On two-dimensional turbulence

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An inequality is derived that demonstrates the growth of the differential scale of an isotropic two-dimensional turbulence. It is shown that the spectrum has a deltalike asymptote, which is not realized physically because of restrictions imposed by the law of angular momentum conservation. The consequence is either the destruction of the turbulence isotropic structure (phase transition) or a transition to stochastic motion, which does not possess correlation characteristics. The properties of inhomogeneous two-dimensional turbulence are discussed briefly.

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Two-dimensional turbulent motion possesses a number of features that distinguish it essentially from the three-dimensional case and that allow a modeling of processes with so-called "negative viscosity"[1] (such processes determine to a large extent the dynamics of the atmosphere and of the ocean). Chief among these features is the possibility of transfer of energy from fine-grained to course-grained components, first demonstrated in Ref. 2 on the basis of dimensionality considerations. Such a behavior of the two-dimensional isotropic spectrum has also been investigated numerically in Refs. 3–5 (the corresponding equations were closed by use of Millionshchikov's hypothesis). Actually, the difference between two- and three-dimensional turbulence also appears in the linear situation, at the final stage of degeneracy, when the nonlinear terms in the equations of hydrodynamics can be neglected.[6] Anomalies in the behavior of two-dimensional turbulence motion (in comparison with three-dimensional) are obviously connected with the change in the dimensionality of the space (it is easy to see, for example, that one-dimensional turbulence is completely impossible in an incompressible fluid). In particular, in the case of zero viscosity in the two-dimensional case, there is an infinite set of the integrals of the motion [integrals of all positive integer powers of (curl $\mathbf{v}$)]. As a consequence of the conservation of the integral of (curl $\mathbf{v})^2$, several general spectral inequalities appear, as established in Ref. 7; these inequalities demonstrate the absence of a cascade transfer of the energy to higher wave numbers.

Similar results are also obtained in the modeling of two-dimensional turbulence by choosing point vortices with subsequent use of the methods of statistical mechanics—vortices of the same sign have a tendency toward merging; this approach dates back to Ref. 8 (see also Ref. 9 and the detailed review of this theme in Ref. 10).

Returning to the equation for the mean energy and the mean square of the curl ($\boldsymbol{\omega} = \mathrm{curl}\,\mathbf{v}$) of isotropic turbulent motion[11]

$$\frac{d\mathscr{E}}{dt} = -\frac{1}{2}\nu\left\langle\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right)^2\right\rangle, \tag{1}$$

$$\frac{d\langle\boldsymbol{\omega}^2\rangle}{dt} = 2\left\langle \omega_i\omega_k\frac{\partial v_i}{\partial x_k}\right\rangle - 2\nu\left\langle\frac{\partial \omega_i}{\partial x_k}\frac{\partial \omega_i}{\partial x_k}\right\rangle \tag{2}$$

(the angular brackets indicate averaging, summation is carried out over repeated subscripts), it is easy to note that the two-dimensional situation differs from the three-dimensional in the absence of the first term in the right-hand side of (2). Just this term corresponds to the increase in $\langle\omega^2\rangle$ and the transfer of energy to smaller scales in three-dimensional turbulent motion, so that we should have $\langle\omega_i\omega_k \partial v_i/\partial x_k\rangle > 0$. However, the proof of this inequality is as yet unknown, and in the final analysis it is a consequence of experimental facts. Thus, the absence of a cascade process in the two-dimensional case already follows from the very form of the equation for $\langle\omega^2\rangle$. Geometrically, this is obviously also connected with the impossibility of producing on a plane a "figure eight" out of a closed non-self-intersecting smooth curve—the process which leads to the decay of large-scale vortices.[6]

The cascade process of energy transfer usually enters as a necessary factor in the determination of the turbulent motion. Therefore, its absence in the two-dimensional situation demonstrates in advance the "incompleteness" of two-dimensional turbulence in comparison with three-dimensional (several other mathematical aspects of the difference between two- and three-dimensional hydrodynamic turbulence are discussed in Ref. 12). In this context, we can cast doubts on the validity of the ordinary description of two-dimensional turbulence, which is based on the use of correlation functions and spectra. However, we shall in what follows start from the ordinary assumption of the validity of such a process, at any rate up until such time as internal contradictions appear.

It is easy to see that the absence of transfer of energy to small scales leads to the preservation of the scale of turbulence or to its growth. It is shown below that some mean scale of two-dimensional turbulence increases, and a possible asymptotic behavior of the spectrum is observed (Sec. 1), while the asymptotic is a delta function. It is shown next that an isotropic δ-like spectrum cannot exist because of limitations imposed by the law of conservation of angular momentum, so that the motion should be reorganized in some fashion. Possible types of rearrangement are discussed in Sec. 2. In the conclusion (Sec. 3), the corresponding features of inhomogeneous two-dimensional turbulence are considered.

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1. GROWTH OF THE SCALE OF TURBULENCE

In the two-dimensional case, the vortex has one nonvanishing component (perpendicular to the plane of the motion) and can be considered to be a scalar. Tranforming to the spectral energy density $E(k,t)$, ($\mathscr{E}=\int_0^\infty E(k,t)dk$ and using the general relation of Ref. 11

$$\left\langle \frac{\partial^m\varphi(\mathbf{x})}{\partial x_j^m}\frac{\partial^n\varphi(\mathbf{x}')}{\partial x_k'^n}\right\rangle=(-i)^m(i)^n\int_0^\infty \exp\{i\mathbf{k}(\mathbf{x}-\mathbf{x}')\}k_j^m k_k^n\Phi(\mathbf{k})\,d\mathbf{k},$$

where $\Phi(\mathbf{k})$ is the spectrum $\langle\varphi(\mathbf{x})\varphi(\mathbf{x}')\rangle$, we reduce (1) and (2) to the form (for brevity, we omit the limits of integration)

$$\frac{d}{dt}\int E\,dk=-2\nu\int k^2E\,dk, \tag{3}$$

$$\frac{d}{dt}\int k^2E\,dk=-2\nu\int k^4E\,dk. \tag{4}$$

Dividing (3) and (4) by the nonzero $\int Edk$ and $\int k^2Edk$, respectively, we obtain (the prime denotes the time derivative)

$$\int E'\,dk\Big/\int E\,dk=-2\nu\int k^2E\,dk\Big/\int E\,dk, \tag{5}$$

$$\int k^2E'\,dk\Big/\int k^2E\,dk=-2\nu\int k^4E\,dk\Big/\int k^2E\,dk. \tag{6}$$

Now, dividing (5) by (6), we obtain

$$\int E'\,dk\int k^2E\,dk\Big/\int E\,dk\int k^2E'\,dk=\left\{\int k^2E\,dk\right\}^2\Big/\int k^4E\,dk\int E\,dk. \tag{7}$$

By virtue of the Cauchy-Hölder inequality, the right side of (7) does not exceed unity; consequently,

$$\int E'\,dk\int k^2E\,dk\Big/\int E\,dk\int k^2E'\,dk\leqslant 1, \tag{8}$$

or, with account of $\int E'dk, \int k^2E'dk<0$ [see Eqs. (3) and (4)]

$$\frac{d}{dt}\left\{\int E\,dk\Big/\int k^2E\,dk\right\}\geqslant 0. \tag{9}$$

The quantity in the curly brackets is equal, to within a constant factor, to the square of the so-called differential scale of turbulence. (see Ref. 11, Sec. 12). Thus, this scale increases in the nonstationary situation. The indicated increase is slower the closer the right side of (7) to unity, i.e., the smaller the range of values of k in which $E(k)$ is significantly different from zero. The stationary situation is achieved, as is seen from (7)–(9), in the case of a δ-like spectrum.

Of course, the inequalities obtained cannot be regarded in any way as a complete proof of the tendency of the two-dimensional isotropic spectrum towards the δ shape. Such a proof could be based only on an equation that contains $\partial E(k,t)/\partial t$ and includes spectral representations of third-order moments. It is physically obvious, however, that the components with large k should be rapidly damped under the action of the viscosity, whereas $F(k)$ at small k evolves practically without account of the viscosity and in correspondence with the stated inequalities. Since the law of conservation of momentum establishes $E(0)=0$, the spectrum can be contracted only at the point $k\neq 0$.

It is easy to see that the indicated results are valid for an arbitrarily small viscosity; however, the situation with $\nu=0$ requires special consideration. There are indications that in the two-dimensional case the solutions of the Navier-Stokes equation goes over into solutions with $\nu=0$ as $\nu\to 0$.[13] The spectral inequalities that lead to the results, that are partially discernible in those reported in the present work, are obtained for the case $\nu=0$ in Ref. 7. Here only the spectrum $E(k)$ is considered, but not its time derivatives; a whole class of functions (containing, in particular, the δ spectrum) turn out to be possible. In this analysis, the δ asymptote is realized almost uniquely. It appears that the indicated "collapse" of the spectrum is generally typical of the two-dimensional situation. At least it has been detected numerically in another connection in Ref. 14 and also, very recently, for two-dimensional turbulence (also numerically) in Ref. 15.

2. CORRELATION FUNCTIONS AND THE PROPERTIES OF MOTION IN THE CASE $E(k)=\delta(k-k^*)$

For the calculation of the correlation functions

$$B_{ij}(\mathbf{r})=\langle u_i(\mathbf{x})u_j(\mathbf{x}+\mathbf{r})\rangle$$

it is necessary to transform to the spectrum, so that

$$B_{ij}(r)=\int_0^\infty\int_0^{2\pi} e^{i\mathbf{kr}}F_{ij}(k)\,k\,dk\,d\varphi. \tag{10}$$

In the two-dimensional case, we have for the spectrum

$$F_{ij}(k)=\frac{E(k)}{2\pi k}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right). \tag{11}$$

Substituting (11) in (10) and using the replacement $k_l\to -i\partial/\partial r_l$, it is easy to obtain the expression for $B_{ij}(r)$. Then, separating the longitudinal and transverse correlation scalars,[11] and after simple calculations, it is easy to verify that the first of these, B_{LL}, is expressed in terms of the Bessel function J_0 and its derivatives; at large k^*r, the result is

$$B_{LL}\sim(2/\pi k^*r)^{1/2}\cos(k^*r-\pi/4), \tag{12}$$

with a power-law fall-off. If at some initial instant of time the correlation functions decrease exponentially, the appearance of a δ spectrum indicates a change in the character of their decrease, a picture that is formally similar to a phase transition in a two-dimensional liquid.[16]

It turns out, however, that the law of conservation of angular momentum imposes such restrictions on the form of the correlation functions at no isotropic δ spectrum exists. Actually, by virtue of the law of conservation of the angular momentum, there is a definite value (usually called the Loĭtsyanskiĭ invariant)[11,17]

$$\int_0^\infty B_{LL}(r)r^3\,dr.$$

In the case considered here, the corresponding integral diverges, as is seen from (12), so that as $E(k)$ approaches $\delta(k-k^*)$ the turbulent motion seems to change its structure; in the opposite case, the angular momentum, having a certain finite value at the initial instant of time (with subsequent conservation of this value), would take on an arbitrarily large value when the spectrum becomes δ-like. The character of the indicated change cannot be established unambiguously by the theory. We shall consider two examples.

a) *Violation of the isotropy and one-dimensionality of the motion in the case of a δ-like spectrum.* A certain periodic structure can correspond to this case, as is shown below. Following Ref. 18, we consider a system of plane vortices, given by the relations

$$u_1=-\frac{\partial}{\partial x_2}(\cos k_1^* x_1 \cos k_2^* x_2),$$
$$u_2=\frac{\partial}{\partial x_1}(\cos k_1^* x_1 \cos k_2^* x_2). \quad (13)$$

Such a system can be tentatively assigned a correlation function

$$B_{ij}(\mathbf{r})=\langle u_i(\mathbf{x})u_j(\mathbf{x}+\mathbf{r})\rangle,$$

by averaging over $\mathbf{x}$, i.e.,

$$\langle\varphi\rangle=\lim_{a,b\to\infty}\frac{1}{ab}\int_{-a}^{a}\int_{-b}^{b}\varphi(\mathbf{x})\,dx_1\,dx_2.$$

After differentiation in (13), the product u_1u_2 can be represented in the form of a sum in which one term does not depend on x while the remaining terms depend on x in periodic fashion. The averaging indicated above yields, correspondingly,

$$B_{12}(r)={}^1/_4 k_1^* k_2^* \sin k_1^* r_1 \sin k_2^* r_2.$$

Taking into account the known representation

$$\frac{1}{2}[\delta(x-\alpha)+\delta(x+\alpha)]=\frac{1}{\pi}\int_0^\infty \cos\omega x\cos\omega\alpha\,d\omega,$$

we can easily show that the spectrum F_{12} contains δ-like components (and, is here anisotropic, of course). Such a change in the structure of the turbulent motion is similar to a phase transition "liquid–crystal" in the sense of a violation of the symmetry of the correlation functions. Here it is especially necessary to emphasize the following circumstances. The correlation function B_{12} written above cannot be obtained as a result of solution of the dynamic equation for isotropic quantities, since it is already anisotropic. This corresponds to the fact that when $E(k)$ tends to $\delta(k-k^*)$ at some instant of time, even the dynamic equations for the isotropic quantities cease to be valid. If the statistical description remains valid in this case, then the indicated equations ought to be replaced by more general ones, in which the spectrum is anisotropic. Then, apparently, the solutions which lead to correlation functions of the type B_{12} should be regarded as the result of the instability of the motion with an isotropic spectrum that is close to δ-like in relation to small perturbations of definite configuration. Actually, one can, it is true, find solutions which lead to different [from (13)] but also periodic structures (the solution (13) corresponds to dividing the plane into identical rectangular cells, in each of which a vortex is placed; the neighboring vortices have opposite signs).

b) *Stochasticity in the absence of mean values.* This possibility appears if the statistical description itself becomes invalid as E approaches $\delta(k-k^*)$. While remaining random, the motion is deprived of its correlation characteristics in their usual sense. The dynamic equation for the latter can be replaced in this case by Navier-Stokes equations.

Obviously, a decrease of the interval over which $E(k)$ differs from zero, in a finite-dimensional approximation of the equations of hydrodynamics, would correspond to a decrease in the number of degrees of freedom of the motion (as is easy to see, a δ spectrum corresponds to only one degree of freedom). Thus an interesting possibility of realization of the description of a continuous dynamical system with stochastic behavior as a system with a finite number of degrees of freedom and possessing a strange attractor appears in the present case (in contrast with three-dimensional turbulence, which possesses a large number of degrees of freedom and which does not admit apparently of such a description[19]).

To conclude this section, we must note the following circumstance. Both types of motion considered above can eventually become unstable with subsequent transition again to the stage of ordinary isotropic turbulence with further repetition of the described picture. Then the two-dimensional turbulence is realized in the form of a certain periodic sequence of regimes, one of which represents isotropic turbulence with a collapsing spectrum, while the other corresponds to one of the two possibilities described above.

3. INHOMOGENEOUS TURBULENCE

In conclusion, we touch upon the problem of inhomogeneous two-dimensional turbulence. The results above generally do not take place here, since generation of eddying on the boundary of the flow is possible. A term $\nu\Delta\omega$ enters on the right in the equation for the vortex, where Δ is the Laplacian. Multiplication by ω with subsequent integration over the volume yields by virtue of the Gauss theorem the quantity $\int \omega\nabla\omega dS$ (S is the boundary) which, generally speaking, does not vanish. However, if we consider planar[1)] (x, z) motion directed along the x axis between plane parallel boundaries (x,y) then, close to the boundary in the viscous sublayer $\langle v_x\rangle\sim z$,[20] in addition, v_x' (the pulsating component) is apparently also $\sim z$.[20] Therefore, all the second derivatives vanish and no vortex is generated on the boundary. The entire dynamics of the turbulence reduces then to vortex and the energy exchange between the averaged and the pulsating motions (and also to viscous damping).

Experimental investigations[21] show that the energy is transferred from the pulsating to the mean motion. These results cannot be connected with Prandtl theory, in which $\langle v_x'v_z'\rangle\sim\partial\langle v_x\rangle/\partial z$, since in this case it is necessary to stipulate a negative sign for the turbulence viscosity, which is physically meaningless. It appears that $\langle v_x'v_z'\rangle$ includes also other quantities which do not depend on the derivatives of the velocity with respect to the coordinates; however their form cannot be established from only theoretical considerations.

It must be kept in mind that the experiments just mentioned[21] were carried out in a channel whose length was not too great. It is physically obvious that in the case of a sufficiently long channel, the anomalous sign of $\langle v_x'v_z'\rangle$ leads eventually to a complete transfer of the energy of the pulsating motion to the mean motion, so that the latter will have the usually Poiseuille laminar profile. Then, by virtue of the instability, a transition again takes place to turbulent motion and so on. Thus, even in the given case (cf. the previous section), an oscillatory regime is established in which the directions of the energy transfer and of the vortex (in k space) change periodically along the direction of the mean motion. The possibility of formation of a laminar profile is then limited by the fact that, as in the case of isotropic turbulence, the stochastic motion can be transformed still earlier into some regular structure or that the meaning of statistical averaging is lost when the stochasticity is conserved (e.g., the Reynolds rules can cease to be valid[20]).

[1)] Such motion can be achieved experimentally in a conducting liquid with the aid of an external magnetic field.[21]

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