Nonlinear relaxation of a beam of relativistic electrons in a plasma: four-plasmon interactions

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We find by direct calculation the cubic collision term in the kinetic equation for Langmuir waves. The structure of this ("four-plasmon") term differs from the usually accepted one; this affects the Langmuir turbulence kinetics in an appreciable energy range. We study the heating of the plasma by a beam of relativistic electrons under conditions when the beam instability is eliminated by the four-plasmon process. Apart from estimates, we also obtain analytical expressions for the beam-electron distribution function and for the Langmuir wave spectrum. A result of this paper, which is of practical importance, consists in an important shift of the limit of applicability of relaxation theory into the region of large ratios n_b/n_0 of the beam and plasma densities.

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1. INTRODUCTION

One of the most effective mechanisms for energy transfer from a beam of relativistic electrons to a plasma is connected with the instability of this system against the excitation of Langmuir waves. The length over which the beam loses an appreciable fraction of its initial energy determines the dimensions of the apparatus necessary for reaching a high efficiency of plasma heating and therefore is an especially important parameter in the study of beam heating. For the theoretical determination of this length and of the Langmuir turbulence spectrum the problem of the mechanism for the stabilization of the instability plays a decisive role. A large number of papers has been devoted to this problem but so far a complete answer to it has not been found. The existing studies of the beam relaxation in the strong Langmuir turbulence regime (see, e.g., Ref. 1) are to a considerable extent phenomenological. The basis of the theory for the case of weak turbulence is reliable, but the relaxation regimes studied in the framework of this theory are feasible only for small ratios n_b/n_0 of the beam and plasma densities. In contemporary beam-heating experiments the ratio n_b/n_0 often exceeds by orders of magnitude the values described by a consistent theory (see, e.g., Ref. 2). Even in the case when the increase in the plasma density in the experimental apparatus will occur for fixed beam densities[1)] the indicated margin vanishes only for the parameters of a hypothetical thermonuclear reactor.[3] It is thus very desirable to extend the theory of the relaxation to larger values of n_b/n_0. In the present paper we develop and in the simplest circumstances realize the possibility of appreciable progress in that direction on the basis of weak turbulence theory. This possibility is connnected with taking into account higher-order nonlinear processes, primarily the scattering of Langmuir waves by induced density fluctuations.

The "simplest conditions" which will be understood in what follows to be satisfied consist in the following: the magnetic field is negligibly small, the excitation of electromagnetic waves is counteracted by their fast removal from the plasma,[2)] and the beam density is not too large—so that the scattering by induced density fluctuations is a third-order process in the Langmuir-wave energy. To solve the problem posed here it is necessary to know the collision term corresponding to this process. The terms of the first three orders in the wave energy in the kinetic equation that describes weak Langmuir turbulence have been evaluated in many papers (see, e.g., Ref. 5 and references given there). However, notwithstanding the fact that ultimately agreement was reached between the results of different authors, the cubic term for the case of a broad Langmuir spectrum was never evaluated correctly. We call here a spectrum "broad" if the spread in the group velocities of the Langmuir waves $\Delta v_{gr}=\Delta\omega/k_0$ is much larger than the quantity $c_s\equiv(T/m_i)^{1/2}$, where $T=T_e+T_i$ is the sum of the electron and ion temperatures and m_i the ion mass. For turbulence excited by a beam of relativistic electrons, $-k_0=\omega_p/c$, the condition

$$\Delta\omega>k_0c_s \quad (1)$$

reduces to the inequality

$$\frac{T_e}{m_ec^2}\frac{T_e}{T}>\frac{m_e}{m_i} \quad (1')$$

and is satisfied in not too cold a plasma ($T_e>0.2$ keV in an isothermal deuterium plasma). Before turning to a study of the relaxation we therefore fill this gap.

2. LANGMUIR-TURBULENCE KINETICS

The cubic collision term of the kinetic equation

$$dN_{\mathbf{k}}/dt=\mathrm{St}_{\mathbf{k}} \quad (2)$$

for the Langmuir waves was generally looked for in the form

$$\mathrm{St}_{\mathbf{k}}^{(3)}=\int d^3k_1d^3k_2d^3k_3\delta(\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3)\delta(\omega_k-\omega_{k_1}+\omega_{k_2}-\omega_{k_3})P_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}(N_{\mathbf{k}_1}N_{\mathbf{k}_2}N_{\mathbf{k}_3}-N_{\mathbf{k}}N_{\mathbf{k}_1}N_{\mathbf{k}_2}+N_{\mathbf{k}}N_{\mathbf{k}_1}N_{\mathbf{k}_3}-N_{\mathbf{k}}N_{\mathbf{k}_2}N_{\mathbf{k}_3}). \quad (3)$$

It was assumed here that there is only one process of third order in the wave energy—the scattering of two waves into two others. The combination of spectral densities (or occupation numbers) of Langmuir waves $N_{\mathbf{k}}$ in the right-hand side of (3) has been written down by analogy with quantum mechanics. The probability $P_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}$ for the process (3) has been determined from the spontaneous emission. In Refs. 6 to 8 formulae were found for it which agreed with one another. This result enables one to admit that the probability for spontaneous emission was reliably established, but has no bearing on the problem of whether it is equal to the probability for induced processes. Doubts about this equality arise for the following reason. The "semi-quantal" method of calculation used in deriving (3) is unsuitable to describe that part of the interaction between weakly damped collective excitations (quasi-particles) which is caused by strongly damped collective excitation. The assumption that this part is small and, hence, that the semi-quantal method is valid was explicitly formulated in Ref. 9, where it was stated that strongly damped modes do not affect the interaction between quasi-particles if their number exceeds by far the thermodynamic-equilibrium value. Applied to Langmuir turbulence, this statement can be reformulated as follows: the presence of a strongly damped ion-sound branch does not affect the interaction of strongly epithermal Langmuir noise. At the same time, for a sufficiently broad Langmuir spectrum this branch is undoubtedly excited: low-frequency beats of the electric field generate ion-sound fluctuations with an amplitude which is quadratic in the field. The scattering of Langmuir waves by these density fluctuations is a process of third order in the wave energy and can compete with the process of (3) independently of the density. These two processes can not coincide since, in contrast to the second one, the first does not conserve the momentum of the Langmuir waves—part of the momentum is transferred to the induced density fluctuations and is absorbed by the resonant ions and electrons (which cause the damping of the ion-sound oscillations). Thus there are at least two four-plasmon processes and the result obtained by the semi-quantal method must be reconsidered.

In the relaxation problem in which we are interested, as also in many other cases which are of practical importance, electron nonlinearities are unimportant. Neglecting them we can start the determination of the collision term from Zakharov's dynamic equations.[10] In the Fourier representation in the coordinates and in the time these equations have the following form:

$$a_{\mathbf{k}\omega}{}^{s}=g_{\mathbf{k}\omega}^{0s}\left[\int d^4k_1 d^4q\,\delta^4(ks-k_1s-q)\,V_{\mathbf{k}\mathbf{k}_1}n_{\mathbf{q}\Omega}a^{s}_{\mathbf{k}_1\omega_1}+\alpha_{\mathbf{k}\omega}{}^{s}\right],$$

$$n_{\mathbf{q}\Omega}=G_{\mathbf{q}\Omega}{}^{0}\left[\frac{1}{2}\sum_{s}\int d^4k\,d^4k_1\,\delta^4(ks-k_1s-q)\,V_{\mathbf{k}\mathbf{k}_1}a_{\mathbf{k}\omega}{}^{s}a^{-s}_{\mathbf{k}_1\omega_1}+\beta_{\mathbf{q}\Omega}\right]. \tag{4}$$

Here

$$a_{\mathbf{k}\omega}{}^{s}=\frac{k}{(8\pi\omega_p)^{1/2}}\begin{cases}\varphi_{\mathbf{k}\omega} & (s=1)\\ \varphi_{\mathbf{k}\omega}{}^{*} & (s=-1)\end{cases},$$

$\varphi_{\mathbf{k}\omega}$ and $n_{\mathbf{q}\Omega}$ are the Fourier transforms of the amplitude of the high-frequency electric potential (φ) and of the ion-density perturbation (n):

$$V_{\mathbf{k}\mathbf{k}_1}=\frac{\omega_p}{2n_0}\frac{\mathbf{k}\mathbf{k}_1}{kk_1},\quad g_{\mathbf{k}\omega}^{0s}=(\omega-\omega_k+is0)^{-1},\quad \omega_k={}^3/_2\omega_p r_D{}^2k^2,$$

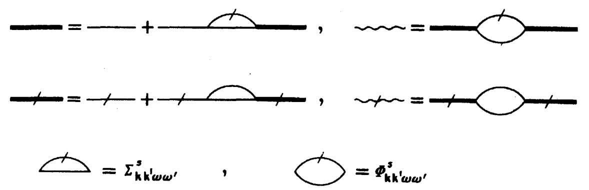
$$G_{\mathbf{q}\Omega}{}^{0}=-\frac{\rho_{\mathbf{q}\Omega}^{(i)}\rho_{\mathbf{q}\Omega}^{(e)}}{\rho_{\mathbf{q}\Omega}^{(i)}+\rho_{\mathbf{q}\Omega}^{(e)}},\quad \rho_{\mathbf{q}\Omega}^{(\sigma)}=\int d^3p\,\frac{\mathbf{q}\partial f_\sigma/\partial\mathbf{p}}{\Omega-\mathbf{q}\mathbf{v}+i0};$$

$f_\sigma(\mathbf{p})$ are the unperturbed electron ($\sigma=e$) and ion ($\sigma=i$) momentum distribution functions, normalized to the plasma density n_0; $\alpha^s_{\mathbf{k}\omega}$ and $\beta_{\mathbf{q}\Omega}$ are small random quantities the introduction of which is justified in the way which is usual for the Wyld diagram technique.[11,12] We introduce the Green functions and pair-correlation functions averaged over the independent random quantities $\alpha^s_{\mathbf{k}\omega}$ and $\beta_{\mathbf{q}\Omega}$:

$$\left\langle\frac{\delta a_{\mathbf{k}\omega}{}^{s}}{\delta\alpha^{s'}_{\mathbf{k}'\omega'}}\right\rangle=\delta_{ss'}g^{s}_{\mathbf{k}\mathbf{k}'\omega\omega'},\quad \langle a^{s}_{\mathbf{k}\omega}a^{-s'}_{\mathbf{k}'\omega'}\rangle=\delta_{ss'}N^{s}_{\mathbf{k}\mathbf{k}'\omega\omega'};$$

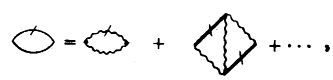
$$\left\langle\frac{\delta n_{\mathbf{q}\Omega}}{\delta\beta_{\mathbf{q}'\Omega'}}\right\rangle=G_{\mathbf{q}\mathbf{q}'\Omega\Omega'},\quad \langle n_{\mathbf{q}\Omega}n_{-\mathbf{q}'-\Omega'}\rangle=F_{\mathbf{q}\mathbf{q}'\Omega\Omega'}.$$

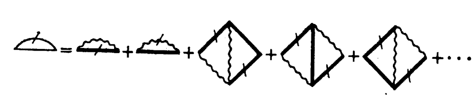
Depicting the correlation functions by wavy lines, the exact Green functions by thick and the unperturbed ones by thin lines and crossing off, to distinguish them, lines corresponding to density fluctuations, we can represent the basic equations of the Wyld technique in the form of the following graphs:



(5)

The symbols $\Sigma^s_{\mathbf{k}\mathbf{k}'\omega\omega'}$ and $\Phi^s_{\mathbf{k}\mathbf{k}'\omega\omega'}$ have been introduced for the mass operator and the compact part of the Langmuir wave correlation function. It is well known that these quantities are given by infinite series of compact diagrams:


(6)


(7)

Similar expansions are true for the compact part of the correlation function and the mass operator of the density fluctuations:


(8)

When the energy of the Langmuir waves is not too large the reciprocal of the nonlinear interaction time is less than the damping rate of the density fluctuations:

$$\gamma_n \ll \nu_s; \tag{9}$$

the latter are purely induced and their correlation function is quadratic in the wave energy. In that case, if we want to evaluate $\Phi^s_{\mathbf{k}\mathbf{k}'\omega\omega'}$ up to and including third-order and $\Sigma^s_{\mathbf{k}\mathbf{k}'\omega\omega'}$ up to and including second-order in the wave energy, the diagrams shown explicitly in (6) to (8) are sufficient. The required kinetic equation for the spectral density of the waves

$$N_{\mathbf{k}}(\mathbf{r},t)=\int d^3\varkappa\,d\omega\,df\,N_{\mathbf{k}+\varkappa/2,\mathbf{k}-\varkappa/2,\omega+f/2,\omega-f/2}e^{i(\varkappa\mathbf{r}-ft)}$$

can with sufficient accuracy be represented in the form

$$dN_{\mathbf{k}}/dt=2\pi\Phi_{\mathbf{k}}+2N_{\mathbf{k}}\,\mathrm{Im}\,\Sigma_{\mathbf{k}}, \tag{10}$$

where

$$\Phi_{\mathbf{k}}=\int d^3\varkappa\,df\,\Phi_{\mathbf{k}+\varkappa/2,\mathbf{k}-\varkappa/2,\omega_k+f/2,\omega_k-f/2}e^{i(\varkappa\mathbf{r}-ft)},$$

$$\Sigma_{\mathbf{k}}=\int d^3\varkappa\,df\,\Sigma_{\mathbf{k}+\varkappa/2,\mathbf{k}-\varkappa/2,\omega_k+f/2,\omega_k-f/2}e^{i(\varkappa\mathbf{r}-ft)}.$$

A simple calculation gives:

$$\mathrm{St}_{\mathbf{k}}=I_{\mathbf{k}}^{(1)}+I_{\mathbf{k}}^{(2)}+I_{\mathbf{k}}^{(3)}+I_{\mathbf{k}}^{(4)},$$

$$I_{\mathbf{k}}^{(1)}=2\,\mathrm{Im}\int d^3k_1 V_{\mathbf{k}\mathbf{k}_1}^2 G^0_{\mathbf{k}-\mathbf{k}_1,\omega_k-\omega_{k_1}} N_{\mathbf{k}_1}N_{\mathbf{k}},$$

$$I_{\mathbf{k}}^{(2)}=2\,\mathrm{Im}\int d^3k_1d^3k_2d^3k_3\delta(\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3)(\omega_k-\omega_{k_1}+\omega_{k_2}-\omega_{k_3}+i0)^{-1}V_{\mathbf{k}\mathbf{k}_1}^2V_{\mathbf{k}_2\mathbf{k}_3}^2G^{0\,2}_{\mathbf{k}-\mathbf{k}_1,\omega_k-\omega_{k_1}}(N_{\mathbf{k}_2}-N_{\mathbf{k}_3})N_{\mathbf{k}_1}N_{\mathbf{k}},$$

$$I_{\mathbf{k}}^{(3)}=2\pi\int d^3k_1d^3k_2d^3k_3\delta(\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3)\delta(\omega_k-\omega_{k_1}+\omega_{k_2}-\omega_{k_3})V_{\mathbf{k}\mathbf{k}_1}^2V_{\mathbf{k}_2\mathbf{k}_3}^2|G^0_{\mathbf{k}-\mathbf{k}_1,\omega_k-\omega_{k_1}}|^2(N_{\mathbf{k}_1}-N_{\mathbf{k}})N_{\mathbf{k}_2}N_{\mathbf{k}_3}, \quad (11)$$

$$I_{\mathbf{k}}^{(4)}=2\,\mathrm{Im}\int d^3k_1d^3k_2d^3k_3\frac{\delta(\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3)}{\omega_k-\omega_{k_1}+\omega_{k_2}-\omega_{k_3}-i0}V_{\mathbf{k}\mathbf{k}_1}V_{\mathbf{k}_1\mathbf{k}_2}V_{\mathbf{k}_2\mathbf{k}_3}V_{\mathbf{k}_3\mathbf{k}}$$
$$\times\{N_{\mathbf{k}_1}N_{\mathbf{k}_3}(N_{\mathbf{k}_2}G^0_{\mathbf{k}_2-\mathbf{k}_1,\omega_{k_2}-\omega_{k_1}}G^0_{\mathbf{k}_3-\mathbf{k}_2,\omega_{k_3}-\omega_{k_2}}+N_{\mathbf{k}}G^0_{\mathbf{k}_1-\mathbf{k},\omega_{k_1}-\omega_k}$$
$$\times G^0_{\mathbf{k}_3-\mathbf{k},\omega_{k_3}-\omega_k})-N_{\mathbf{k}}N_{\mathbf{k}_2}(N_{\mathbf{k}_3}G^0_{\mathbf{k}_2-\mathbf{k}_3,\omega_{k_2}-\omega_{k_3}}G^0_{\mathbf{k}_3-\mathbf{k},\omega_{k_3}-\omega_k}$$
$$+N_{\mathbf{k}_1}G^0_{\mathbf{k}_1-\mathbf{k},\omega_{k_1}-\omega_k}G^0_{\mathbf{k}_1-\mathbf{k}_2,\omega_{k_1}-\omega_{k_2}})\}.$$

The terms $I_{\mathbf{k}}^{(1)}$ and $I_{\mathbf{k}}^{(2)}$ are generated by the first of the graphs in (7), $I_{\mathbf{k}}^{(3)}$ by the first from (6) and by the second from (7), and $I_{\mathbf{k}}^{(4)}$ by the second from (6) and by the last three from (7). The quantity $I_{\mathbf{k}}^{(1)}$ describes the induced scattering of Langmuir waves by ions, $I_{\mathbf{k}}^{(2)}$ the correction to this process with allowance for the effect of the Langmuir turbulence on the density fluctuation Green function, $I_{\mathbf{k}}^{(3)}$ the scattering of Langmuir waves by induced density fluctuations, and $I_{\mathbf{k}}^{(4)}$ the scattering of two Langmuir waves into two other ones.

In the case of a broad spectrum the quantities $I_{\mathbf{k}}^{(1)}$, $I_{\mathbf{k}}^{(2)}$, and $I_{\mathbf{k}}^{(4)}$ can easily be estimated by appropriately shifting the integration path into the complex plane and using the expansion of the function $G^0_{\mathbf{q}\Omega}$ in the parameter $qc_s/|\Omega|\ll 1$ in the region $\mathrm{Im}\,\Omega>0$:

$$G_{\mathbf{q}\Omega}^0\approx n_0q^2/m_i\Omega^2.$$

Doing this we find:

$$\frac{I_{\mathbf{k}}^{(1)}}{N_{\mathbf{k}}}\sim\omega_p\frac{W}{n_0T}\left(\frac{k_0c_s}{\Delta\omega}\right)^2,\quad \frac{I_{\mathbf{k}}^{(2)}}{N_{\mathbf{k}}}\sim\omega_p\left(\frac{W}{n_0T}\right)^2\frac{\omega_p}{\Delta\omega}\left(\frac{k_0c_s}{\Delta\omega}\right)^4\sim\frac{I_{\mathbf{k}}^{(4)}}{N_{\mathbf{k}}}.$$

Here W is the Langmuir wave energy density. By virtue of (1), $I_{\mathbf{k}}^{(2)}$ and $I_{\mathbf{k}}^{(4)}$ are negligibly small compared with $I_{\mathbf{k}}^{(1)}$ in the whole range of applicability of the weak-turbulence theory. On the other hand, in the equations for $I_{\mathbf{k}}^{(1)}$ and $I_{\mathbf{k}}^{(3)}$ we can change to the so-called differential approximation which is well known for induced scattering by ions. This change is accomplished by the substitutions

$$\mathrm{Im}\,G_{\mathbf{q}\Omega}^0\to\frac{\pi n_0}{m_i}\delta'\left(\frac{\Omega}{q}\right),$$
$$|G_{\mathbf{q}\Omega}^0|^2\to\delta\left(\frac{\Omega}{q}\right)\int d\frac{\Omega}{q}|G_{\mathbf{q}\Omega}^0|^2=\frac{\pi n_0\zeta}{m_i^2c_s^3}\delta\left(\frac{\Omega}{q}\right), \quad (12)$$

where $\zeta\gtrsim 1$ is a dimensionless coefficient which can be estimated to be equal to the ratio of the frequency $\Omega_s=k_0c_s$ of the ion-sound oscillations to their damping ν_s:

$$\zeta=\begin{cases}\Omega_s/\nu_s & (T_e\gg T_i)\\ \sqrt{\pi} & (T_i\gg T_e)\end{cases}.$$

After the substitutions (12) the collision term (11) takes the form

$$\mathrm{St}_{\mathbf{k}}=\frac{2\pi n_0}{m_i}\int d^3k_1\delta'(\omega_k-\omega_{k_1})|\mathbf{k}-\mathbf{k}_1|^2V_{\mathbf{k}\mathbf{k}_1}^2N_{\mathbf{k}}N_{\mathbf{k}_1}$$
$$+\frac{2\pi^2n_0^2\zeta}{m_i^2c_s^3}\int d^3k_1d^3k_2d^3k_3\delta(\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3)$$
$$\times\delta(\omega_k-\omega_{k_1})\delta(\omega_{k_2}-\omega_{k_3})|\mathbf{k}-\mathbf{k}_1|V_{\mathbf{k}\mathbf{k}_1}^2V_{\mathbf{k}_2\mathbf{k}_3}^2N_{\mathbf{k}_2}N_{\mathbf{k}_3}(N_{\mathbf{k}_1}-N_{\mathbf{k}}). \quad (13)$$

The reciprocal time of scattering by induced density fluctuations can be estimated to be equal to

$$\frac{I_{\mathbf{k}}^{(3)}}{N_{\mathbf{k}}}\sim\omega_p\left(\frac{W}{n_0T}\right)^2\frac{\omega_p}{\nu_s}\left(\frac{k_0c_s}{\Delta\omega}\right)^2. \quad (14)$$

When

$$W/n_0T\ll\nu_s/\omega_p \quad (15)$$

this process is slower than the induced scattering by ions, but in the region

$$\nu_s/\omega_p\ll W/n_0T\ll\nu_s\Delta\omega/\Omega_s\omega_p \quad (16)$$

the cubic term becomes the main one. The second inequality in (16) follows from the condition (9) that the density fluctuations be "static." When $T_i\gtrsim T_e$ (when $\nu_s\sim\Omega_s$) this inequality is the same as the condition for the applicability of weak-turbulence theory; when $T_e\gg T_i$ there is a region

$$\frac{\nu_s}{\Omega_s}\frac{\Delta\omega}{\omega_p}\lesssim\frac{W}{n_0T}\ll\left(\frac{\nu_s}{\Omega_s}\right)^{1/2}\frac{\Delta\omega}{\omega_p},$$

where the turbulence is still weak, but the sound is no longer static. This region is described by the standard kinetic equations for the three-wave decay interaction of Langmuir and ion-sound waves. These equations are applicable also in the wider region

$$\nu_s;\ \gamma_n\ll\Omega_s,$$

which intersects (9). In the case of static sound they reduce to a single kinetic equation for Langmuir waves[4] with a collision term which agrees with (11) and disagrees with (3).

We note that for a broad Langmuir spectrum the probability, evaluated in Refs. 6 to 8, is practically the same as the probability for scattering by induced density fluctuations (as the scattering by two waves into two others is weak). Nonetheless our conclusion that the cubic collision term may prevail over the quadratic one under conditions (1) and (9) contradicts the conclusions of these papers. The cause of the disagreement is Refs. 6 and 7 contain an error the estimate of the cubic term, while in Ref. 8 the region of applicability of the kinetic equation was established inaccurately (while the cubic term and accordingly the width of the "jet" were correctly estimated).

In concluding this section we show how the collision term (11) can be simplified for a narrow Langmuir spectrum ($\Delta\omega\ll k_0c_s$). In that case the ion-sound branch is practically not excited and the cubic term must have the form (3). A direct calculation confirms this conclusion: when $\Delta\omega\ll k_0c_s$ we can put $G^0_{q\Omega}=-n_0/T$ in the formulae for $I_{\mathbf{k}}^{(2)}$, $I_{\mathbf{k}}^{(3)}$, $I_{\mathbf{k}}^{(4)}$, after which the sum of all cubic terms in (11) takes the form (3) with the well known probability

$$P_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}=T^{-2}\pi n_0^2(V_{\mathbf{k}\mathbf{k}_1}V_{\mathbf{k}_2\mathbf{k}_3}+V_{\mathbf{k}_1\mathbf{k}_2}V_{\mathbf{k}_3\mathbf{k}})^2.$$

3. BASIC EQUATIONS

We consider in what follows the problem of quasi-stationary heating of a plasma, which uniformly occupies a half-space, by a concentrated beam of relativistic electrons injected along the normal to its boundary. Under such conditions there remains in the left-hand side of the kinetic Eq. (2) only the term containing the derivative of the spectral function $N_{\mathbf{k}}$ in the injection direction z. At distances from the boundary of the plasma which appreciably exceed the length of the nonlinear interaction of the waves, we can neglect also that term, for after the spectrum has been established the scale along which its parameters change becomes of the order of the beam-electron quasi-linear diffusion length, which is large when the elimination of the instability is nonlinear. Therefore, for the determination of the wave spectrum we must set the collision term equal to zero and solve the resulting equation. After that we can determine the spatial dependence of the beam parameters and of the spectrum from the diffusion equation.

To state the problem more precisely we recall that the the instability of a low-density relativistic beam can be eliminated on account of the diffusion of its electrons[13-16] or of the induced scattering of the Langmuir waves by the plasma particles—the ions[17-23] or the electrons.[14] The reciprocal times for the processes listed here are proportional to the wave energy and because of this it does not enter into the conditions that one of them will predominate over the others. Like the energy, the growth rate for the beam instability (to which the energy is proportional when there is stabilization present) also does not enter into these conditions and hence neither does the beam density n_b. Below, in view of what is of practical interest, we assume that the fastest process at small values of n_b/n_0 is the induced scattering of Langmuir waves by ions; the instability is assumed to be kinetic and the plasma to be not too cold [see (1′)].

The expressions for the probabilities of the various processes, for the beam relaxation length, and for the other quantities in dimensional variables turn out to be very cumbersome. To simplify the formulae it is convenient to introduce natural units for scaling. We shall scale: the length by the characteristic length for the resonant Langmuir waves interacting with the beam; the frequency by the dispersive correction to the frequency of these waves; the energy of the Langmuir turbulence by an energy of the order of the threshold for the modulational instability; the plasma density perturbation n, and also the beam density n_b, by the magnitude of the perturbation necessary for the capture of resonant waves; and the velocity and momentum of the beam electrons by the initial values of these quantities. To make precise the numerical coefficients we give the appropriate formulae:

$$[x]-\frac{v_0}{\omega_p},\quad [n,n_b]-\frac{3n_0T_e}{m_ev_0^2},\quad [v]-v_0\approx c,$$

$$[t]-\frac{2}{3}\frac{m_ev_0^2}{\omega_pT_e},\quad [W]-\frac{6n_0TT_e}{m_ev_0^2},\quad [p]-p_0\approx\eta m_ec.$$

Here η is the ratio of the initial energy of the beam electrons to the rest energy m_ec^2.

In these units the diffusion equation for the beam electrons and the collision term (13) have the following form:

$$\cos\theta\frac{\partial f}{\partial z}=\frac{\partial}{\partial p_\alpha}D_{\alpha\beta}\frac{\partial f}{\partial p_\beta},$$

$$D_{\alpha\beta}=\frac{6\pi}{l_0}\int d^3k\frac{k_\alpha k_\beta}{k^2}N_{\mathbf{k}}\delta(1-\mathbf{kv}),\quad l_0=\frac{(\eta m_ec^2)^2}{TT_e};\tag{17}$$

$$\mathrm{St}_{\mathbf{k}}=(2\gamma_{\mathbf{k}}-\nu_e+\tilde{\gamma}_{\mathbf{k}}-\Gamma_{\mathbf{k}})N_{\mathbf{k}}+u_{\mathbf{k}},$$

$$\tilde{\gamma}_{\mathbf{k}}=2\pi g^2\int d^3k_1\delta'(k^2-k_1^2)\left(\frac{\mathbf{kk}_1}{kk_1}\right)^2N_{\mathbf{k}_1};\tag{18}$$

$$u_{\mathbf{k}}-\Gamma_{\mathbf{k}}N_{\mathbf{k}}=2\pi\int d^3k_1\delta(k^2-k_1^2)\left(\frac{\mathbf{kk}_1}{kk_1}\right)^2F_{\mathbf{k}-\mathbf{k}_1}(N_{\mathbf{k}_1}-N_{\mathbf{k}});$$

$$F_{\mathbf{q}}=\zeta g\int d^3k_2d^3k_3\delta(\mathbf{k}_2-\mathbf{k}_3+\mathbf{q})\delta(k_2^2-k_3^2)\left(\frac{\mathbf{k}_2\mathbf{k}_3}{k_2k_3}\right)^2N_{\mathbf{k}_2}N_{\mathbf{k}_3},\tag{19}$$

$$g\equiv\frac{k_0c_s}{\omega_{k_0}}=\frac{2}{3}\left(\frac{m_e}{m_i}\frac{T}{T_e}\frac{m_ec^2}{T_e}\right)^{1/2}\ll 1.\tag{20}$$

We have included in the collision term the Langmuir-wave damping ν_e caused by the Coulomb collisions of the plasma electrons with the ions[3)] and the beam-instability growth rate

$$\gamma_{\mathbf{k}}=\frac{\pi n_b}{\eta}\int d^3p\frac{\mathbf{k}}{k^2}\frac{\partial f}{\partial\mathbf{p}}\delta(1-\mathbf{kv}).\tag{21}$$

Here and in (17) $f(\mathbf{p})$ is the momentum distribution function for the beam electrons, normalized to unity. For a beam with an angular spread $\Delta\theta$ the following estimate holds

$$\gamma\sim\gamma_0/\Delta\theta^2\quad(\gamma_0=\pi n_b/\eta).\tag{22}$$

4. ESTIMATES

As we have already noted above, we shall assume that the instability of a low density beam is eliminated by induced scattering of the Langmuir waves by the ions. The reciprocal time of this process can be estimated to equal

$$\tilde{\gamma}\sim g^2W.\tag{23}$$

For comparison we give here the reciprocal time for the induced scattering by electrons:

$$\gamma_e\sim(T_e/m_ec^2)^{5/2}W.$$

The scattering by electrons will be faster than by electrons if

$$\frac{T_e}{m_ec^2}<\left(\frac{m_e}{m_i}\frac{T}{T_e}\right)^{1/3}.$$

($T_e<20$ keV for an isothermal deuterium plasma). Characteristic lengths for these two processes are given by the same estimates as the times since the group velocity of the Langmuir waves is of the order of unity.

The Langmuir spectrum when there is induced scattering by ions is jet-like[8] with one jet containing an appreciable fraction of the Langmuir waves close to the maximum growth rate. It therefore follows from the energy-balance condition that $\tilde{\gamma}\sim\gamma$ and that

$$W\sim\gamma/g^2.\tag{24}$$

The idea of the jetlike spectrum is justified as long as the width of the jet ($\Delta\psi$) is less than the angular width ($\Delta\theta$) of the excitation region. One can easily show (see

Ref. 8) that

$$\Delta\psi\sim(\Gamma/\gamma)^{1/2}\Delta\theta,\quad \Gamma\sim g\zeta W^2. \tag{25}$$

Here Γ is the reciprocal time of the four-plasmon process (in the case considered—scattering by induced density fluctuations). The condition that the jet be narrow is equivalent to the inequality $\Gamma<\gamma$, which is satisfied when $\gamma<\zeta^{-1}g^3$, i.e., when

$$\gamma_0<\zeta^{-1}g^3\Delta\theta^2. \tag{26}$$

In dimensional variables inequality (26) has the form

$$\frac{n_b}{n_0}<\eta\frac{T_e}{m_ec^2}\zeta^{-1}g^3\Delta\theta^2$$

(when ion sound is damped by electrons in a deuterium plasma with $T_e=10$ keV we have $n_b/n_0<3\times10^{-7}\,\eta\Delta\theta^2$).

The length l_θ over which the angular spread doubles can be estimated using the diffusion equation (17): $l_\theta\sim\Delta\theta^3 l_0/W$. When the relatively weak condition

$$\Delta\theta>\left(\frac{1}{\eta^4}\frac{T}{T_e}\frac{T}{m_ec^2}\right)^{1/4}$$

is satisfied, this length is larger than the length for induced scattering by electrons and all the more for induced scattering by ions.

The energy flux towards lower frequencies, caused by the induced scattering by ions, manages to get absorbed thanks to Coulomb collisions and does not lead to the formation of a Langmuir condensate only at a small excess above the critical value[4]: $\gamma-\nu_e\lesssim\nu_e$. In practice this condition means that after the initial angular spread ($\Delta\theta_0$) has been doubled the beam instability is stopped by the collisions. The beam manages to lose a fraction $\sim\Delta\theta_0$ of its initial energy. This fraction is large only when $\Delta\theta_0\sim1$. For a beam with an angular spread of order unity the condition (26) has the form $\gamma_0<\zeta^{-1}g^3$.

If

$$\gamma_0\gg\zeta^{-1}g^3, \tag{27}$$

induced scattering by ions is a relatively slow process for a beam with any angular spread. The instability is eliminated by elastic scattering of the plasmons by density fluctuations.[5] The reciprocal time for this process must satisfy the condition

$$\Gamma\gtrsim\gamma. \tag{28}$$

If inequality (28) were satisfied with an infinitely large margin the spectrum would be isotropic. For the case of an isotropic spectrum the beam would absorb Langmuir waves over a time γ_0^{-1}.[24] With increasing ratio γ/Γ a hump will develop on the spectral function of the plasmons in the region where the growth rate is positive, and a dip in the region where it is negative. The magnitude of the hump (dip) is determined by the condition that there be a balance between the isotropic (N) and the anisotropic (N') parts of the spectrum:

$$\gamma N\sim\Gamma N'. \tag{29}$$

For the steady-state spectrum this quantity is such that the additional excitation of waves connected with the hump and the dip exactly compensates for the damping of the isotropic part of the spectrum and also for the losses connected with the Coulomb collisions and with the induced scattering by the ions:

$$\gamma N'\Delta\theta\sim(\gamma_0+\nu_e+\tilde{\gamma})N. \tag{30}$$

The relations (29) and (30) enable us to find N and N'. Let

$$\gamma_0\gtrsim\nu_e\gtrsim\tilde{\gamma}. \tag{31}$$

The first condition is necessary in order that the instability not stop when there is a small angular spread and that the beam can transfer a considerable fraction of its energy to the plasma; the second condition guarantees the absence of a Langmuir condensate.[6] Using (31) we can retain only the first term in the right-hand side of (30). After that it follows from (30) and (29) that:

$$N'/N\sim\gamma/\Gamma\sim\Delta\theta. \tag{32}$$

Assuming that the spectrum is confined to the region $k\sim1$ and has a width $\Delta\omega\sim1$ the quantity Γ is given as before by the estimate (25) and

$$N\sim W\sim(\gamma_0/\zeta g\Delta\theta^3)^{1/2}. \tag{33}$$

The change in the angular spread in the beam is described by the diffusion equation $d\Delta\theta^2/dz\sim D\sim W/l_0$ and proceeds according to the relation

$$\Delta\theta^{7/2}=\Delta\theta_0^{7/2}+z/l, \tag{34}$$

where

$$l\sim(\zeta g/\gamma_0)^{1/2}l_0 \tag{35}$$

is the length for the angular relaxation of the beam. At distances $\sim l$ from the boundary of the plasma the instability is stopped by the isotropization of the beam. The fraction of energy lost by the beam is connected with its angular spread through the relation $\varepsilon\sim\Delta\theta^2\nu_e/\gamma_0$ and $\varepsilon\sim\nu_e/\gamma_0$ at the moment when the instability is stopped. The efficiency of the plasma heating is high when $\nu_e\sim\gamma_0$. In that case the second of conditions (31) is the same as the inequality given in footnote 5, and gives a lower bound for the angular spread of the beam:

$$\Delta\theta>g/(\zeta\gamma_0)^{1/2}. \tag{36}$$

By virtue of (27) this limitation is satisfied even when $\Delta\theta\ll1$.

In the present section we have assumed the existence of a stable stationary spectrum confined to the region $k\sim1$ and with a width $\Delta\omega\sim1$. It is not possible to verify this assumption by estimates. We therefore turn to an exact solution of the problem.

5. SELF-SIMILARITY

It is possible to advance appreciably towards an analytical solution of the beam relaxation problem at $\Delta\theta\ll1$ when the anisotropic part $N_{\mathbf{k}}'$ of the plasmon spectral density is small compared with the isotropic part $N_{\mathbf{k}}$:

$$N_{\mathbf{k}}=N_k+N_{\mathbf{k}}';\quad N_k=\frac{1}{4\pi}\int do_{\mathbf{k}}N_{\mathbf{k}};\quad |N_{\mathbf{k}}'|\ll N_k. \tag{37}$$

here $do_{\mathbf{k}}$ indicates an element of solid angle in **k**-space.

As the anisotropic correction is concentrated in a narrow (of the order of $\Delta\theta$) region of angles close to

$$\theta_k=\arccos(1/k),$$

all possible integrals of $N_{\mathbf{k}}'$ over angles contain an additional small factor $\Delta\theta$ and can be neglected with terms of order $N_{\mathbf{k}}'/N_k$ retained. Up to the largest anisotropic terms the equation $St_{\mathbf{k}}=0$ has the form

$$(2\gamma_{\mathbf{k}}-\Gamma_k)N_k-\Gamma_k N_{\mathbf{k}}'+u_k=0. \qquad (38)$$

Quantities with a scalar index indicate, as in (37), the isotropic parts of the corresponding functions:

$$\Gamma_k=8\pi^2\int_0^k dk_1\frac{k_1}{k}\left(1-2\frac{k_1^2}{k^2}\right)^2 F_{2k_1}, \quad u_k=\Gamma_k N_k,$$

$$F_{2k_1}=\frac{1}{2}\pi^2\zeta g\int_{k_1}^\infty dk_2k_2\left(1-2\frac{k_1^2}{k_2^2}\right)^2 N_{k_2}^2. \qquad (39)$$

Equation (38) enables us to express $N_{\mathbf{k}}'$ in terms of N_k:

$$N_{\mathbf{k}}'=2\frac{\gamma_{\mathbf{k}}-\gamma_k}{\Gamma_k}N_k. \qquad (40)$$

Integrating the exact equation $St_{\mathbf{k}}=0$ over the angles gives us an equation for N_k:

$$(2\gamma_k-\nu_e+\tilde{\gamma}_k)N_k+\frac{1}{4\pi}\int do_{\mathbf{k}}2\gamma_{\mathbf{k}}N_{\mathbf{k}}'=0. \qquad (41)$$

Here $\tilde{\gamma}_k$ is the isotropic part of $\tilde{\gamma}_{\mathbf{k}}$:

$$\tilde{\gamma}_k=\frac{4}{3}\pi^2g^2\frac{d}{dk}k^2N_k. \qquad (42)$$

Equations (40) and (41) are the analogs of the estimates (29) and (30). Substituting (40) in (41) leads to the relation

$$\gamma_k^{\text{eff}}N_k=0,$$

$$\gamma_k^{\text{eff}}\equiv 2\gamma_k-\nu_e+\tilde{\gamma}_k+\frac{1}{\pi\Gamma_k}\int do_{\mathbf{k}}\gamma_{\mathbf{k}}^2. \qquad (43)$$

Using the condition for the extrinsic stability of the spectrum[7] we can write (43) in the form

$$\gamma_k^{\text{eff}}=0, \text{ when } N_k>0,$$

$$\gamma_k^{\text{eff}}\leqslant 0, \text{ when } N_k=0. \qquad (44)$$

A general formula for the isotropic part γ_k of the growth rate was obtained in Ref. 24. For a relativistic beam with a small energy spread ($\Delta E\ll 1$) this formula can be simplified considerably:

$$\gamma_k=-k^{-2}\gamma_0\Theta(k-1). \qquad (45)$$

Here $\theta(x)$ is the unit step function:

$$\Theta(x)=\begin{cases}1, & \text{when } x>0\\ 0, & \text{when } x<0\end{cases},$$

and γ_0 is given by Eq. (22).

Deviations from the simple relation (45) occur only in a narrow region

$$|k-1|\leqslant\Delta E/\eta^2$$

and do not play any significant role in what follows.

The calculation of the isotropic part of the square of the growth rate for an axially symmetric beam leads to the following formula:

$$\int\gamma_{\mathbf{k}}^2do_{\mathbf{k}}=\frac{1}{\pi}\frac{(k^2-1)^{1/2}}{k^5}\Theta(k-1)$$

$$\times\int_0^\infty d\theta d\theta_1\frac{\partial u(\theta)}{\partial\theta}\frac{\partial u(\theta_1)}{\partial\theta_1}(\theta+\theta_1)\left[K'\left(\frac{|\theta-\theta_1|}{\theta+\theta_1}\right)-E'\left(\frac{|\theta-\theta_1|}{\theta+\theta_1}\right)\right]. \qquad (46)$$

Here K' and E' are complete elliptic integrals of the first and second kind of the complementary modulus

$$u(\theta)=2\pi\int_0^\infty dpf(\mathbf{p})p.$$

Formula (46) is applicable everywhere except in a narrow region that is unimportant for what follows,

$$|k-1|\leqslant(\Delta\theta+\Delta E/\eta^2\Delta\theta)^2.$$

The distribution function for the beam electrons is determined from the diffusion equation. In the present case it can be written in the form

$$\frac{\partial f}{\partial z}=\frac{D_\parallel}{p^2}\frac{\partial}{\partial p}p^2\frac{\partial f}{\partial p}+\frac{D_\perp}{p^2}\frac{1}{\theta}\frac{\partial}{\partial\theta}\theta\frac{\partial f}{\partial\theta}, \qquad (47)$$

where

$$D_\parallel=\frac{12\pi^2}{l_0}\int_1^\infty dk\frac{N_k}{k},$$

$$D_\perp=\frac{12\pi^2}{l_0}\int_1^\infty dk\left(k-\frac{1}{k}\right)N_k. \qquad (48)$$

If we use the fact that the energy spread ΔE is small we can obtain from (47) a closed equation for the function $u(\theta,z)$:

$$\frac{\partial u}{\partial z}=\frac{D_\perp}{\theta}\frac{\partial}{\partial\theta}\theta\frac{\partial u}{\partial\theta}. \qquad (49)$$

Equation (49) has a self-similar solution

$$u(\theta,z)=\frac{1}{\Delta\theta^2}u_1\left(\frac{\theta}{\Delta\theta}\right). \qquad (50)$$

The factor in front of u_1 is chosen such that the normalization condition

$$\int_0^\infty d\xi\xi u_1(\xi)=1 \qquad (51)$$

is satisfied. Substituting (50) into (49) and using (51) we prove easily that

$$d\Delta\theta^2/dz=D_\perp, \qquad (52)$$

$$u_1(\xi)={}^1/_2e^{-\xi^2/4}. \qquad (53)$$

The distribution (53) is established after the angular spread has increased by an appreciable factor for any initial distribution with $\Delta\theta_0\ll 1$. Together with $u(\theta,z)$, the growth rate of the beam instability also turns out to be a universal function. In particular, Eq. (46) takes the form

$$\int\gamma_{\mathbf{k}}^2do_{\mathbf{k}}=\frac{A\gamma_0}{\Delta\theta^3}\frac{(k^2-1)^{1/2}}{k^5}\Theta(k-1), \qquad (54)$$

where

$$A=\frac{3}{2(2\pi)^{1/2}}\int_0^1\frac{dx(1-x^2)[K'(x)-E'(x)]}{(1+x^2)^{5/2}}\approx 0{,}19. \qquad (55)$$

The Langmuir spectrum also satisfies a universal equation; to find it, it is convenient to introduce an auxiliary function $N(\omega)$:

$$N_k=\frac{W}{4\pi a}k^2N(k^2), \tag{56}$$

normalized by the condition

$$\int_0^\infty dkk^4N(k^2)=a. \tag{57}$$

Substitution of (56) in (39) gives

$$\Gamma_k=\frac{\pi^2 g\zeta W^2}{16a^2k^5}\Gamma(k^2),\quad \Gamma(\omega)=\int_0 d\omega_1(\omega-2\omega_1)^2F(\omega_1),$$
$$F(\omega_1)=\int_{\omega_1}^\infty d\omega_2(\omega_2-2\omega_1)^2N^2(\omega_2). \tag{58}$$

The quantity a which so far has been arbitrary can be defined such that the relation

$$W=\left(\frac{2}{\pi\Delta\theta}\right)^{3/2}\left(\frac{A\gamma_0}{\zeta g}\right)^{1/2}a \tag{59}$$

holds. The equation for γ_k^{eff} then turns out to be

$$\gamma_k^{\text{eff}}=2\gamma_0\left[\frac{(k^2-1)^{1/2}}{\Gamma(k^2)}-\frac{1}{k^3}\right]\Theta(k-1)-\nu_e+\frac{\pi g^2W}{3a}\frac{d}{dk}k^4N(k^2). \tag{60}$$

When inequality (36) is satisfied with a large margin we are able to neglect in (60) the third term describing the induced scattering by ions. When that process is absent the Langmuir spectrum is obviously concentrated in the region $k>1$ and we can write Eqs. (44) in the form

$$\Gamma(\omega)=\bar\Gamma(\omega),\ \text{when}\ N(\omega)>0; \tag{61}$$
$$\Gamma(\omega)\geqslant\bar\Gamma(\omega),\ \text{when}\ N(\omega)=0\quad(\omega>1), \tag{62}$$

where

$$\bar\Gamma(\omega)=\frac{(\omega-1)^{1/2}}{\omega^{-3/2}+\nu_e/2\gamma_0}. \tag{63}$$

The function $N(\omega)$ defined by Eqs. (61) and (62) does not change in the relaxation process. Knowing this function we can use Eq. (57) to evaluate the quantity a and substituting it into (59) find the energy of the Langmuir waves as a function of the angular spread in the beam. The change of the angular spread is determined by Eq. (52) with the diffusion coefficient

$$D_\perp=\left(\frac{18A\gamma_0}{\pi\zeta g\Delta\theta^3}\right)^{1/2}\frac{b}{l_0},\quad b=\int_1^\infty d\omega(\omega-1)N(\omega). \tag{64}$$

Integration of this equation leads to Eq. (34) with a more exact expression for the beam relaxation length:

$$l=\frac{4}{7}\left(\frac{\pi\zeta g}{18A\gamma_0}\right)^{1/2}\frac{l_0}{b}. \tag{65}$$

6. CONSTRUCTION OF THE SPECTRUM

Differentiating six times with respect to ω reduces Eq. (61) to an ordinary fourth-order differential equation:

$$\hat L^2N^2(\omega)=-\frac{d^6\bar\Gamma(\omega)}{d\omega^6},$$
$$\hat L=\left(\omega\frac{d}{d\omega}\right)^2+7\omega\frac{d}{d\omega}+14. \tag{66}$$

The general solution of Eq. (66) has the form

$$N^2(\omega)=c_1\left(\frac{\omega}{\omega_M}\right)^\alpha+c_2\left(\frac{\omega}{\omega_M}\right)^\alpha\ln\frac{\omega_M}{\omega}+\text{c.c.}+\tilde N^2(\omega). \tag{67}$$

Here $\tilde N(\omega)$ is a particular solution which is constructed in the standard way using solutions of the homogeneous equation; $\alpha=\frac{1}{2}(-7+i\sqrt7)$ is one of the roots of the characteristic equation $\alpha^2+7\alpha+14=0$; c_1 and c_2 are complex constants. The real parameter ω_M could have been included in c_1 and c_2, but our form of (67) is more convenient for what follows.

In the general case the Langmuir spectrum consists of a set of spherical layers in each of which it has the form (67) with its own constants c_1 and c_2. With a single layer are connected six real constants: $\operatorname{Re}c_1$, $\operatorname{Im}c_1$, $\operatorname{Re}c_2$, $\operatorname{Im}c_2$, ω_m, and ω_M, where ω_m and ω_M are the lower and upper boundaries of the layer. Substitution of (67) in $\Gamma(\omega)$ yields a function which differs from $\bar\Gamma(\omega)$ in each of the layers by a fifth degree polynomial (different in each layer). The conditions that the six coefficients of this polynomial must vanish in all layers enable us to determine all parameters of the spectrum uniquely. The function $N(\omega)$, being uniquely defined inside the layers and zero outside them, undergoes in general a discontinuity on the boundary of each layer. The discontinuities vanish when one takes into account induced scattering by the ions. This scattering imparts to layer boundaries a finite width $\delta\omega$:

$$\delta\omega\sim(g^3/\zeta\gamma_0\Delta\theta^3)^{1/2}\ll1.$$

In the case of interest to us the function $\bar\Gamma(\omega)$ is given by Eq. (63) and the spectrum consists of a single spherical layer. The particular solution of Eq. (66) can be written in the form

$$\tilde N^2(\omega)=\begin{cases}-\dfrac{1}{\omega^2}\hat R_2\dfrac{d}{d\omega}\hat R_1\dfrac{1}{\omega^2}\dfrac{d\bar\Gamma(\omega)}{d\omega}, & \omega_m<\omega<\omega_M\\ 0,\quad \omega<\omega_m\ \text{or}\ \omega>\omega_M\end{cases}. \tag{68}$$

Here $\hat R_1$ and $\hat R_2$ are linear operators whose action is given by the formulae

$$\hat R_1\chi(\omega)=\chi(\omega)+\frac{4}{\sqrt7}\operatorname{Im}(\alpha+3)\,\omega^{\alpha+3}\int_{\omega_m}^{\omega}\frac{d\omega_1}{\omega_1^{\alpha+4}}\chi(\omega_1), \tag{69}$$

$$\hat R_2\chi(\omega)=\chi(\omega)+\frac{8}{\sqrt7}\operatorname{Im}(\alpha+3)\,\omega^{\alpha+2}\int_{\omega}^{\omega_M}\frac{d\omega_1}{\omega_1^{\alpha+3}}\chi(\omega_1). \tag{70}$$

Substitution of (67) with the particular solution (68) into (61) leads to the following set of equations:

$$\frac{c_1}{\alpha+p}+\frac{c_2}{(\alpha+p)^2}+\text{c.c.}=\Gamma_p,\quad p=1,2,3;$$
$$\left(\frac{\omega_m}{\omega_M}\right)^\alpha\left[\frac{c_1+c_2\ln(\omega_M/\omega_m)}{\alpha+p}+\frac{c_2}{(\alpha+p)^2}\right]+\text{c.c.}=\Gamma_p,\quad p=4,5,6. \tag{71}$$

Here

$$\Gamma_1=\Gamma_2=0,\quad \Gamma_3=\tilde F(\omega_M)/\omega_M^3,$$
$$\Gamma_4=3\frac{\tilde F(\omega_m)}{\omega_m^3}+\frac{4}{\sqrt7}\operatorname{Im}(\alpha+5)I(\omega_m,\omega_M),$$
$$\Gamma_5=3\frac{\tilde F(\omega_m)}{\omega_m^3}+\frac{4}{\sqrt7}\operatorname{Im}(\alpha+4)I(\omega_m,\omega_M),$$
$$\Gamma_6=\frac{5}{2}\frac{\tilde F(\omega_m)}{\omega_m^3}-\frac{15}{8}\frac{\bar\Gamma(\omega_m)}{\omega_m^6}+\frac{1}{\sqrt7}\operatorname{Im}(3\alpha+11)I(\omega_m,\omega_M),$$

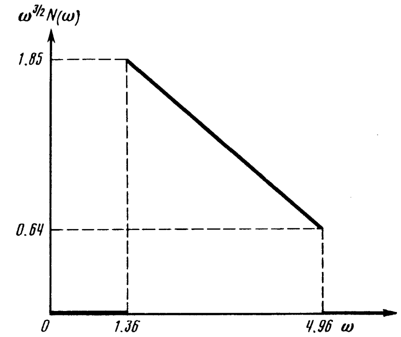


FIG. 1. Energy density of the Langmuir waves with respect to frequency as function of the dispersive correction when the beam instability is stabilized by the four-plasmon process. All variables are dimensionless. The damping of the Langmuir waves is small compared to the growth rate of the instability of a spread-out beam: $\nu_e \ll \gamma_0$.

$$F(\omega)=\hat{R}_1 \frac{1}{\omega^2}\frac{d\Gamma(\omega)}{d\omega}, \quad I(\omega_m, \omega_M)=\omega_m^{\alpha}\int_{\omega_m}^{\omega_M}\frac{d\omega}{\omega^{\alpha+3}}\frac{dF(\omega)}{d\omega}.$$

The constants c_1, c_2, c_1^*, c_2^* occurring linearly in Eqs. (71) can easily be eliminated from these equations. The remaining set of two equations with two unknowns ω_m and ω_M can be solved numerically. In the particular case $\nu_e/2\gamma_0 \ll 1$ we have

$$\omega_m=1.36, \quad \omega_M=4.96.$$

Figure 1 shows the graph of the function $\omega^{3/2}N(\omega)$ (which is proportional to the wave energy density as function of frequency) corresponding to this case. The constants a and b turn out to be

$$a=2.25, \quad b=1.61.$$

Knowing a and b we easily find the energy of the Langmuir waves and the beam relaxation length:

$$W=0.50(\gamma_0/\zeta g\Delta\theta^3)^{1/2}, \quad l=0.34(\zeta g/\gamma_0)^{1/2}l_0.$$

In dimensional variables these formulae have the form

$$W=3.76 n_0 T\left(\frac{n_b}{\zeta\eta n_0}\right)^{1/2}\left(\frac{m_i c^2}{T}\right)^{1/4}\frac{T_e}{m_e c^2}\frac{1}{\Delta\theta^{3/2}},$$

$$l=0.27\frac{c}{\omega_p}\frac{(\eta m_e c^2)^2}{T_e T}\left(\frac{\zeta\eta n_0}{n_b}\right)^{1/2}\left(\frac{T}{m_i c^2}\right)^{1/4}.$$

When the parameter ν_e/γ_0 increases the result does not change qualitatively up to values $\nu_e/\gamma_0 \sim 1$.

7. STABILITY

It is convenient to calculate the function $\Gamma(\omega)$ outside the region of the spectrum using the formulae

$$\Gamma(\omega)=\Gamma(\omega_M)+(\omega-\omega_M)\frac{d\Gamma(\omega_M)}{d\omega_M}+\frac{1}{2}(\omega-\omega_M)^2 \times\left[\frac{d^2\Gamma(\omega_M)}{d\omega_M^2}+\omega_M^4 N^2(\omega_M-0)\right], \quad \omega>\omega_M;$$

$$\Gamma(\omega)=\omega^3\left\{\frac{\Gamma(\omega_m)}{\omega_m^3}+(\omega-\omega_m)\frac{d}{d\omega_m}\frac{\Gamma(\omega_m)}{\omega_m^3}+\frac{1}{2}(\omega-\omega_m)^2\left[\frac{d^2}{d\omega_m^2}\frac{\Gamma(\omega_m)}{\omega_m^3}+\omega_m N^2(\omega_m+0)\right]\right\}, \quad \omega<\omega_m.$$

Using these formulae one easily verifies the extrinsic stability of the solution found [inequality (62)].

The problem of the stability can also be solved in a positive sense: it turns out that an extrinsically stable spectrum must be stable. For simplicity we consider this problem in the framework of the equation

$$\partial N(k^2, t)/\partial t=\gamma_k^{\text{eff}} N(k^2, t)+\varepsilon_k. \quad (72)$$

The term ε_k describing a noise source is taken into account in (72) to exclude the physically unrealistic possibility that the function $N(k^2)$ vanishes. When we neglect induced scattering by ions, the effective growth rate γ_k^{eff} can be written in the form

$$\gamma_k^{\text{eff}}=\begin{cases} 2\gamma_0(k^2-1)^{1/2}\left(\dfrac{1}{\hat{\Gamma}N^2(k^2)}-\dfrac{1}{\Gamma(k^2)}\right) & k>1 \\ -\nu_e, \quad k<1 \end{cases}.$$

We have introduced here the notation $\Gamma(k^2)=\hat{\Gamma}N^2(k^2)$ underlining the linear dependence of the functional $\Gamma(k^2)$ on $N^2(k^2)$. The operator $\hat{\Gamma}$ is symmetric and positive-definite. One can easily check this using the representation

$$\Gamma(\omega)=\int_0^\infty d\omega_2 G(\omega, \omega_2)N^2(\omega_2),$$

$$G(\omega, \omega_2)=\int_0^\infty d\omega_1(\omega-2\omega_1)^2(\omega_2-2\omega_1)^2\Theta(\omega-\omega_1)\Theta(\omega_2-\omega_1).$$

Let $N_0(k^2)$ be a stationary solution of (72), and $\Gamma_0(k^2)$ the value of $\Gamma(k^2)$ corresponding to that solution. The function $N_0(k^2)$ is close to the spectrum found above by virtue of its extrinsic stability and the fact that ε_k is small. Expressing $\hat{\Gamma}(k^2)$ in terms of $N_0(k^2)$ we can rewrite Eq. (72) in the form

$$\frac{\Gamma\Gamma_0}{2(k^2-1)^{1/2}}N_0^2\frac{\partial}{\partial t}\left(\frac{N^2}{N_0^2}-1-\ln\frac{N^2}{N_0^2}\right)=-2\gamma_0(N^2-N_0^2) \times\hat{\Gamma}(N^2-N_0^2)-\varepsilon_k\frac{N+N_0}{NN}(N-N_0)^2, \quad k>1, \quad (73)$$

$$\frac{\partial}{\partial t}(N-N_0)^2=-2\nu_e(N-N_0)^2, \quad k<1.$$

After integration over k^2 the stability of the stationary spectrum against perturbations with

$$|\Gamma(\omega)-\Gamma_0(\omega)|\ll\Gamma_0(\omega)$$

follows at once from Eq. (73). The time to establish the spectrum is of the order ν_e^{-1}.

8. REGION OF APPLICABILITY

As the maximum of the energy release usually occurs in the concluding stage of the relaxation, the losses of the beam are determined by the nonlinear mechanism which predominates when $\Delta\theta \sim 1$. The four-plasmon interaction is such a mechanism when

$$\eta g^2\frac{\nu_e}{\omega_p}<\frac{n_b}{n_0}<\eta\frac{\nu_e}{\omega_p}$$

[the lower limit is connected with the condition that the induced scattering of Langmuir waves by ions be weak, and the upper one with the requirement (9) that the induced density fluctuations be static]. When one studies the four-plasmon interaction the range of values of n_b

described by the relaxation theory is broadened by a factor g^{-2} (by a factor 100 for an isothermal deuterium plasma with a temperature of 15 keV).

We have assumed above that there is no magnetic field, but the results obtained are valid also in not too strong magnetic fields. As the whole spectrum is concentrated in the resonance region of frequencies, the field does not affect the relaxation as long as the "magnetic" correction to the frequency of the resonant plasmons is small compared with the "thermal" one, i.e., when $\beta \gg 1$, where $\beta = 8\pi n_0 T_e/H^2$ is the ratio of the gas kinetic pressure of the electrons to the pressure of the magnetic field. This case is of interest in connection with planned experiments on beam plasma heating in systems with "wall" confinement (see, e.g., Refs. 26 and 3). The case $\beta \ll 1$ which occurs in most contemporary experiments is not described by the estimates we obtained. It is, however, clear that the study of the scattering by induced density fluctuations enables us to broaden considerably the region of applicability of the relaxation theory also in that case. Such a study can easily be performed at $\nu_e < \gamma_0 \beta^{3/2}$, when the Langmuir spectrum is nearly ergodic. At stronger damping of the Langmuir waves the constant-frequency surfaces are not fully occupied; this complicates the analytical solution of the problem, but we can easily write down estimates also in this case.

We also neglected above electromagnetic waves—we were dealing with a transparent plasma. When these waves are suppressed the spectral density of their energy turns out, owing to scattering by induced plasma density fluctuations, to be a function of the frequency only, and furthermore the same function as for Langmuir waves. The ratio of the total energies is then equal to that of the phase volumes. Hence, the energy of the Langmuir waves is larger than that of the electromagnetic waves by a factor $(m_e c^2/T_e)^{3/2}$ and the latter have practically no effect on the relaxation.

The author is grateful to L. S. Pekker and V. P. Nagornyĭ for their help with the computer calculations.

1) The increase in the beam energy necessary for plasma heating can then be guaranteed by increasing their duration.

2) See Ref. 4 for details about the conditions for the unimportance of the electromagnetic branches.

3) The nature and constancy of the Langmuir wave damping are not very important from the formal point of view.

4) One should note that when there is a magnetic field present or when electromagnetic waves are present the region of applicability of the existing relaxation theory in terms of the parameters γ/ν_e turns out to be much broader.[21-23]

5) The possibility of eliminating the beam instability by elastic scattering of Langmuir waves was first noted in Ref. 24.

6) It will become clear in what follows that no condensate appears under the weaker condition $\gamma_0 \gtrsim \tilde{\gamma}$ which is automatically valid in the range (27) for a beam with not too small an angular spread.

7) There is an explanation of the term "extrinsic" in Ref. 25.

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