{2}{\pi}\tilde{g}(q)g_1(q) - \frac{2}{\pi}\tilde{g}(q)\int\frac{dq'}{2\pi}\tilde{g}(q') + \frac{2}{\pi}\tilde{g}(q)\int\frac{dq'}{2\pi}g_2(q')$$

with initial conditions in the form

$$g_1(\xi, q)|_{\xi=0} = g_0' + g_\perp\cos q, \quad g_2(\xi, q)|_{\xi=0} = g_0'', \quad \tilde{g}(\xi, q)|_{\xi=0} = \tilde{g}_0. \tag{18}$$

Just as in the two-dimensional case, the form of the solution depends essentially on the sign of $g_0$. Substituting in the system (17) the solutions in the form

$$g(\xi, q) = c(\xi) + c'(\xi)\cos q, \tag{19}$$

we can easily verify (see Appendix A) that at $c_1(0) + c_2(0) = g_0 < 0$ ($T > T^*$) there is realized inside the layers the two-dimensional situation described in Sec. II:

$$(c_1 + c_2)(\xi\to\infty) \approx g_0\exp(-\tilde{g}_0/|g_0|) \equiv g_\infty,$$
$$\tilde{c}(\xi\to\infty) \approx \tilde{g}_0\exp\left(-\frac{2}{\pi}|g_\infty|\xi\right) \to 0, \tag{20}$$

and the coupling of the layers remains weak

$$c_1'(\xi\to\infty) \approx g_\perp\exp\left(-\frac{2}{\pi}|g_\infty|\xi\right) \to 0,$$
$$c_2'(\xi\to\infty) \simeq \frac{g_\perp{}^2}{4|g_\infty|}\left(1 + 2\frac{\tilde{g}_0}{|g_\infty|}\right) \ll |g_\infty|, \tag{21}$$
$$\tilde{c}(\xi\to\infty) \approx \tilde{g}_0\frac{g_\perp}{|g_\infty|}\exp\left(-\frac{2}{\pi}|g_\infty|\xi\right) \ll \tilde{c}(\xi) \to 0.$$

In the case $g_0 > 0$ solutions of type (19) will not do, since the coupling $c'$ between the layers becomes of the order of $c$ (see Appendix A) and account must be taken of the next-order harmonics, i.e., the problem becomes essentially three-dimensional. In this case one might expect the solutions to be of the moving-pole type, as is the case in the theory of quasi-one-dimensional metals[19]:

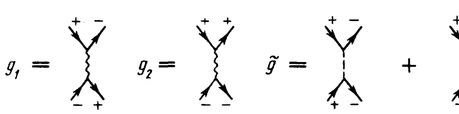

FIG. 7. Diagram representation of the vertices $g_1^{ik}$, $g_2^{ik}$, and $\tilde{g}^{ik}$.

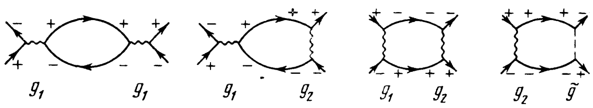

FIG. 8. Diagrams that renormalize the vertex $g_1^{ik}$.

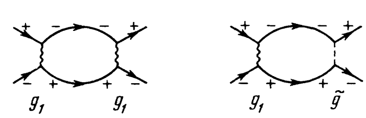

FIG. 9. Diagrams that renormalize the vertex $g_2^{ik}$.

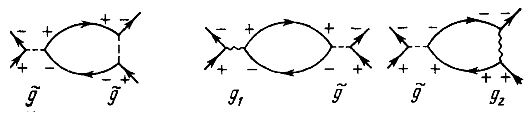

FIG. 10. Diagrams that renormalize the vertex $\tilde{g}^{ik}$.

$$g(\xi, q) \approx 1/(\xi_0(q)-\xi),$$

where $\xi_0(q) \approx \xi_0 - aq^2$ at small momenta. Reasoning as in Ref. 19, one might assume that the pole singularity in the integral terms of the system (17) can be integrated and the principal terms are the first of the first equation, which yields

$$g_1(\xi, q) \approx \frac{\pi}{\xi_0(q)-\xi},$$

and the first term in the third equation, which leads to

$$\tilde{g}(\xi, q) \approx \mathrm{const}/(\xi_0(q)-\xi)^2.$$

Since $\tilde{g}(\xi, q)$ has a stronger singularity than $g_1(\xi, q)$, those terms of (17) which contains integrals of $\tilde{g}(\xi, q)$ turn out to be more singular as $\xi \to \xi_0$ than the non-integral terms. Thus, a solution of the "moving-pole" type is found to be unstable to inclusion of $\tilde{g} \neq 0$; this is one more manifestation of the known destruction of the long-range order by a random field at $d \leqslant 4$. It is interesting to note that a formal analysis of the system (17) at $d > 4$ shows that the "moving pole" solution is stable, for in this case the integration is with respect to $d^a q'$, where $a = d-2 > 2$ and the singularities of the integral terms are indeed weaker.

To understand what happens in our case at $g_0 > 0$ we start from the fact that with increasing $\xi$ an ever increasing number of layers begins to interact. In the coordinate representation we seek the solution in the form

$$g^{ik}(\xi) \approx \exp(-\lambda(\xi) l_{ik}),$$

where $l_{ik}$ is the distance between the layers $i$ and $k$ at $l_{ik} \gg 1/\lambda$. It turns out that such a solution actually exists and is of the form (see Appendix B)

$$g_1^{ik}(\xi) \approx a_1 \exp(-\lambda(\xi) l_{ik}), \quad g_2^{ik}(\xi) \approx -a_2 \exp(-2\lambda(\xi) l_{ik}), \qquad (22)$$
$$\tilde{g}^{ik}(\xi) \approx a_2 l_{ik} \exp(-\lambda(\xi) l_{ik}).$$

Here

$$\lambda(\xi) = \lambda_0 - \frac{1}{\pi} a_1 \xi, \qquad (23)$$

$\lambda_0$ and $a_{1,2}$ are certain positive parameters that must be determined from the initial conditions ($\lambda_0 \sim 1, a_1 \sim g_0$). Thus, at $T < T^*$ the interaction between the layers takes the form of an attenuating exponential in $l_{ik}$, whose attenuation length increases with increasing $\xi$ in accord with (23).

So long as $\xi$ is small enough and $\lambda(\xi) \simeq \lambda_0 \sim 1$ only the nearest layers interact and equalization takes place of the constants of the interaction between the nearest layers and inside the layers, as described in Appendix A. But when $\lambda(\xi)$ becomes much smaller than unity, the solution in the inner region $l_{ik} \ll 1/\lambda(\xi)$ is independent of $l_{ik}$ and assumes a form that corresponds in the $q$-representation to a "standing pole" (see Appendix C):

$$g_1(\xi) \approx \frac{3\pi}{4} \frac{\lambda(\xi)}{\xi_0-\xi}, \quad g_2(\xi) \simeq -\frac{3\pi}{8} \frac{\lambda(\xi)}{\xi_0-\xi}, \qquad (24)$$
$$\tilde{g}(\xi) \simeq \frac{\pi}{4} \frac{1}{\xi_0-\xi}.$$

The following remark is in order here. The solutions (24) are meaningful only if $\xi_0 - \xi$ vanishes earlier than $\lambda(\xi)$. It is seen from (22) that if $\lambda(\xi)$ vanishes earlier, all the layers of the system begin immediately to interact simultaneously. (Such a solution of the system (16) with charges independent of $i$ and $k$ does formally exist:

$$g_1 = \frac{3\pi}{4} \frac{1}{L(\xi_0-\xi)}, \quad g_2 = -\frac{3\pi}{8} \frac{1}{L(\xi_0-\xi)},$$
$$\tilde{g} = \frac{\pi}{4} \frac{1}{\xi_0-\xi},$$

where $L$ is the size of the system.) Such a situation seems quite strange from the physical point of view, and we assume therefore that $\lambda(\xi) \neq 0$ at $\xi < \xi_0$.

It is known[18,19] that the presence of a moving pole of the type $[\xi_0(q) - \xi]^{-1}$ points to a transition into a spatially ordered phase: a certain $q_0$ exists at which a pole appears first ($q_0 = 0$ in the case of a ferromagnetic transition). The fact that in our case we have precisely a standing pole (the pole appears simultaneously for all $q$) indicates that below $T^*$ there occurs a transition into a spatially fully disordered state (of the spin-glass type).

All the foregoing pertains only to the case when the random field has an anisotropy order $n \geqslant 3$. At $n=1$ and 2 ($T^* < T$) it is necessary to take into account near $T^*$ the vortices, and it is not clear what happens in this case.

## CONCLUSION

We have considered an $XY$ magnet with random anisotropy of $h(\mathbf{x}) \cos n\theta$. A quantitative analysis was carried for $n \geqslant 3$, for in this case we can neglect the vortical excitations in the analysis of the effects connected with the random fields. A qualitative examination of the situation at $n=1$ and 2 points to the absence of any phase transitions whatever in a $2D$ system. At $n \geqslant 3$ there exists a temperature region $T^* < T < T_c$ where the random field is inessential. We have shown that at $T < T^*$ the interaction of the thermal fluctuations with random field leads to an effective "heating" of the $2D$ system to a universal effective temperature $T^*$, so that a phase with randomly quenched moments exists. It was found convenient in this case to use a representation of our model in terms of fermion variables that describe soliton excitations of the initial boson field; this confirms the importance of taking into account excitations of topological character in strongly disordered systems.

To study the properties of a three-dimensional system we chose a quasi-two-dimensional model with weak interlayer interaction ($J_\perp / J = g_1 \ll 1$) of special form $J_\perp \cos(\theta_{i+1} - \theta_i)$. The scale dimensionality of this term is $2 - n^2\alpha(T)$, whereas the dimensionality of the ordinary interaction $J\cos(\theta_{i+1} - \theta_i)$ equals $2 - \alpha(T)$. The factor $n$ therefore decreases the scale dimensionality of our interaction, and this should also weaken the effect of the interaction between the layers. Nonetheless, this interaction turns out to be quite substantial at $T < T^*$ and leads to a solution of the "standing-pole" type in the fast-parquet equations. This is a weighty argument favoring the existence of a spin-glass state at $T < T^*$, although our method only indicates the character of the instability of the high-temperature phase, without making an investigation of the low-temperature structure possible.

It must be noted that in the employed special model it would be possible to disregard excitations of the vortex-line type, since the temperature $T^*$ turned out to be lower than the vortex-formation temperature $T_c$. In a real system the interaction between layers is of the form $J\cos(\theta_{i+1}-\theta_i)$ and is significant at all $T < T_c$, so that effects connected with randomness, vortices, and three-dimensional interaction cannot be separated. Since, however, the instability that leads to the phase transition into the spin-glass state was observed for a substantially weakened interlayer interaction $J_\perp \cos n(\theta_{i+1} - \theta_i)$, it is natural to propose that it exists in a real system with an interlayer interaction $J\cos(\theta_{i+1}-\theta_i)$ and with random fields $h(\mathbf{x})\cos n\theta$. To cast light on this question we must know the properties of the vortex lines in the system with random field (we point out in this connection a recent paper,[23] in which it is shown that in an Ising magnet with random field the "surface tension" of the domain wall vanishes at $d \leqslant 3$). We emphasize in conclusion that our results (apart from the value of the characteristic temperature $T^*$) cannot be transferred directly to the case of isotropically distributed random fields $[h_1(\mathbf{x})\cos(n\theta) + h_2(\mathbf{x})\sin(n\theta);\ \overline{h_i(\mathbf{x})h_k(\mathbf{x}')} = \delta_{ik}\tilde{g}\delta(\mathbf{x}-\mathbf{x}')]$, for in this case the parquet equations take a different form and require a separate treatment.

In conclusion we wish to thank L. P. Gor'kov, S. P. Obukhov, and V. L. Pokrovskiĭ for numerous helpful discussions.

## APPENDIX A

Substitution of solutions in the form

$$g(\xi, q) = c(\xi) + c'(\xi)\cos q \tag{A1}$$

in the system (17) yields two systems of equations

$$\begin{cases} \dfrac{dc_1}{d\xi} = \dfrac{1}{\pi}c_1^2 - \dfrac{2}{\pi}c_2\tilde{c} + \dfrac{1}{2\pi}(c_1')^2, \\ \dfrac{dc_2}{d\xi} = -\dfrac{1}{\pi}c_1^2 - \dfrac{2}{\pi}c_1\tilde{c}, \\ \dfrac{d\tilde{c}}{d\xi} = \dfrac{2}{\pi}(c_1+c_2)\tilde{c} - \dfrac{2}{\pi}\tilde{c}^2 + \dfrac{1}{\pi}\tilde{c}'c'; \end{cases} \tag{A2}$$

$$\begin{cases} \dfrac{dc_1'}{d\xi} = \dfrac{2}{\pi}(c_1+c_2)c_1' - \dfrac{1}{\pi}c_1'c_2' - \dfrac{1}{\pi}c_2'\tilde{c}', \\ \dfrac{dc_2'}{d\xi} = -\dfrac{1}{2\pi}(c_1')^2 - \dfrac{1}{\pi}c_1'\tilde{c}', \\ \dfrac{d\tilde{c}'}{d\xi} = \dfrac{2}{\pi}(c_1+c_2)\tilde{c}' + \dfrac{2}{\pi}\tilde{c}c_1' - \dfrac{2}{\pi}\tilde{c}\tilde{c}'. \end{cases} \tag{A3}$$

Adding the first two equations of (A2) we obtain

$$\begin{aligned} &\frac{d}{d\xi}(c_1+c_2) = -\frac{2}{\pi}(c_1+c_2)\tilde{c} + \frac{1}{2\pi}(c_1')^2, \\ &\frac{d\tilde{c}}{d\xi} = \frac{2}{\pi}(c_1+c_2)\tilde{c} - \frac{2}{\pi}\tilde{c}^2 + \frac{1}{\pi}\tilde{c}'c_1'. \end{aligned} \tag{A4}$$

Accurate to the small terms $(c_1')^2$ and $\tilde{c}'c_1'$, these are Eqs. (5) and have at $(c_1+c_2)|_{\xi=0} \equiv g_0 < 0$ the solutions

$$\begin{aligned} &(c_1+c_2)(\xi\to\infty) = g_0\exp(-\tilde{g}_0/|g_0|) \equiv g_\infty, \\ &\tilde{c}(\xi\to\infty) = \tilde{g}_0\exp\left(-\frac{2}{\pi}|g_\infty|\xi\right) \to 0. \end{aligned} \tag{A5}$$

Substituting (A5) in the system (A3) we obtain

$$\begin{aligned} &c_1'(\xi\to\infty) = g_\perp\exp\left(-\frac{2}{\pi}|g_\infty|\xi\right) \to 0, \\ &c_2'(\xi\to\infty) = -\frac{g_\perp^2}{4|g_\infty|}\left(1+2\frac{\tilde{g}_0}{|g_\infty|}\right) \ll g_\infty, \\ &\tilde{c}'(\xi\to\infty) = \tilde{g}_0\frac{g_\perp}{|g_\infty|}\exp\left(-\frac{2}{\pi}|g_\infty|\xi\right) \ll \tilde{c}(\xi) \to 0, \end{aligned} \tag{A6}$$

i.e., the coupling between the layers remains weak and we can confine ourselves in the expansion to the first harmonic (A1).

On the other hand, if $g_0 > 0$, $c$ and $c'$ become of the same order of magnitude when the solutions (A4) approach the zero-charge asymptotic form (7). Since the principal terms in Eqs. (A3) are the first terms in the first and third equations, the $c'(\xi)$ first increase like $\exp(2g_0\xi/\pi)$ at $g_0 > 0$, and this growth continues until the negative terms in the equations become dominant. In this case $c$ and $c'$ become quantities of the same order and expression (A1) no longer holds, since it is necessary to take into account the higher harmonics.

## APPENDIX B

We seek solutions of the system (16) in the form

$$g_\alpha^{ik}(\xi) \sim \exp(-\lambda_\alpha(\xi)l_{ik})$$

where $l_{ik} \gg 1/\lambda$. We first verify easily by direct integration that

$$\begin{aligned} \int_{-\infty}^{+\infty} dx\exp(-\lambda|x_1-x|-\lambda|x-x_2|) &= (1/\lambda+|x_1-x_2|)\exp(-\lambda|x_1-x_2|) \\ &\approx |x_1-x_2|\exp(-\lambda|x_1-x_2|), \end{aligned} \tag{B1}$$

$$\int_{-\infty}^{+\infty} dx|x_1-x|\exp(-\lambda|x_1-x|-\lambda|x-x_2|) \approx {}^1/_2|x_1-x_2|^2\exp(-\lambda|x_1-x_2|). \tag{B2}$$

Owing to the derivatives, the left-hand sides of (16) contain terms $\sim l_{ik}\exp(-\lambda l_{ik})$. Therefore, for such a term to appear in the right-hand side of the second equation (where there is no summation over the layers), we must choose $\tilde{g}^{ik} \sim l_{ik}\exp(-\tilde{\lambda}l_{ik})$. In addition, it can be seen from the second and third equations that $\lambda_2(\xi) = 2\lambda_1(\xi)$ and $\lambda_1(\xi) = \tilde{\lambda}(\xi)$. We seek therefore the solution in the form

$$\begin{aligned} &g_1^{ik}(\xi) = a_1\exp(-\lambda(\xi)l_{ik}), \quad g_2^{ik}(\xi) = -a_2\exp(-2\lambda(\xi)l_{ik}), \\ &\tilde{g}^{ik}(\xi) = \tilde{a}l_{ik}\exp(-\lambda(\xi)l_{ik}). \end{aligned} \tag{B3}$$

Substituting (B3) in the system (16) and retaining the terms principal in $l_{ik}$ and $\exp(-\lambda l_{ik})$, we obtain

$$-a_1\frac{d\lambda}{d\xi} = \frac{1}{\pi}a_1^2, \quad 2a_2\frac{d\lambda}{d\xi} = -\frac{2}{\pi}a_1\tilde{a}, \quad -\tilde{a}\frac{d\lambda}{d\xi} = \frac{2}{\pi}\tilde{a}a_1\cdot\frac{1}{2}. \tag{B4}$$

From this we easily find that

$$\lambda(\xi) = \lambda_0 - \frac{1}{\pi}a_1\xi, \quad \tilde{a} = a_2.$$

## APPENDIX C

Since the layer interaction attenuates exponentially at $l_{ik} \gtrsim 1/\lambda$, the summation in the terms such as $\sum_l g^{il} g^{lk}$ can be cut off at $1/\lambda$. Assuming that $g^{ik}$ in the inner region $l_{ik} \ll 1/\lambda$ are independent of $l_{ik}$, we seek the solutions in the form

$$g_a(\xi) = \frac{b_a(\xi)}{\xi_0 - \xi}. \tag{C1}$$

We substitute (C1) in (16):

$$\frac{b_1}{(\xi_0-\xi)^2} + \frac{1}{(\xi_0-\xi)}\frac{db_1}{d\xi} = \frac{1}{\pi}\frac{b_1^2}{\lambda(\xi)}\frac{1}{(\xi_0-\xi)^2} - \frac{2}{\pi}\frac{b_2\tilde{b}}{(\xi_0-\xi)^2},$$

$$\frac{b_2}{(\xi_0-\xi)^2} + \frac{1}{(\xi_0-\xi)}\frac{db_2}{d\xi} = -\frac{1}{\pi}\frac{b_1^2}{(\xi_0-\xi)^2} - \frac{2}{\pi}\frac{b_1\tilde{b}}{(\xi_0-\xi)^2}, \tag{C2}$$

$$\frac{\tilde{b}}{(\xi_0-\xi)^2} + \frac{1}{(\xi_0-\xi)}\frac{d\tilde{b}}{d\xi} = \frac{2}{\pi}\frac{\tilde{b}b_1}{\lambda(\xi)(\xi_0-\xi)^2} - \frac{2}{\pi}\frac{\tilde{b}^2}{(\xi_0-\xi)^2} + \frac{2}{\pi}\frac{\tilde{b}b_2}{(\xi_0-\xi)^2}.$$

We choose $b_{1,2}(\xi) = c_{1,2}\lambda(\xi)$ and $\tilde{b} = \tilde{c}$ ($c_{1,2}$ and $\tilde{c}$ are numbers). We then obtain near the pole $\xi_0$ the system

$$c_1 = \frac{1}{\pi}c_1^2 - \frac{2}{\pi}c_2\tilde{c}, \quad c_2 = -\frac{2}{\pi}c_1\tilde{c},$$

$$\tilde{c} = -\frac{2}{\pi}\tilde{c}^2 + \frac{2}{\pi}c_1\tilde{c} + \frac{2}{\pi}c_2\tilde{c}.$$

This system has a solution

$$c_1 = \frac{3\pi}{4}, \quad c_2 = -\frac{3\pi}{8}, \quad \tilde{c} = \frac{\pi}{4}.$$

A second solution, in which $\tilde{c} < 0$, has no physical meaning, since $\tilde{g}$ is essentially a positive quantity.


[1]S. F. Edwards and P. W. Anderson, J. Phys. **F5**, 965 (1975).

[2]V. L. Berezinskiĭ, Zh. Eksp. Teor. Fiz. **59**, 907 (1970); **61**, 1144 (1971) [Sov. Phys. JETP **32**, 493 (1971); **34**, 610 (1972)].

[3]J. M. Kosterlitz and D. J. Thouless, J. Phys. **C6**, 1181 (1973). J. M. Kosterlitz, *ibid.* **C7**, 1046 (1974).

[4]J. V. Jose, L. P. Kadanoff, S. Kirkpatrick, and D. A. Nelson, Phys. Rev. **B16**, 1217 (1977).

[5]P. B. Wiegmann, J. Phys. **C11**, 1583 (1978).

[6]S. Coleman, Phys. Rev. **D11**, 2088 (1975).

[7]S. Mandelstamm, *ibid.*, p. 3026.

[8]A. B. Zamoldchikov, Preprint ITEF-91, 1976.

[9]J. J. Villain, J. Phys. **C10**, 4793 (1977); **C11**, 745 (1978).

[10]E. Fradkin, B. A. Huberman, and S. H. Shenker, Phys. Rev. **B18**, 4789 (1978).

[11]G. Toulouse, Comm. Phys. **2**, 115 (1977).

[12]Y. Imry, and S. K. Ma, Phys. Rev. Lett. **35**, 1399 (1975).

[13]R. A. Pelcovits, E. Pytte, and J. Rudnick, *ibid.* **40**, 476 (1978).

[14]A. I. Bray and M. A. Moore, J. Phys. **C12**, 79 (1979).

[15]M. V. Feĭgel'man and A. M. Tsvelik, Zh. Eksp. Teor. Fiz. **77**, 2524 (1979) [Sov. Phys. JETP **50**, 1222 (1979)].

[16]E. Pytte and J. Rudnick, Phys. Rev. **B19**, 3603 (1979).

[17]Vik. S. Dotsenko and M. V. Feĭgelman, J. Phys. C14, L823 (1981).

[18]L. P. Gor'kov and I. E. Dzyaloshinskiĭ, Zh. Eksp. Teor. Fiz. **67**, 379 (1974) [Sov. Phys. JETP **40**, 198 (1975)].

[19]S. P. Obukhov, *ibid.* **72**, 1051 (1977) [**45**, 550 (1977)].

[20]P. G. de Gennes, Phys. Lett. **38A**, 339 (1972).

[21]A. Houghton, R. D. Kenway, and S. C. Ying, Phys. Rev. **B23**, 298 (1981).

[22]John L. Cardy and S. Ostlund, Preprint, 1981.

[23]E. Pytte, Y. Ymry, and D. Mukamel, Phys. Rev. Lett. **46**, 1173 (1981).


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