On the freezing-in integrals and Lagrange invariants in hydrodynamic models

S. S. Moiseev, R. Z. Sagdeev, A. V. Tur, and V. V. Yanovskiĭ

Institute of Space Research, Academy of Sciences of the USSR
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The topologically nontrivial solutions (called topological solitons in field theory) in the hydrodynamics of a plasma are studied. A procedure for obtaining new freezing-in integrals and Lagrange invariants on the basis of the already known invariants is proposed. The relations between these invariants are studied. Exact solutions describing topologically nontrivial spherical vortices are obtained.

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A large number of physical phenomena are described by equations that admit of a hydrodynamic interpretation. In these cases an important role is played in the quantitative and qualitative analyses by the freezing-in integrals and the Lagrange invariants that arise within the framework of the specific equations. Normally, the search for them is essentially based on the form of the equations describing the continuous medium. The invariants obtained then change when the hydrodynamic model is changed. It seems natural in constructing them to proceed from the canonical set of Lagrange invariants that arise in the formulation of the variational principle. The existence of such a set can be verified on the basis of the following simple arguments. By applying the canonical transformations, we can always go over to the initial coordinates and the initial generalized momenta, which are Lagrange invariants. In the present paper, proceeding from a certain set of Lagrange invariants, we construct topological invariants. The latter reflect the entanglement of the field lines of the frozen-in quantities, and can play the role of a topological charge for the localized motions. Furthermore, the interrelationship between the Lagrange invariants and the freezing-in integrals is elucidated. The relations obtained are applicable to any hydrodynamic model for a nondissipative medium. As an example, we give the invariants of a compressible fluid and the two-fluid model of a plasma in the adiabatic case. In conclusion, we consider the problems connected with the use of the invariants obtained. In particular, we find the steady-state solutions of the two-fluid hydrodynamics of a plasma that describe spherical plasma vortices, interest in which has lately risen in connection with the study of the feasibility of thermonuclear fusion in the "spheromak" type of configurations.[1] Plasma waves reminiscent of the Rossby waves are reported in the nonstationary case.

1. It is well known that the quantities satisfying in the Euler variables the equation

$$\partial I/\partial t+(\mathbf{v}\nabla)I=0, \tag{1}$$

which reflects the conservation of the quantity I in its transport by the fluid particles, are called Lagrange invariants. Another quantity, studied in the present paper, namely, the freezing-in integral, is given by the equation

$$d\mathbf{J}/dt=(\mathbf{J}\nabla)\mathbf{v}. \tag{2}$$

Equation (2) describes the freezing in of the J-field lines in the fluid. Below we consider a number of relations that do not depend on the specific form of the equations describing the velocity field. In deriving them we use only the continuity equation for the density ρ (or some other quantity).

Let there exist, besides the Lagrange invariants $\mathbf{x}_0$ (the initial coordinates of the Lagrange particles), a set of other Lagrange invariants I_1, I_2, and I_3. Let us treat them as a new Lagrange system of coordinates, and go over from the coordinates x_1^0, x_2^0, and x_3^0 to I_1, I_2, and I_3. Let us, in doing, this, go over from x_0 to the Euler variables $\mathbf{x}$, and from them to I_1, I_2, and I_3, i.e., effect the transition $\mathbf{x}_0\to\mathbf{x}\to\mathbf{I}$. It is evident that, since the volume elements in both the old and the new coordinates are Lagrange invariants, the Jacobian of the transition is also a Lagrange invariant. Thus, we obtain a new Lagrange invariant:

$$I_4=\frac{1}{\rho}\frac{\partial(I_1,I_2,I_3)}{\partial(x_1,x_2,x_3)}. \tag{3}$$

By repeatedly applying the relation (3), we can construct Lagrange conservation laws from three or more known Lagrange invariants.

Another relation that allows us to construct Lagrange invariants form a known freezing-in integral is

$$I'=\mathbf{J}\nabla I. \tag{4}$$

Indeed, that I' is a Lagrange invariant can be directly verified by computing dI'/dt with allowance for (1) and (2). But physical arguments allow us to predict the answer. Let us, for this purpose, consider the surface $I=\text{const}$, and choose a contour element lying on this surface. It is clear that, owing to the conservation of the Lagrange invariant I, the contour will remain on the surface $I=\text{const}$ in any motion. The direction of the area element $d\overline{\mathbf{s}}$ enclosed by the contour element coincides with the direction of ∇I. Moreover, noting that the equation for ∇I coincides with the equation describing $\rho d\mathbf{s}$, let us choose a const, such that $\nabla I=\text{const}\times\rho d\mathbf{s}$. Then the relation (4) can be interpreted as expressing the conservation of the flux of the frozen-in quantity, thereby proving that the quantity I' is a Lagrange invariant.

Let us now proceed to construct frozen-in quantities from given Lagrange invariants. We use in the construction the physical meaning of the freezing-in inte-

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grals. The relation (2) can be interpreted as follows. Let us assume that at some moment of time t_0 the vector J coincides in direction with a length element $dl(t_0)$ of a streamline. Then this element will subsequently continue to coincide in direction with $dl(t)$, remaining in constant proportion, i.e.,

$$\mathbf{J}(t)/|\mathbf{J}(t_0)|=dl(t)/|dl(t_0)|.$$

The last relation reflects the existence of some similitude between the frozen-in quantities and the dl field. We can easily construct on the basis of the above-given physical interpretation a frozen-in quantity from two Lagrange invariants: $I_1(\mathbf{x},t)$ and $I_2(\mathbf{x},t)$. Indeed, let us consider in the space of the initial coordinates $\mathbf{x}_0$ the surfaces $I_1(\mathbf{x}_0)=I_1^0$ and $I_2(\mathbf{x}_0)=I_2^0$ of fixed values of the invariants. In the case of a general configuration they intersect along some curve. Let us choose the element dl along the intersection of these surfaces, so that it is an element of the streamline specified by the invariants I_1 and I_2. It is evident that the vector $\nabla I_1\times\nabla I_2$ is oriented along the tangent to dl, i.e., it always coincides with this element in direction, maintaining the similitude in the process as a result of the nondependence of these vectors on the time in this coordinate system. Going over to the Euler system of coordinates, we obtain the expression for the freezing-in integral:

$$\mathbf{J}=\frac{1}{\rho}[\nabla I_1\times\nabla I_2]. \tag{5}$$

Thus, the existence of Lagrange invariants allowing us to construct quantities that are frozen in the medium, and satisfy (2). Naturally, this can be verified by differentiating (5) with respect to time. Because Eq. (2) is linear in J, the sum of the frozen-in quantities is again a frozen-in quantity. Furthermore, the multiplication of the quantity J by a Lagrange invariant does not destroy its frozen-in character. It is easy to understand on the basis of these simple arguments that the quantity constructed from three Lagrange invariants I_1, I_2, and I_3 is frozen in the medium:

$$J_\alpha=\frac{1}{\rho}\varepsilon_{\alpha\beta\gamma}\varepsilon_{ikl}I_i\frac{\partial I_k}{\partial x_\beta}\frac{\partial I_l}{\partial x_\gamma}, \tag{6}$$

where the subscripts assume the values 1, 2, and 3.

The constructed quantity also has an independent meaning, e.g., in terms of n-fields.[2] An even more general expression for the frozen-in quantities has the form

$$J_\alpha=\frac{1}{\rho}\varepsilon_{\alpha\beta\gamma}\mathbf{I}\left[\frac{\partial \mathbf{I}'}{\partial x_\beta}\times\frac{\partial \mathbf{I}''}{\partial x_\gamma}\right].$$

Here I, I′, and I″ are different, and it is only essential that their components be Lagrange invariants.

It should be noted that, like the quantities (5), all the known fields that are frozen in the medium (e.g., $\mathbf{J}=\mathbf{H}/\rho$, $\rho^{-1}\,\mathrm{curl}\mathbf{v}$, etc.) possess the property $\mathrm{div}\rho\mathbf{J}=0$. The constructed fields (6) in the case of the general configuration do not possess this property, and are an example, from this point of view, of frozen-in quantities of a new type. In this case the two forms of freezing-in differential relations, namely, (2) and

$$\partial\rho\mathbf{J}/\partial t=\mathrm{rot}[\rho\mathbf{J}\times\mathbf{v}], \tag{7}$$

are not equivalent, since from (2) we have for ρJ an equation of the form

$$\partial\rho\mathbf{J}/\partial t=\mathrm{rot}[\rho\mathbf{J}\times\mathbf{v}]-\mathbf{v}\,\mathrm{div}\,\rho\mathbf{J},$$

which, for $\mathrm{div}\rho\mathbf{J}\neq0$, does not coincide with (7).

The conservation of the flux ρJ follows from (2), and, therefore, as the differential equation reflecting the conservation of the flux ρJ, we must take Eq. (2), which preserves its form also in the $\mathrm{div}\rho\mathbf{J}\neq0$ case. A consequence of the nonvanishing of $\mathrm{div}\rho\mathbf{J}$ is the nonvanishing of the flux of the frozen-in quantity through a closed surface and the emergence thereby of a charge analog for such fields. For apparent reasons, no general circulation theorem exists. If the Lagrange invariants in (6) are dependent [i.e., if there exists a relation $\varphi(I_1,I_2,I_3)=0$], then the $\mathrm{div}\rho\mathbf{J}$ vanishes, and we return to frozen-in quantities of the ordinary type.

Besides the above-noted differences, (5) and (6) also possess certain topological differences within the limits of one and the same class. But before investigating them, let us introduce the topological characteristics of the above-constructed freezing-in integrals. To obtain these characteristics, we consider the purely rotational field of the quantities ρJ (i.e., in the case when I_1 and I_3 are independent). In this case the introduction of the vector potential A ($\rho\mathbf{J}=\mathrm{curl}\mathbf{A}$) is admissible. Let us consider the quantity

$$I^\tau=\int_{V_0}\rho\mathbf{J}\mathbf{A}\,dv. \tag{8}$$

Here the integration is over the volume enclosed by the surface S_0, the normal to which is orthogonal to J (i.e., $\mathbf{n}\cdot\mathbf{J}=0$) at all times. The above-introduced quantity characterizes the entanglement of the field lines of the frozen-in quantities, and is conserved in time.

In proving this assertion, we use for the vector potential A an equation that follows from the specific form of J, (5) and (6):

$$\partial\mathbf{A}/\partial t=[\mathbf{v}\times\mathrm{rot}\,\mathbf{A}]+\nabla\psi. \tag{9}$$

The form of the function ψ is unimportant. Using (9) and (2), we obtain

$$d\mathbf{J}\mathbf{A}/dt=(\mathbf{J}\nabla)(\psi+\mathbf{A}\mathbf{v}).$$

It is easy to show, using the equation obtained, that

$$dI^\tau/dt=\int_{S_0}\rho J(\psi+\mathbf{A}\mathbf{v})\,ds=\int_{S_0}\rho\mathbf{J}\mathbf{n}(\psi+\mathbf{A}\mathbf{v})\,ds,$$

which vanishes as a result of the fact that $\mathbf{J}\cdot\mathbf{n}=0$ by definition.

The physical interpretation of the Lagrange invariants (7) is entirely similar to the interpretation of their particular cases in a compressible fluid and in magnetohydrodynamics.[3] The following natural question arises: What other conservation laws of a similar type are possible in hydrodynamic problems? To answer this question, let us consider the form of the differential equation that gives rise to such conservation laws. It is clear that, if

$$d\mathbf{J}\mathbf{B}/dt=(\mathbf{J}\nabla)\mathbf{B},$$

then the quantity

$$\int_{V_0} \rho \mathbf{J}\mathbf{B} dv$$

with the same limitation on the direction (i.e., for $\mathbf{J} \cdot \mathbf{n}=0$) is conserved in the course of the evolution. The equation given above gives rise to a limitation on the admissible **B** fields. Indeed, it follows from the equation that **B** should satisfy the identity

$$\mathbf{J}\frac{\partial \mathbf{B}}{\partial t}+\mathbf{J}(\mathbf{v}\nabla)\mathbf{B}-\mathbf{v}(\mathbf{J}\nabla)\mathbf{B}=(\mathbf{J}\nabla)\varphi,$$

where φ is arbitrary. It is easy to verify that this relation is satisfied by $\mathbf{B}_1=\nabla\psi$ and $\mathbf{B}_2=I_\alpha\nabla I'_\alpha$ (I_α and I'_α are Lagrange invariants). On the basis of the linearity of the identity, and with allowance for the invariance of the equation under addition to B of terms orthogonal to J, we obtain the form of the B fields that give rise to conservation laws:

$$\mathbf{B}=\nabla\varphi+I_\alpha\nabla I_\alpha'+g[\mathbf{v}\times\mathbf{J}]+g_\alpha[\nabla I_\alpha\times\mathbf{J}];$$

here g, g'_α, and φ are arbitrary functions of the coordinates and the time. Thus, we arrive at the laws of conservation of the quantities

$$I=\int_{V_0} \mathbf{B}\mathbf{J}\rho dv \tag{10}$$

in time if $\mathbf{J}\cdot\mathbf{n}=0$ (n is the normal to the surface enclosing the volume V_0). The value of this invariant is the same for an entire class of fields differing in their potential parts and in their parts orthogonal to J. In a compressible fluid and in magnetohydrodynamics, for example, the flow velocity **v** coincides with the form of the B field given above. Consequently, the existence of Lagrange invariants allows us to obtain the conservation laws for (10) and (7) (which reflect a conserved topological structure of the fields under investigation in nondissipative media).

Let us now return to the topological differences between the introduced freezing-in quantities (5) and (6). It is clear that the lines of intersection of the two surfaces determined by the Lagrange invariants cannot be linked. Because of this, the invariant (7) for the quantities (5) is equal to zero (we are discussing the case in which the invariants give a one-to-one mapping from $\mathbf{x}_0\to\mathbf{I}$). For the freezing-in integral (6), the invariant (7) may be nonzero. Thus, the field lines of the freezing-in integral (5) are a set of unentangled lines, while in the case of (6) the field lines can form nodes and be linked with each other.

We derive the topological invariants of the fields for which a vector potential A cannot be introduced (i.e., when all the invariants I_1, I_2, and I_3 are independent), using a quantity called the degree of mapping[3,4]:

$$I^\tau=\frac{1}{4\pi}\int_S d\theta d\varphi \mathbf{j}\left[\frac{\partial \mathbf{j}}{\partial\theta}\times\frac{\partial \mathbf{j}}{\partial\varphi}\right], \tag{11}$$

where $\mathbf{j}=\mathbf{J}/|\mathbf{J}|$. This quantity is equal to the number of times the vector **j** goes around a unit sphere as **r** runs over S. In the course of the temporal evolution, the field of the frozen-in quantity is deformed by the flow of the continuous medium in a continuous (i.e., homotopic) fashion. It is well known that the degree of mapping is conserved in a homotopy.[4,5] Thus, I^T is conserved in time.

The invariants obtained are particularly useful in the study of particle-like solutions (localized vortices or flows). Indeed, on account of the conservation of I^T, the number of linkages under the initial condition at $t=0$ is conserved in time, and, consequently, the topological classification of the initial conditions carries over to the corresponding fields at an arbitrary moment of time (i.e., to the solutions of the corresponding equations), which allows us in a number of cases to affirm the global stability of the solutions in nondissipative media.

Thus, to construct the freezing-in integrals, the Lagrange invariants reflecting the topological structure of the freezing-in fields, we only need some initial set of Lagrange invariants. In the canonical form, such a set is easily obtained when the corresponding equations are formulated in terms of the variational principle. Indeed, in the variational formulation, based on the use of the Lagrange multipliers, of the equations of motion of nondissipative media,[6] as the coordinates, we use the trivial Lagrange invariants $x_{0k}(\mathbf{x},t)$ (the initial positions of the Lagrange particles), with the Lagrange invariants λ_k that acquire the meaning of generalized momenta in the Hamiltonian formulation serving as their conjugates. Consequently, it is natural to take as the initial set of Lagrange invariants the set that arises in the variational principle. Besides the invariants obtained with their help, a number of invariants similar to the Poincaré-Cartan invariants in classical mechanics (e.g., the fluxes of the frozen-in quantities, etc.) can be constructed.

It should be noted that the relations (3)–(7) can also be used to construct new exact solutions from known particular solutions. Below we illustrate the application of the relations obtained in some specific cases.

2. Let us investigate what invariants arise in a compressible fluid in the adiabatic case. For this purpose, let us use the variational principle constructed in Ref. 6 for a compressible fluid, and giving the initial Lagrange invariants $\mathbf{x}_0(\mathbf{x},t)$, $\lambda_k(\mathbf{x},t)$, and s (s is the entropy, the $\mathbf{x}_0$ are the initial coordinates, and λ_k is the analog of the generalized velocity). In these variables the flow velocity **v** has the form

$$\mathbf{v}=-\nabla\alpha+\beta\nabla s+\lambda_k\nabla x_0^k. \tag{12}$$

The relation (12) allows us to go back to the flow velocity of the fluid from $\mathbf{x}_0$, λ_k, and s. Using the relation (3) and the above set of Lagrange invariants as the initial set, we can easily construct other Lagrange invariants:

$$I_{ij}^{(1)}=\frac{1}{\rho}\nabla s[\nabla\lambda_i\times\nabla\lambda_j],\quad I_{ij}^{(2)}=\frac{1}{\rho}\nabla s[\nabla x_{0j}\times\nabla\lambda_i],$$
$$I_{ijk}=\frac{1}{\rho}\nabla\lambda_i[\nabla\lambda_k\times\nabla\lambda_j], \tag{13}$$
$$I_{ijk}^{(3)}=\frac{1}{\rho}\nabla\lambda_i[\nabla\lambda_j\times\nabla x_{0k}],\quad I_{ijk}^{(4)}=\frac{1}{\rho}\nabla\lambda_i[\nabla x_{0k}\times\nabla x_{0j}].$$

It is clear from the construction of (3) that such invariants are not independent; some of them are trivial. In Ref. 7 expressions of the type (3) are used to construct invariants corresponding to a more particular class of flows (i.e., flows without the entanglement of the field lines of the frozen-in quantities), and contained, in particular, in (13). It is easy to notice that the quantities (11) contain the well-known Ertel integrals.[8] Let us not dwell on the trivial scheme for constructing Lagrange invariants of higher order in the derivatives [by repeatedly using (3) and (13)], and proceed to construct the freezing-in integrals. We use in the construction the relation (5) and the initial set of invariants $\mathbf{x}_0$, λ_k, and s. Then we obtain the following set of quantities frozen in the fluid:

$$\mathbf{J}_i^{(1)}=\frac{1}{\rho}[\nabla s\times\nabla\lambda_i],\quad \mathbf{J}^{(2)}=\frac{1}{\rho}[\nabla s\times\nabla x_{0i}],$$
$$\mathbf{J}_{ij}^{(3)}=\frac{1}{\rho}[\nabla\lambda_i\times\nabla x_{0j}],\quad \mathbf{J}_{ij}^{(4)}=\frac{1}{\rho}[\nabla x_{0i}\times\nabla x_{0j}],\tag{14}$$
$$\mathbf{J}_{ik}^{(5)}=\frac{1}{\rho}[\nabla\lambda_i\times\nabla\lambda_k].$$

The frozen-in quantities (14) correspond to fields whose lines of force do not form nodes and linkages, a fact which follows from the derivation of (5).

The quantities frozen in a fluid of a more general topological type can easily be obtained, using (6):

$$J_\alpha=\frac{1}{\rho}\,\varepsilon_{\alpha\beta\gamma}\varepsilon_{ikl}a_i\frac{\partial a_k}{\partial x_\beta}\frac{\partial a_l}{\partial x_\gamma},\tag{15}$$

where the vectors a stand for vectors whose coordinates coincide with either x_{0i} or s,λ_i. Among the frozen-in quantities (15) are, in particular, ρ^{-1} curlv, which, as is well known, is frozen in a fluid,[9] and frozen-in fields that are not purely rotational, i.e., for which $\operatorname{div}\rho\mathbf{J}\neq 0$. The above-obtained quantities (15) allow us to introduce a number of new Lagrange invariants:

$$I'=\nabla s\mathbf{J}.\tag{16}$$

And, finally, the conservation of the fluxes of the corresponding quantities naturally follows from (14) and (15). The topological invariants corresponding to the frozen-in quantities (14) and (15) are constructed with the aid of the relations (10) and (13). They contain, in particular, the Moffatt invariant,[3] which characterizes the entanglement of the vortex lines of the velocity field.

Similarly, we can construct the Lagrange invariants and the freezing-in integrals in a rotating fluid, in magnetohydrodynamics, and in a superfluid liquid.

3. As the next example, let us consider the invariants that arise in the two-fluid hydrodynamics of a plasma. In the process we shall illustrate another possibility of using the above-obtained relations, avoiding the formulation in canonical terms. To obtain the initial set, we use the system of equations of the two-fluid hydrodynamics of a plasma without dissipation in the adiabatic case. Computing the curl of the equations of motion, we easily obtained the freezing-in integral[10]

$$\frac{d}{dt}\left\{\frac{\operatorname{rot}\mathbf{v}_\alpha+e_\alpha\mathbf{H}/m_\alpha c+\boldsymbol{\Omega}_\alpha}{n_\alpha}\right\}=\left(\frac{\operatorname{rot}\mathbf{v}_\alpha+e_\alpha\mathbf{H}/m_\alpha c+\boldsymbol{\Omega}_\alpha}{n_\alpha}\nabla\right)\mathbf{v}_\alpha.\tag{17}$$

Here $\boldsymbol{\Omega}_\alpha$ is the angular velocity of the α components as a whole (the remaining symbols have their conventional meanings).

Thus, we have for each component a freezing-in integral and a Lagrange invariant s_α (the entropy), which we use as the initial set. Applying the relation (4), we obtain another Lagrange invariant:

$$I_1^\alpha=\nabla s_\alpha\frac{\operatorname{rot}\mathbf{v}_\alpha+e_\alpha\mathbf{H}/m_\alpha c+\boldsymbol{\Omega}_\alpha}{n_\alpha},\tag{18}$$

which is close in meaning to the Ertel integral for a compressible fluid.[8] Using the relations (4) and (5) successively, we can construct freezing-in integrals and Lagrange invariants of higher order in the derivatives, e.g.,

$$\mathbf{J}_1^\alpha=\frac{1}{n_\alpha}\left[\nabla s_\alpha,\nabla\left(\nabla s_\alpha\frac{\operatorname{rot}\mathbf{v}_\alpha+e_\alpha\mathbf{H}/m_\alpha c+\boldsymbol{\Omega}_\alpha}{n_\alpha}\right)\right],\tag{19}$$

etc.

The topological characteristics of the field $\operatorname{curl}\mathbf{v}_\alpha+e_\alpha\mathbf{H}/m_\alpha c+\boldsymbol{\Omega}_\alpha$ are determined by the invariant (8):

$$I_\alpha^\tau=\int\limits_{V_0}\left(\operatorname{rot}\mathbf{v}_\alpha+\frac{e_\alpha}{m_\alpha c}\mathbf{H}+\Omega_\alpha\right)\left(v_\alpha+\frac{e_\alpha}{m_\alpha c}\mathbf{A}+\Omega_\alpha\right)dv,\tag{20}$$

where the integration is over the volume, the normal to whose surface is orthogonal to $(\operatorname{curl}\mathbf{v}_\alpha+e_\alpha\mathbf{H}/m_\alpha c+\boldsymbol{\Omega}_\alpha)$ at all times.

The invariant (20) reflects the entanglement of the field lines of the curl of the generalized momentum of each of the components of the plasma.

4. Below we illustrate in the particular case of the two-fluid hydrodynamics the computational advantages of using the relations obtained. Let us first of all discuss the steady-state solutions. From the relation (2) it is easy to note the existence of a one-parameter class of steady-state solutions satisfying the equation

$$\mathbf{J}=c\mathbf{v},\tag{21}$$

where c is an arbitrary constant or a function of the conserved quantities. In the case under consideration, to Eq. (21) corresponds the set of equations[10]

$$\operatorname{rot}\mathbf{v}_\alpha+e_\alpha\mathbf{H}/m_\alpha c=c_\alpha n_\alpha\mathbf{v}_\alpha.\tag{22}$$

Supplementing (22) with the Maxwell equation

$$\operatorname{rot}\mathbf{H}=\frac{4\pi}{c}\sum e_\alpha n_\alpha\mathbf{v}_\alpha,\tag{23}$$

in the incompressible case, we obtain a closed set of equations describing the stationary flows and fields possessing a "generalized" helicity. Below we shall seek H and v in the form

$$\mathbf{H}=\operatorname{rot}\operatorname{rot}(\mathbf{x}\psi)+\operatorname{rot}(\mathbf{x}\varphi),$$
$$\mathbf{v}=\operatorname{rot}\operatorname{rot}(\mathbf{x}F_\alpha)+\operatorname{rot}(\mathbf{x}\Phi_\alpha).$$

From the set of equations (20) and (21) we easily obtain the equations for the potentials ψ, φ, and F_α,Φ_α:

$$\Delta F_\alpha+c_\alpha n_\alpha\Phi_\alpha=\frac{e_\alpha}{m_\alpha c}\varphi,\quad c_\alpha n_\alpha F_\alpha-\frac{e_\alpha}{m_\alpha c}\psi=\Phi_\alpha,$$
$$\Delta\psi+\frac{4\pi}{c}\sum e_\alpha n_\alpha\Phi_\alpha=0,\quad \varphi=\frac{4\pi}{c}\sum e_\alpha n_\alpha F_\alpha.\tag{24}$$

Eliminating φ and Φ_α from (24), we obtain

$$\Delta F_\alpha+c_\alpha^2n_\alpha^2F_\alpha=\frac{e_\alpha}{m_\alpha c}\left(c_\alpha n_\alpha\psi+\frac{4\pi}{c}\sum e_\beta n_\beta F_\beta\right)$$
$$\Delta\psi-\frac{\omega_p^2}{c^2}\psi+\frac{4\pi}{c}\sum e_\alpha c_\alpha n_\alpha^2F_\alpha=0.\tag{25}$$

Here we have introduced the notation

$$\frac{\omega_p^2}{c^2}=\frac{4\pi}{c}\sum\frac{e_\alpha^2 n_\alpha}{cm_\alpha}.$$

The simplest particular solutions to (25) can easily be found by assuming that $F_\alpha = b_\alpha F$ and $\psi = \varkappa F$ (where b_α and $\varkappa$ are some arbitrary constants). Substitution into the system (25) leads to a system of equations for b_α and $\varkappa$ of the form [the consistency condition for (25)]

$$b_\alpha c_\alpha^2 n_\alpha^2 - \varkappa\frac{e_\alpha c_\alpha n_\alpha}{m_\alpha c} - \frac{4\pi}{c}\frac{e_\alpha}{m_\alpha c}\sum_\beta e_\beta n_\beta b_\beta = \beta b_\alpha,$$
$$\frac{4\pi}{c}\sum e_\beta c_\beta n_\beta^2 b_\beta - \left(\beta+\frac{\omega_p^2}{c^2}\right)\varkappa = 0. \tag{26}$$

In its turn, the value of β is computed from the solvability condition for (26):

$$\begin{vmatrix} \left(\beta+\frac{4\pi}{c}\frac{e_1^2 n_1}{m_1 c}-c_1^2 n_1^2\right); & \frac{4\pi}{c}\frac{e_1 e_2 n_2}{m_1 c}; & \frac{e_1 c_1 n_1}{m_1 c} \\ \frac{4\pi}{c}\frac{e_1 e_2 n_1}{m_2 c}; & \left(\beta+\frac{4\pi}{c}\frac{e_2^2 n_2}{m_2 c}-c_2^2 n_2^2\right); & \frac{e_2 c_2 n_2}{m_2 c} \\ \frac{4\pi}{c}e_1 c_2 n_1^2; & \frac{4\pi}{c}e_2 c_2 n_2^2; & -\left(\beta+\frac{\omega_p^2}{c^2}\right) \end{vmatrix}=0,$$

and the system (25) reduces to the single equation

$$\Delta F+\beta F=0.$$

This equation contains β, which can be either positive or negative, depending on the values of c_1 and c_2. A detailed analysis of the sign of β can easily be carried out. The (c_1, c_2) plane splits up into regions with $\beta > 0$ and $\beta < 0$. Let us, without giving the exact boundaries of the regions, point out, for example, that, for c_1 and c_2 lying inside the circle of radius ω_{pe}/cn (i.e., for $c_1^2 + c_2^2 < \omega_{pe}^2/n^2c^2$), $\beta > 0$. Equation (25) has been thoroughly studied for both $\beta > 0$ and $\beta < 0$. Let us consider the $\beta > 0$ case, i.e., the case in which c_1 and c_2 belong to the corresponding region.

In a spherical region occupied in a vacuum, the solution has the form

$$F=F_0 r^{-1/2} J_{3/2}(\beta^{1/2} r)\cos\theta. \tag{27}$$

By choosing the boundary conditions

$$J_{3/2}(\beta^{1/2} r)=0,\quad 3H_0=-F_0\varkappa\frac{d}{dR}(R^{-1/2}J_{3/2}(\beta^{1/2}R)),$$

we effect a matching with the solution to the external problem (in the vacuum) with a field that is constant at $r\to\infty$ ($\mathbf{H}|_{r\to\infty}=H_0\mathbf{z}$). Thus, the vortex obtained can exist in a constant magnetic field. The field lines and the streamlines (Fig. 1) coil into a family of tori embedded in each other. On the outermost torus, only the $v_{\theta\alpha}$ and H_θ, the velocity and field components, are nonzero. It should be noted that the overwhelming majority of the lines of force of the magnetic field and the streamlines of the plasma, being in a closed volume ($r \leq R$), are not closed, coiling into a torus with an irrational ratio of the rotational speeds (which corresponds to the case of the general configuration). Such flows are of interest even from the purely hydrodynamic point of view as an example of topologically nontrivial motions. Notice that the vortex naturally becomes structurally complicated when it is chosen as the nth root of the equation $J_{3/2}(\beta^{1/2}R)=0$. The configuration of the magnetic field of the indicated vortex coincides with that of the magnetic field of the well-known magnetohydrodynamic vortex.[11]. Furthermore, such vortices can exist in plasma fluxes, and then, depending on c_i and c_e, two cases are possible: when $\mathbf{v}_i$ and $\mathbf{v}_e$ at the boundary coincide in direction and when they are oppositely directed. In the latter case such vortices can be observed in the countercurrents of the ionic and electronic components of the plasma. Besides the above-described vortices (27), other configurations are possible:

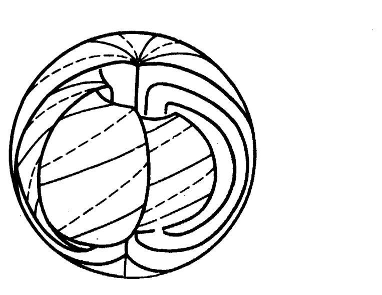

FIG. 1. The continuous curves depict the magnetic lines of force; the dashed curves, the fluid streamlines.

$$F=F_{0m}\frac{1}{r^{1/2}}Z_{n+1/2}(\beta^{1/2}r)P_{mn}(\cos\theta)\cos(m\varphi+\psi_m).$$

But such vortices are refinements of significantly more complicated fields and flows with swirls. (Here the $Z_{n+1/2}$ are Bessel functions and the P_{mn} are associated Legendre functions.) By computing the invariant (8), we can verify that it is nonzero in the case of the solution (27). Indeed, the vortex lines of the generalized momentum are entangled (see Fig. 1). Because of the entanglement and the freezing-in of the curl of the generalized momentum, the vortex obtained is topologically stable, i.e., it cannot go over continuously into steady flow. But the decay becomes possible upon the inclusion of viscous terms, which lead to closure again. Thus, such a vortex is stable, at least during the period of viscous flow.

Next, let us give the second type of vortices, which illustrates another possibility of using (2). Going over in (2) to cylindrical coordinates (z, φ, r), and setting $v_{\alpha\varphi}=0$, $\partial/\partial\varphi=0$, and $\mathbf{J}=\mathbf{J}_\varphi\varphi_0$, we obtain

$$\frac{d}{dt}\left(\frac{J_\varphi}{r}\right)=0.$$

This relation actually indicates the transition of the freezing-in integrals into the Lagrange invariants in the two-dimensional case. The requirement that $\partial/\partial\varphi=0$ does in fact indicate the effective two-dimensionality of the problem. In the two-fluid hydrodynamics of a plasma, this equation leads to a steady-state form that can be written as

$$(\operatorname{rot} v_\alpha)_\varphi+e_\alpha H_\varphi/m_\alpha c=n_\alpha r\gamma_\alpha. \tag{28}$$

Using the Maxwell equation (23) and Eq. (28) in the incompressible case, we obtain the system of equations

$$-\frac{\partial^2 H_\varphi}{\partial z^2}-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial rH_\varphi}{\partial r}\right)+\frac{\omega_p^2}{c^2}H_\varphi=\frac{4\pi}{c}\sum n_\alpha^2 e_\alpha\gamma_\alpha r,$$
$$\sum e_\alpha n_\alpha\Phi_\alpha=\delta-rH_\varphi,\quad v_r^\alpha=\frac{1}{r}\frac{\partial\Phi_\alpha}{\partial z},\quad v_z^\alpha=-\frac{1}{r}\frac{\partial\Phi_\alpha}{\partial r}, \tag{29}$$

where δ and γ_α are arbitrary constants. The solution to (29) in a spherical region ($r^2+z^2\leq R^2$) has the form

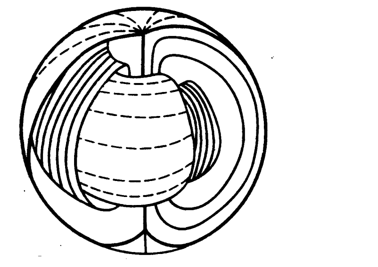

FIG. 2.

$$H_\varphi = r\left[\frac{4\pi}{c}\sum_\alpha n_\alpha^2 e_\alpha \gamma_\alpha - c_1\left(\frac{\omega_p^2}{c^2}\xi^{-1}\operatorname{ch}\left(\frac{\omega_p}{c}\xi^{1/2}\right) - \frac{\omega_p \xi^{1/2}}{c}\operatorname{sh}\left(\frac{\omega_p}{c}\xi^{1/2}\right)\right)\right], \quad \xi = r^2+z^2. \tag{30}$$

Choosing

$$c_1 = \frac{4\pi}{c}\sum n_\alpha^2 e_\alpha \gamma_\alpha \left[\frac{\omega_p^2 R^2}{c^2}\operatorname{ch}\left(\frac{\omega_p}{c}R\right) - \frac{\omega_p R}{c}\operatorname{sh}\left(\frac{\omega_p R}{c}\right)\right]^{-1},$$

we obtain $H_{\varphi|R}=0$, and the current is directed along the tangent to the surface of the sphere:

$$J|_R = 2\frac{\partial H_\varphi}{\partial \xi}\bigg|_R (-r\mathbf{z}_0 + z\boldsymbol{\rho}_0).$$

Figure 2 shows the streamlines and constant-H_φ surfaces. The configuration of the vortex obtained is reminiscent of the hydrodynamic Hill vortex whose field is oriented along the curl of the velocity. The solutions obtained preserve their form when the presence of the gravitational field is taken into account: Only the pressure computed from the Bernoulli integral then changes. It should be noted that the vortex (30) is topologically unstable, since the lines of the curl of the generalized momentum are not entangled in its case.

Recently there has been an upsurge in interest in such flows,[1,12] since they can be realized in both cosmic and laboratory plasmas. The states with curl$\mathbf{v} \neq 0$ have also been intensively discussed in connection with the problem of controlled fusion[13] (spheromaks).

Thus far, we have used the relations obtained to analyze the steady-state solutions. But they are useful in the nonstationary cases also, e.g., in such areas as the derivation of the dispersion relations in the presence of inhomogeneities. As an example, let us derive the dispersion law for waves in the two-fluid plasma hydrodynamics model, which are analogous to the Rossby hydrodynamic waves. We shall, for simplicity, assume that $n_\alpha = \text{const}$. As the ground state we use

$$\mathbf{H}_0 = (0, 0, H_0 y/y_0), \quad \mathbf{v}_0^\alpha = (v_0^\alpha, 0, 0),$$

i.e., the field is oriented along the z axis and inhomogeneous along y, while the constant current is directed along x. The nonzero perturbations

$$v_x,\ v_y,\ H_z \sim \exp(ik_x x + ik_y y - i\omega t).$$

In the absence of v_z and v_{0z} the freezing-in integral goes over (as a result of the two-dimensionality) into a Lagrange invariant:

$$\frac{d}{dt}\left((\operatorname{rot}\mathbf{v}_\alpha)_z + \frac{e_\alpha}{m_\alpha c}H_z\right) = 0. \tag{31}$$

Taking the Maxwell equation (23) and Eq. (31) into account, we easily obtain a dispersion equation of the form

$$(k_x v_{0e}-\omega)(k_x v_{0i}-\omega)k^2\left(k^2+\frac{\omega_{pi}^2}{c^2}+\frac{\omega_{pe}^2}{c^2}\right) - (k_x v_{0e}-\omega)\frac{k_x\omega_{Hi}}{y_0}\left(k^2+\frac{\omega_{pe}^2}{c^2}\right) + (k_x v_{0i}-\omega)\frac{k_x\omega_{He}}{y_0}\left(k^2+\frac{\omega_{pi}^2}{c^2}\right) - \frac{k_x^2\omega_{He}\omega_{Hi}}{y_0^2} = 0.$$

Taking the smallness of m_e/m_i into account, and neglecting the motion of the ions, i.e., setting $v_{0i}=0$, we obtain

$$\omega^2 k^2\left(k^2+\frac{\omega_{pe}^2}{c^2}\right) - \omega\frac{k^2 k_x}{y_0}\left(\omega_{He}+y_0\left(k^2+\frac{\omega_{pe}^2}{c^2}\right)v_{0e}\right) - \frac{k_x^2}{y_0^2}\omega_{Hi}\left(\omega_{He}+y_0\left(k^2+\frac{\omega_{pe}^2}{c^2}\right)v_{0e}\right) = 0. \tag{32}$$

From (32) we easily obtain two vibrational branches ω_+ and ω_-. The dispersion in the long-wave region is given by the relation

$$\omega_+ = \frac{k_x(\omega_{He}+y_0\omega_{pe}^2 v_{0e}/c^2)^{1/2} c\,\omega_{Hi}^{1/2}}{y_0\omega_{pe}(k_x^2+k_y^2)^{1/2}},$$

$$\omega_- = -\frac{k_x(\omega_{He}+y_0\omega_{pe}^2 v_{0e}/c^2)^{1/2} c\,\omega_{Hi}^{1/2}}{y_0\omega_{pe}(k_x^2+k_y^2)^{1/2}}$$

when

$$k^2 r_{de}^2 \ll \frac{v_{Te}^2}{c^2}\frac{\omega_{Hi}}{(\omega_{He}+y_0\omega_{pe}^2 v_{0e}/c^2)}.$$

And in the shorter-wave region

$$k^2 r_{de}^2 \gg \frac{v_{Te}^2}{c^2}\frac{\omega_{Hi}}{(\omega_{He}+y_0\omega_{pe}^2 v_{0e}/c^2)}$$

we have

$$\omega_+ = \frac{k_x(\omega_{He}+y_0(k^2+\omega_{pe}^2/c^2)v_{0e})}{y_0(k^2+\omega_{pe}^2/c^2)},$$

$$\omega_- = -\frac{k_x\omega_{Hi}}{y_0(k_x^2+k_y^2)}.$$

The branch ω_- is reminiscent of the Rossby-wave dispersion, with the rotation frequency replaced by the Larmor ion frequency and ω_+, by the flow mode.

In conclusion, let us note that for lack of space a number of important consequences of the relations obtained, such as the modification of the "B minimum" principle in a plasma in the presence of vortex motions, two-dimensional stationary flows, solitons on the vortex tubes of frozen-in quantities, etc., have not been analyzed in the present paper. Furthermore, the relations obtained turn out to be useful in the kinetic theory of a plasma as well. Indeed, as is well known, the solutions to the system of Vlasov equations are connected with the solutions to the system of hydrodynamic Benny equations,[14] from which similar invariants can easily be obtained. These invariants allow us to introduce a topological classification of the solutions to the system of kinetic equations in accordance with the solutions to the system of hydrodynamic Benny equations, and also obtain some exact solutions to the system of Vlasov equations.

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