Theory of reorientation transitions in plates

M. I. Kaganov and A. V. Chubukov

Moscow State University
(Submitted 17 November 1981)
Zh. Eksp. Teor. Fiz. **82**, 1617–1627 (May 1982)

A phenomenological theory of reorientation transitions in a plate of thickness d is constructed. The effect of the surface is described by introducing a surface energy containing the surface-anisotropy constant k_s^*. The character of the phase diagrams is determined by the quantity $\sigma \sim k_s^* (\mu M_0/\Theta)(k_s^*/k_2)$, where k_2 is the second anisotropy constant, μ is the Bohr magneton, M_0 is the magnetic moment per unit volume, and Θ is the Curie temperature. It is shown that in plates of finite thickness the phase diagrams differ substantially from the bulk diagrams. In particular, increasing the plate thickness can eliminate the phase transitions. The thermodynamic characteristics for second-order phase transitions and the fluctuation level are calculated. A criterion for the applicability of the Landau theory is obtained.

PACS numbers: 75.30.Kz, 75.40.Dy

INTRODUCTION

Magnetic orientational transitions, frequently called spin-reorientation transitions, are typical examples of transitions of the order-order type. These transitions manifest themselves most strongly in rare-earth magnets.[1] Owing to the substantial difference between the anisotropy energy and the exchange energy, spin-orientation transitions are well described phenomenologically. The purpose of the present study is to investigate the role of surface energy in the construction of the phase diagram of the magnet.

We start from a description that is valid far from the Curie point and presupposes that the problem can be reduced to a study of the anisotropy energy:

$$M_0^{-2}F_v(\theta)=f_v(\theta)=-k_1\sin^2\theta-k_2\sin^4\theta, \tag{1}$$

where M_0 is the magnetic moment per unit volume, θ is the angle between the direction of the magnetic moment and the chosen axis of the crystal, k_1 and k_2 are the first and second anisotropy constants, with $k_1=\alpha\tau$, $\tau=(T-T_c)/T_c \ll 1$ (we set $\alpha>0$ for the sake of argument); k_2 can be regarded as constant in the considered temperature region. The inhomogeneous exchange interaction is taken into account in natural fashion in the thermodynamic potential by the term $c(d\theta/dx)^2$, where $c\sim a^2\Theta/\mu M_0$ (a is the distance between the atoms, μ is the Bohr magneton, and Θ is the Curie temperature).

The problem was solved earlier[2] for a half-space. We consider here a plate of finite thickness ($|x|\leq d$). In analogy with Ref. 2 we introduce the surface energy[1]

$$M_0^{-2}F_{\text{sur}}=k_s\sin^2(\theta_s-\varphi), \tag{2}$$

where k_s is the surface anisotropy constant (measured in centimeters), θ_s is the value of the angle θ on the sample surface, and the angle φ specifies the direction of the easy-magnetization axis on the surface. We assume that the surface-anisotropy constants are the same on both sides of the plate, in which case a symmetrical distribution of the magnetization ($\theta(x)=\theta(-x)$) is the most convenient. This allows us to restrict ourselves to a "half-plate," putting

$$d\theta/dx|_{x=0}=0. \tag{3}$$

The total free energy of the half-plate is thus

$$F=M_0^2\int_0^d\left[c\left(\frac{d\theta}{dx}\right)^2-k_1\sin^2\theta-k_2\sin^4\theta\right]dx+M_0^2k_s\sin^2(\theta_s-\varphi), \tag{4}$$

and the Lagrange-Euler equation for the functional (4) takes the form

$$cd^2\theta/dx^2+(k_1+2k_2\sin^2\theta)\sin\theta\cos\theta=0. \tag{5}$$

The boundary condition is obtained from the requirement that the variation with respect to θ_s

$$\left.\frac{d\theta}{dx}\right|_{x=d}=\frac{k_s}{2c}\sin 2(\varphi-\theta_s) \tag{6}$$

vanish.

Using (3), we can reduce the first integral of (5) to the form

$$c^{1/2}d\theta/dx=\operatorname{sign}(\varphi-\theta_s)[f_v(\theta)-f_v(\theta_0)]^{1/2}. \tag{7}$$

Substitution of (7) in (4) transforms the functional F into a function of θ_0 and θ_s. Introducing the notation

$$\sin^2\theta_0=u,\quad \sin^2\theta_s=v,\quad k_s^2/c=\tilde{k}_s,$$

we obtain

$$\frac{F}{M_0^2c^{1/2}}=\int_u^v\left\{\frac{(u-x)[k_1+k_2(u+x)]}{x(1-x)}\right\}^{1/2}dx-dc^{-1/2}u(k_1+k_2u)+\tilde{k}_s^{1/2}$$
$$\times\{\sin^2\varphi+v\cos 2\varphi-[v(1-v)]^{1/2}\sin 2\varphi\} \tag{8}$$

[it is seen directly from (7) that $k_1+k_2(u+v)\leq 0$]. Although the extremum conditions are unwieldy,

$$\left\{\frac{(u-v)[k_1+k_2(u+v)]}{v(1-v)}\right\}^{1/2}+\tilde{k}_s^{1/2}\left\{\cos 2\varphi-\frac{1-2v}{2[v(1-v)]^{1/2}}\sin 2\varphi\right\}=0, \tag{9}$$

$$2dc^{-1/2}=\int_u^v\{x(1-x)(x-u)[-k_1-k_2(x+u)]\}^{-1/2}dx, \tag{10}$$

they lend themselves to qualitative and quantitative analysis. For the solution to be stable we must have

$$\frac{\partial^2F}{\partial v^2}\geq 0,\quad \frac{\partial^2F}{\partial v^2}\frac{\partial^2F}{\partial u^2}-\left(\frac{\partial^2F}{\partial u\partial v}\right)^2\geq 0. \tag{11}$$

The problem has essentially been reduced to finding the stable minimum of the function (8) of the two variables u and v.

1. PHASE TRANSITIONS IN A PLATE FOR A FIRST-ORDER PHASE TRANSITION IN A BULK SAMPLE ($k_2>0$)

We consider first the case $\varphi=\pi/2$. From (3), (5) and (6) it follows directly that the extremum conditions

 0038-5646/82/050937-06$04.00

are satisfied for the homogeneous states $\theta\equiv 0$ (we call such a state phase I) and $\theta\equiv\pi/2$ (phase II). If $k_1>0$, the absolute minimum of the potential corresponds to phase II [this is seen from (1)], so that in this case the phase transitions can be observed only at $k_1<0$.

We determine now the stability regions of the homogeneous state. We start with the case $\theta\equiv 0$. For a small deviation $\delta\theta(\mathbf{r})$ the linearized Lagrange-Euler equation

$$c\nabla^2\delta\theta(\mathbf{r})-|k_1|\delta\theta(\mathbf{r})=0 \quad (k_1<0) \tag{12}$$

must be supplemented by the boundary conditions

$$\frac{d}{dx}\delta\theta(\mathbf{r})|_{x=0}=0, \quad \frac{d}{dx}\delta\theta(\mathbf{r})|_{x=d}=\frac{k_s}{c}\delta\theta(\mathbf{r}). \tag{13}$$

Representing $\delta\theta(\mathbf{r})$ in the form $f(x)\exp(i\varkappa\cdot\rho)$, where $\varkappa=(\varkappa_z, \varkappa_y)$ and $\rho=(\rho_z, \rho_y)$ are two-dimensional vectors, we have from (12) and (13)

$$f(x)=\mathrm{ch}(x[|k_1|/c+\varkappa^2]^{1/2}), \tag{14}$$

and from the boundary conditions we obtain an equation for $|\varkappa|$:

$$\mathrm{th}\left(\varkappa^2d^2+\frac{|k_1|d^2}{c}\right)^{1/2}=\frac{dk_s}{c}\left(\varkappa^2d^2+\frac{|k_1|d^2}{c}\right)^{-1/2}. \tag{15}$$

The condition for the stability of phase I consists in the absence of real solutions of Eq. (15), for otherwise the solution $\theta\equiv 0$ is not a state of local minimum of the function (8). This condition, as follows from (15), is satisfied only if

$$\delta>\beta_1\,\mathrm{arth}\,\beta_1=\delta_{00}(k_1), \quad \beta_1=(\tilde{k}_s/k_1)^{1/2}<1. \tag{16}$$

The subscripts "00" indicate that $u=v=0$.

For phase II, a similar analysis leads to a stability region

$$\delta<\beta_2\,\mathrm{arctg}\,\beta_2\equiv\delta_{11}(k_1),$$
$$\beta_2=[\tilde{k}_s/(|k_1|-2k_2)]^{1/2}, \quad |k_1|\geqslant 2k_2. \tag{17}$$

In Figs. 1–5 these are the regions located respectively above the line 1 (region of stability of the state $\theta\equiv 0$) and below the line 2 (stability region of the state $\theta\equiv\pi/2$). The locations of the lines 1 and 2 on the (k_1, δ) plane depend, as will be shown below, on the ratio $k_s^2/ck_2=\sigma$, which is in fact the measure of the surface anisotropy. We emphasize the following: since $k_s=k_s^*a$, and k_s^* is a dimensionless quantity of the same nature as k_1, it

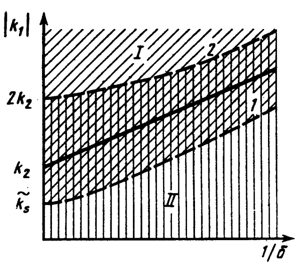

FIG. 1. Phase diagram in the variables k_1 and $1/\delta$ for the case $\sigma<1$. Thick line—first-order transition between phases I and II, 1, 2) lability boundaries of these phases; oblique shading—stability region of phase I; vertical shading—stability region of phase II.

follows that

$$\sigma\sim k_s^*\left(\frac{\mu M_0}{\Theta}\right)\left(\frac{k_s^*}{k_2}\right)$$

can have in essence any positive value, since the surface-anisotropy constant depends on the surface finish and can vary in a wide range.

To determine the character of the transitions we must find the stability region of the inhomogeneous state (phase III), i.e., solve Eqs. (9) and (10), which take at $\varphi=\pi/2$ the somewhat simpler forms

$$(v-u)[|k_1|-k_2(v+u)]/v(1-v)=\tilde{k}_s, \tag{18}$$

$$2dc^{-1/2}=\int_u^v\{x(1-x)(x-u)[|k_1|-k_2(x+u)]\}^{-1/2}dx. \tag{19}$$

With allowance for (18) and (19), the conditions (11) lead to the following inequalities:

$$u(|k_1|/k_2-u)+v(\sigma-1)\geqslant 0, \tag{20}$$

$$u^{-1}v^{1/2}k_2[u^2+v(\sigma-1)]\{(1-v)(v-u)[|k_1|-k_2(u+v)]\}^{-1/2}$$
$$-\frac{1}{2}[u(|k_1|-k_2u)+vk_2(\sigma-1)](I_1+I_2)\geqslant 0, \tag{21}$$

where

$$I_1=\int_1^{v/u}y^{1/2}\{(y-1)(1-yu)^3[|k_1|-k_2u(1-y)]\}^{-1/2}dy,$$

$$I_2=k_2\int_1^{v/u}(1+y)\{y(y-1)(1-yu)[|k_1|-k_2u(1+y)^3]\}^{-1/2}dy. \tag{22}$$

The integrals in (22) cannot be expressed in terms of elementary functions, so that a direct determination of the stability region of the inhomogeneous solutions is difficult.

If it is assumed, however, that at least one of the lability boundaries of the inhomogeneous state coincides with the lability boundary of the corresponding homogeneous phase,[2)] the character of the transition upon intersections of curves 1 and 2 can be determined by investigating the stability of the inhomogeneous states $u, v\ll 1$ and $1-u, 1-v\ll 1$, of which at least one must, as shown above, of necessity determine the phase III near its lability boundary. If any one of these states is stable, then passage through curve 1 or 2 is accompanied by a second-order phase transition. Thus, substituting in (20) and (21) the solution of the system of equations (18) and (19) with $u, v, \ll 1$ we obtain the condition for the stability of the inhomogeneous state

$$\sigma\geqslant 3\beta_1^2\left\{8\left[3(1+\beta_1^2)\left(1-\frac{\mathrm{arth}\,\beta_1}{\beta_1}\frac{(1-\beta_1^2)^2}{1+\beta_1^2}\right)\right]^{-1}-1\right\}. \tag{23}$$

At $\sigma<1$ the inequality (23) has no solutions, i.e., there are no stable states $u, v\ll 1$. In this case line 1 is therefore the boundary of the lability region of phase I relative to a transition into phase II (see Fig. 1).

If $1<\sigma<3$, then line 1 on the section $|k_1|<|k_{10}|$, where

$$|k_{10}|\approx\begin{cases}\tilde{k}_s, & \sigma\gtrsim 1\\ 12\tilde{k}_s/5(3-\sigma), & \sigma\lesssim 3\end{cases}, \tag{24}$$

is a second-order phase-transition line. The corresponding phase diagrams are shown in Figs. 2 and 3. It is seen that a second-order transition is possible for plates of thickness $d>c\delta_{00}(k_{10})/k_s$ [see (16)].

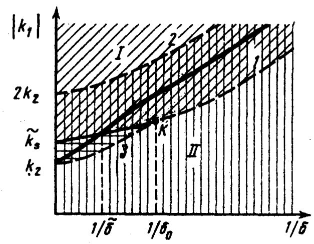

FIG. 2. Phase diagram in the variables k_1 and $1/\delta$ for the case $1<\sigma<2$. Thin line—second-order phase transition; 3) lability boundary of phase III; horizontal shading—stability region of phase III; the thick line in the region $\delta>\tilde{\delta}$ corresponds to equality of the potentials of phases II and III; $\delta=\delta_0\equiv\delta_{00}(k_{10})$ determines the tricritical point of the phase transition (the point K). The remainder is the same as in Fig. 1.

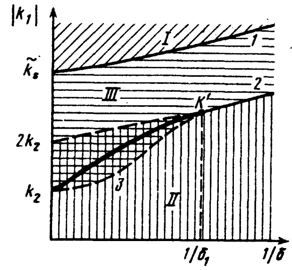

FIG. 4. Case $\sigma>3$. Notation same as in Figs. 1 and 2; $\delta_1=\delta_{11}(k_{11})$ (point K'); two second-order phase transitions are realized at $\delta<\delta_1$.

If $\sigma\geqslant 3$, inequality (23) is satisfied for arbitrary β_1 (but $\beta_1\leqslant 1$ as before), i.e., a second-order phase transition must take place between phases I and III (Fig. 4).

For the stability of the state $1-u$, $1-v\ll 1$ we have similarly

$$\sigma>3\beta_2^2\left\{8\left[3(1-\beta_2^2)\left(\frac{\operatorname{arctg}\beta_2}{\beta_2}\frac{(1+\beta_2^2)^2}{1-\beta_2^2}-1\right)\right]^{-1}+1\right\}. \tag{25}$$

A solution of inequality (15) exists only at $\sigma>3$, i.e., line 2 on the segment $|k_1|>|k_{11}|$, where

$$|k_{11}|\approx\begin{cases}12\tilde{k}_s/5(\sigma-3), & \sigma\gtrsim 3\\ 5\tilde{k}_s/\sigma, & \sigma\gg 1\end{cases} \tag{26}$$

is a second-order phase-transition line. This corresponds to the fact that at $\sigma>3$ second-order phase transitions take place for plates of thickness $d<c\delta_{11}(k_{11})/k_s$ [see (17)].

We consider now the phase diagrams plotted in the variables $1/\delta$ and k_1 (Figs. 1–4). If $\sigma<1$, a second-order phase transition takes place between the homogeneous phases I and II; the transition line is given by the equation

$$|k_1|=k_2+\tilde{k}_s/\delta,$$

and the distance between lines 1 and 2 determines the width of the hysteresis loop (Fig. 1). At $\sigma>1$, besides the instability regions of phases I and II, there exists a region where the inhomogeneous phase III is stable. The lability boundary of this state is the line $\delta=\delta_2(k_1)$ [it corresponds in Figs. 2–4 to line 3, which is defined by Eqs. (18) and (19) and by the inequality (21), if the latter is replaced by an equality]. Analytic expressions for the function $\delta_2(k_1)$ can be obtained only in limiting cases:

$$\delta_2(k_1)=\begin{cases}\sigma^{1/2}\ln|(|k_1|-k_2)(\sigma-1)/\tilde{k}_s|, & |k_1|\approx k_2, \quad \sigma>1,\\ \delta_{00}(k_1)-A(k_1-k_{10})^2, & k_1\approx k_{10}, \quad 1<\sigma<3,\\ \delta_{11}(k_1)-B(k_1-k_{11})^2, & k_1\approx k_{11}, \quad \sigma>3.\end{cases} \tag{27}$$

The coefficients A and B are of the order of unity (we do not present the expressions for them).

We note that at $3>\sigma>2$ (Fig. 3) the phase II becomes unstable in the region $\delta>\delta^*$ with respect to a transition into phase III, and in the region $\delta<\delta^*$ with respect to a transition into phase I; in this case $\delta^*=\delta^*(\sigma)$ is defined by the condition

$$\delta^*=\begin{cases}\pi(\sigma-2)^{-1/2}/\sqrt{2}, & \sigma\gtrsim 2,\\ (3-\sigma)/3, & \sigma\lesssim 3.\end{cases} \tag{28}$$

Particular interest attaches to the case $\sigma>3$ (Fig. 4). At $\delta<\delta_1$ the boundary of the lability of phase III coincides here with the line $\delta=\delta_{11}(k_{11})$, i.e., at small plate thicknesses the reorientation proceeds as two second-order phase transitions via an intermediate inhomogeneous phase.

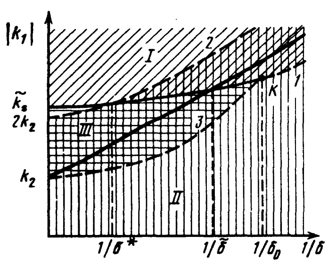

FIG. 3. Case $2<\sigma<3$. The notation is the same as in Figs. 1 and 2. $\delta=\delta^*$ is defined by Eq. (28).

We consider now the case when the easy magnetization axes make some angle φ with each other, and $\pi/4<\varphi<\pi/2$ (if $\varphi<\pi/4$ the substitutions

$$\varphi^*=\pi/2-\varphi, \quad \tilde{\theta}=\pi/2-\theta, \quad k_1^*=-k_1-2k_2$$

bring us back to the considered case). As seen from (4) the surface energy contains a term linear in the order parameter, i.e., its action is somehow similar to the action of an external magnetic field. Only first-order phase transitions are therefore possible in the system. An investigation of Eqs. (9) and (10) [with allowance for (11)] shows that when the plate thickness is decreased the hysteresis loop becomes narrower, so that at a certain $\delta=\delta_{cr}(\varphi,\sigma)$ the loop "collapses" and the phase transition vanishes: u and v become single-valued functions of temperature having no singularities, and go simultaneously through the value $u=v=\sin^2\varphi$ at the point $|k_1|=2k_2\sin^2\varphi$. For $\delta_{cr}(\varphi,\sigma)$ we then obtain the following results: at $\sigma<3$

$$0=\delta_{cr}(\pi/2,\sigma)\leqslant\delta_{cr}(\varphi,\sigma)\leqslant\delta_{cr}(\pi/4,\sigma)=\sigma^{1/2}\operatorname{arctg}\sigma^{1/2},$$

and in the limiting cases

$$\delta_{cr}=\begin{cases}\dfrac{24(\pi/2-\varphi)}{3-\sigma}\sigma, & \dfrac{\pi}{2}-\varphi\ll 3-\sigma,\\ \sigma-\dfrac{64}{9}\left(\varphi-\dfrac{\pi}{4}\right)^2, & \varphi-\dfrac{\pi}{4}\ll\sigma^{1/2}\ll 1.\end{cases} \tag{29}$$

At $\sigma > 3$

$$\delta_1 \leqslant \delta_{cr}(\varphi, \sigma) \leqslant \delta_{cr}(\pi/4, \sigma) = \sigma^{1/2} \operatorname{arctg} \sigma^{1/2},$$

where $\delta = \delta_1(\sigma)$ corresponds to the tricritical phase-transition point in the case $\varphi = \pi/2$ (see Fig. 4).

The expressions for $\delta_{cr}(\varphi, \sigma)$ in the limiting cases $\varphi \approx \pi/2$ and $\varphi \approx \pi/4$ are unwieldy and will not be given here.

2. PHASE TRANSITIONS IN A PLATE FOR A SECOND-ORDER PHASE TRANSITION IN A BULKY SAMPLE ($k_2 < 0$)

In this case the anisotropy energy is

$$f_0(\theta) = -k_1 \sin^2 \theta + |k_2| \sin^4 \theta. \tag{30}$$

We shall assume that $\pi/4 < \varphi < \pi/2$ (if $\varphi < \pi/4$, we can arrive at the considered case in the same manner as in Sec. 1).

We turn first to the case $\varphi = \pi/2$ and consider the stability regions of phases I and II. For phase I we obtain, naturally, the same result as in the case $k_2 < 0$ [see (16)], and for phase II the stability region is

$$\delta < \beta_2' \operatorname{arctg} \beta_2' \equiv \delta_{11}'(k_1),$$
$$\beta_2' = [\tilde{k}_s/(2|k_2| - k_1)]^{1/2}, \quad k_1 < 2|k_2|. \tag{31}$$

The investigation is carried out next in full analogy with Sec. 1 and shows that in this case the situation does not differ from that in the bulk, i.e., at any plate thickness the reorientation proceeds as two second-order phase transitions via an intermediate inhomogeneous phase III (Fig. 5).

If $\varphi \neq \pi/2$, an investigation of expressions (9)–(11) causes u and v at any finite plate thickness to be single valued functions of k_1, having no singularities and passing simultaneously through the value $u = v = \sin^2\varphi$ at the point $k_1 = 2|k_2|\sin^2\varphi$, i.e., no phase transition whatever takes place in the plate.

Actually, of course, in thick plates there should be anomalies on the plots of u and v vs k_1, inasmuch as abrupt changes of the values of u and v take place in the narrow temperature interval $2|k_2| > k_1 > 0$.

3. THERMODYNAMIC CHARACTERISTICS

We write down the analytic expressions for the thermodynamic characteristics in the regions adjacent to the second-order phase transition lines. The conditions (23) and (25), which are needed for the existence of such transitions, will be assumed satisfied. In the region directly adjacent to the line $\delta = \delta_{00}(k_1)$ (which determines the transition temperature $T = T_{cd}$ as a function of the plate thickness), we have

$$\theta_0 = \frac{(2\alpha c)^{1/2}}{k_s T_c^{1/2}} N(\Delta)(T - T_{cd})^{1/2}|_{\delta=\text{const}} = 2P(\Delta)(\delta_{00} - \delta)^{1/2}|_{k_1=\text{const}},$$
$$\langle M_z \rangle = M_0 \theta_0 \psi(\Delta), \quad \theta_s = \theta_0 \operatorname{ch} \Delta,$$
$$\Delta C_p = M_0^2 \frac{T_{cd}\alpha^2 c^{3/2}}{k_s^2 T_c^2} M(\Delta), \tag{32}$$
$$\chi^+(T - T_{cd}) = 2\chi^-(T_{cd} - T) = M_0^2 \frac{2T_c c^{1/2}}{\alpha} \frac{\psi^2(\Delta)}{1 + \psi(2\Delta)},$$

where α is defined in (1); χ^+ and χ^- are the magnetic susceptibilities to the right and the left of the transition point; ΔC_p is the heat-capacity jump at the transition point (all the quantities are given per unit area); the angle θ is reckoned from the z axis, and the averaging is over the plate thickness. The quantities M, N, and P are functions of Δ and take the form

$$N(\Delta) = L(\Delta)[1 + \psi(2\Delta)]^{1/2} \operatorname{th} \Delta, \quad M(\Delta) = N^2(\Delta)[1 + \psi(2\Delta)],$$
$$P(\Delta) = L(\Delta)/(\Delta \operatorname{th} \Delta)^{1/2}, \quad \psi(x) = x^{-1} \operatorname{sh} x, \tag{33}$$
$$L(\Delta) = \{\psi(4\Delta) - 1 - \sigma^{-1} \operatorname{th}^2 \Delta [3 + 4\psi(2\Delta) + \psi(4\Delta)] \operatorname{sign} k_2\}^{-1/2}.$$

The parameter Δ depends on the plate thickness: $\Delta \tanh \Delta = \delta$. In the limiting cases $\delta \gg 1$ and $\delta \ll 1$ these expressions become much simpler. Thus, for the quantity $N(\Delta)$ used in Sec. 5 we have

$$N(\Delta) = \begin{cases} \frac{1}{2}\left(\frac{3\sigma}{\sigma \mp 3}\right)^{1/2}\left[1 - \frac{\delta}{2}\left(\frac{4\sigma}{5(\sigma \mp 3)} + \frac{1}{3}\right)\right], & \delta \ll 1, \\ \left(\frac{2\sigma}{\sigma \mp 1}\right) e^{-\delta}, & \delta \gg 1, \end{cases}$$

where the upper and lower signs pertain to the cases $k_2 > 0$ and $k_2 < 0$, respectively.

We turn now to the region adjacent to the line $\delta = \delta_{11}(k_1)$, which determines the transition point $T = T'_{cd}$. In this case the thermodynamic characteristics take the form ($\tilde{\theta} = \pi/2 - \theta$)

$$\tilde{\theta}_0 = \frac{(2\alpha c)^{1/2}}{k_s T_c^{1/2}} N_1(\Delta_1)(T_{cd}' - T)^{1/2}|_{\delta=\text{const}} = 2P_1(\Delta_1)(\delta - \delta_{11})^{1/2}|_{k_1=\text{const}},$$
$$\langle M_y \rangle = M_0 \tilde{\theta}_0 \psi_1(\Delta_1), \quad \tilde{\theta}_s = \tilde{\theta}_0 \cos \Delta_1,$$
$$\Delta C_p = M_0^2 T_{cd}' \alpha^2 c^{3/2} M_1(\Delta_1)/T_c^2 k_s^2, \tag{34}$$
$$\chi^+(T_{cd}' - T) = 2\chi^-(T - T_{cd}') = 2M_0^2 \frac{T_c}{\alpha} c^{1/2} \frac{\psi_1^2(\Delta_1)}{1 + \psi_1(2\Delta_1)}.$$

Then

$$N_1(\Delta_1) = \operatorname{tg} \Delta_1 [1 + \psi_1(2\Delta_1)]^{1/2} L_1(\Delta_1), \quad M_1(\Delta_1) = N_1^2(\Delta_1)[1 + \psi_1(2\Delta_1)],$$
$$P_1(\Delta_1) = L_1(\Delta_1)/(\Delta_1 \operatorname{tg} \Delta_1)^{1/2}, \quad \Delta_1 \operatorname{tg} \Delta_1 = \delta, \quad \psi_1(\Delta_1) = x^{-1} \sin x, \tag{35}$$
$$L_1(\Delta_1) = \{[\psi_1(4\Delta_1) - 1] \operatorname{sign} k_1 - \sigma^{-1} \operatorname{tg}^2 \Delta_1 [3 + 4\psi_1(2\Delta_1) + \psi_1(4\Delta_1)] \operatorname{sign} k_2\}^{-1/2},$$

We point out in conclusion the characteristic divergence of the heat capacity C_p near the critical points of the second-order phase transition ($k_1 \approx k_{10}$ and $k_1 \approx k_{11}$). Thus, as the point K is approached (Figs. 2 and 3) along the second-order phase transition line we have ($k_1 \approx k_{10}$)

$$M_0^{-2} c^{-1/2} C_p = \begin{cases} \left(\frac{2}{3} \frac{1}{k_1 - k_{10}}\right)^{1/2}, & |k_{10}| \approx \frac{12}{5} \frac{\tilde{k}_s}{3 - \sigma}, \quad \sigma \lesssim 3, \\ e^{\delta}\left[\frac{\tilde{k}_s^3}{29(k_1 - k_{10})}\right]^{1/2}, & |k_{10}| \approx \tilde{k}_s, \quad \sigma \gtrsim 1 \end{cases} \tag{36}$$

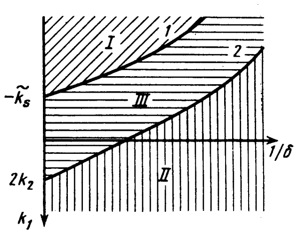

FIG. 5. Phase diagram in the variables k_1 and $1/\delta$ for the case $k_2 > 0$. The transition from the state $\theta \equiv 0$ to the state $\theta \equiv \pi/2$ proceeds at any plate thickness in the form of two second-order phase transitions. The notation is the same as in Figs. 1 and 2.

in full accord with the general theory (see Ref. 3, § 150). Similar results were obtained when the point K' is approached (Fig. 4).

4. PHASE TRANSITIONS INDUCED BY AN EXTERNAL MAGNETIC FIELD

We shall show that effects perfectly similar to those considered above appear when a study is made of reorientation transitions induced by an external magnetic field applied to the plate-surface plane. We consider by way of example the case when the external field is directed along the easy magnetization axis. We assume that $k_1<0$ and $\varphi=\pi/2$; then

$$M_0^{-2}F=\int_0^d\left[c\left(\frac{d\theta}{dx}\right)^2+|k_1|\sin^2\theta-2\tilde{H}\sin\theta\right]dx+k_s\cos^2\theta_s, \tag{37}$$

where $\tilde{H}=H/2M_0$ (H is the external magnetic field). We note that two second-order phase transitions take place in the volume at the points $\tilde{H}=0$ and $\tilde{H}=|k_1|$.

The functional (37) is investigated in analogy with Sec. 1. The phase diagrams plotted in the variables $h=1-\tilde{H}/|k_1|$ and $1/\delta$ for different values of the parameter $\beta_1=(k_s/|k_1|)^{1/2}$ are shown in Fig. 6. The state $\theta\equiv 0$ is stable everywhere at $h>1$ (above the line 1), and the state $\theta\equiv\pi/2$ is stable everywhere at

$$\delta<(\beta_1/h^{1/2})\,\mathrm{arctg}(\beta_1/h^{1/2})\equiv\delta_{11}^h\quad(h>0) \tag{38}$$

(i.e., under the line 2).

Without repeating all the arguments of Secs. 1 and 2, we present final expressions for the characteristic points on the diagrams of Fig. 6:

$$\begin{aligned}&\delta_0^h=\beta_1\,\mathrm{arth}\,\beta_1,\quad \beta_1<1,\\ &\delta_1^h\approx\begin{cases}\beta_1^2(1-2\beta_1^4/3), & \beta_1\ll 1,\\ 0.8(1+0.8/\beta_1^2), & \beta_1\gg 1,\end{cases}\\ &\delta_2^h\approx\begin{cases}3(\beta_1^2-6)/28, & \beta_1\gtrsim\sqrt{6},\\ \beta_1^2, & \beta_1\gg 1.\end{cases}\end{aligned} \tag{39}$$

The point $\delta=\delta_2^h$ exists only at $\beta_1>\sqrt{6}$. We note that at thicknesses $\delta<\delta_2^h$ (or $d<c\delta_2^h/k_s$) the reorientation proceeds as one first-order phase transition.

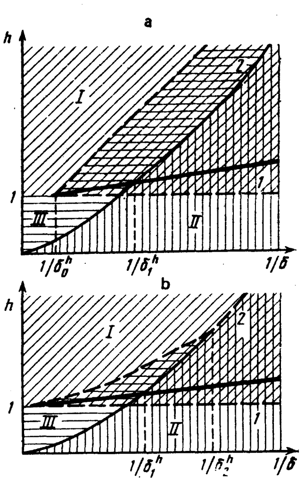

FIG. 6. Phase diagrams in the variables h and $1/\delta$ for the limiting cases $\beta_1<1$ (a) and $\beta_1<\sqrt{6}$ (b). The notation is the same as in Figs. 1 and 2.

5. ROLE OF FLUCTUATIONS

We investigate the regions near the second-order phase transitions. Let, say, $k_2<0$. To ascertain the role of the fluctuations, we consider them in the phase $\theta_0\equiv 0$. Then $\delta\theta_{\mathrm{fl}}(\mathbf{r})=\theta(\mathbf{r})$, and from (4) we have in the Gaussian approximation

$$\begin{aligned}&M_0^{-2}F_{\mathrm{fl}}=\int f_{\mathrm{fl}}\,dS,\\ &f_{\mathrm{fl}}=2\int_0^d\left[c\left(\frac{d}{dx}\delta\theta_{\mathrm{fl}}\right)^2+|k_1|\delta\theta_{\mathrm{fl}}^2\right]dx-2k_s(\delta\theta_{s\,\mathrm{fl}})^2.\end{aligned} \tag{40}$$

A potential of this type was investigated earlier[4] in a description of a transition in a plate from the paramagnetic to the ferromagnetic state. Just as in Ref. 4, we expand the fluctuation solution in a Fourier series with respect to the complete system of functions $\chi_n^{s,a}$ [which are defined in Eqs. (55) and (58) of Ref. 4]:

$$\delta\theta_{\mathrm{fl}}(\mathbf{r})=\sum_n \eta_n^s\chi_n^s+\eta_n^a\chi_n^a. \tag{41}$$

For the fluctuation part of the free energy we obtain, taking into account only the "dangerous" symmetrical fluctuation with $n=1$,

$$F_{\mathrm{fl}}=SdcM_0^2|\eta_1^s|^2(|k_1|/c-(\mu_1^s)^2+\varkappa^2)Q, \tag{42}$$

where η_1^s is the coefficient of $\chi_1^s=\cosh\mu_1^s x[\exp(i\boldsymbol{\varkappa}\cdot\boldsymbol{\rho})]$ ($\boldsymbol{\varkappa}$ and $\boldsymbol{\rho}$ are two-dimensional vectors in the yz plane, μ_1^s is determined from the dispersion equation $\mu_1^s d\tanh\mu_1^s d=\delta$ (see Ref. 4), and

$$Q=1+\delta[(\mu_1^s d)^2-\delta^2]^{-1}.$$

Then

$$\langle|\eta_1^s|^2\rangle=T\{2QM_0^2Sdc[|k_1|/c-(\mu_1^s)^2+\varkappa^2]\}^{-1}, \tag{43}$$

and the correlation radius of the dangerous fluctuation in the yz plane is

$$r_c=[|k_1|/c-(\mu_1^s)^2]^{-1/2} \tag{44}$$

(if $d=\infty$ we have $r_c=[c/|k_1|-\tilde{k}_s]^{1/2}$, cf. Ref. 2). Taking (44) into account, we obtain from (43)

$$\langle\langle|\eta_1^s|^2\rangle\rangle=T/2QM_0^2dc$$

($\langle\langle\ldots\rangle\rangle$ denotes averaging over the correlation area in the yz plane, besides the statistical averaging).

Below the phase transition point we have

$$\theta(\mathbf{r})=\theta_{\mathrm{eq}}(x)+\delta\theta(\mathbf{r}),$$

where $\theta_{\mathrm{eq}}(x)$ is determined by the solution of Eq. (5) [with account taken of (3) and (6)]: Near the transition point we have

$$\theta_{\mathrm{eq}}(x)=\theta_0\,\mathrm{ch}\,\mu_1^s x,$$

and in this case θ_0 is determined from (32). We expand $\theta_{\mathrm{eq}}(x)$ and $\delta\theta_{\mathrm{fl}}(\mathbf{r})$ in a Fourier series in terms of the same functions $\chi_n^{s,a}$:

$$\theta_{\mathrm{eq}}(x)=\sum_n\alpha_n^s\chi_n^s+\alpha_n^a\chi_n^a,\qquad \delta\theta_{\mathrm{fl}}(\mathbf{r})=\sum_n\gamma_n^s\chi_n^s+\gamma_n^a\chi_n^a.$$

The condition for the validity of the Landau theory is then

$$|\alpha_n^{s,a}|\gg|\gamma_n^{s,a}|.$$

Substituting $\theta(\mathbf{r})=\theta_{\mathrm{eq}}(x)+\delta\theta_{\mathrm{fl}}(\mathbf{r})$ in (4), we can readily show that although the fluctuations in the inhomogeneous phase differ from those in the homogeneous one, the dangerous fluctuation is as before the symmetrical one with $n=1$, and the mean square of this fluctuation satisfies the classical "rule of two," i.e.,

$$\langle|\gamma_1^s|^2\rangle=T\{2QSdcM_0^2[2(\mu_1^s)^2-2|k_1|/c+\varkappa^2]\}^{-1},$$

so that after averaging over the correlation area we obtain

$$\langle\langle|\gamma_1^s|^2\rangle\rangle=\langle\langle|\eta_1^s|^2\rangle\rangle.$$

Taking into account the anomalous increase of only the dangerous fluctuation and the fact that $(\mu_1^s)^2=|k_1|/c$ at the transition point, we find that this condition for the applicability of the Landau theory is equivalent to the inequality

$$\theta_0^2\gg\langle\langle|\eta_1^s|^2\rangle\rangle=T/2QdcM_0^2.$$

At $\delta\gg 1$, taking θ_0 from (32) and (33), we find that the fluctuations are dangerous at

$$|\Delta k_1|\leqslant\frac{1}{2}\frac{T_{cd}\mu M_0}{\Theta^2}\bar{k}_s(\sigma+1)k_2$$

[cf. Ref. 2, Eq. (36)]. If $\delta\ll 1$, the fluctuations are dangerous at

$$|\Delta k_1|\leqslant\frac{\sigma+3}{6\sigma}\frac{T_{cd}\mu M_0}{\Theta^2}\bar{k}_s\frac{1}{\delta}. \quad (45)$$

It is seen that the fluctuation region expands with decreasing δ, a natural result, since with decreasing thickness d the behavior of the plate becomes more and more similar to that of a two-dimensional magnet. A similar result is arrived at by examination of the fluctuations in a second-order phase transition from the state $\theta\equiv\pi/2$.

It follows from (45) that the critical plates for the model are those with thickness $d\approx d_{t1}=aT_c/k_s^*$. It is clear therefore that the condition $d\gg d_{t1}$ is equivalent in essence to the condition that the description be macroscopically noncontradictory. It is important, however, that even for plates with $d\gg d_{t1}$ the fluctuation region is much wider than for transitions in the bulk, when the Levanyuk-Ginzburg criterion takes the form

$$\Delta k_1\gg\left(\frac{T_c}{\Theta}\right)^2\left(\frac{\mu M_0}{\Theta}\right)k_2^2.$$

We note in conclusion the following: It is known that allowance for the magnetoelastic interaction can cause the transition to involve the elastic subsystem of the crystal. This, however, leads to renormalization of the constants in the expression for the free energy (see Ref. 5), but does not change the character of the transition. Although this question was not investigated for the case of samples with finite dimensions, it is clear nevertheless that no substantial changes can occur (especially if the magnetoelastic coupling is small). In addition, an estimate of the fluctuations in the case of an inhomogeneous order parameter is of independent interest, since it can be used to consider entirely different objects (e.g., to study superfluid transitions of helium in capillaries[6]).

[1)]It is assumed that the magnetic moment is parallel to the sample plane both on the plate surface and in its interior (cf. Ref. 2).

[2)]We disregard the improbable existence of a stability region of a phase III such that u and v have values different from 0 and 1 in all of its points. The impossibility of the existence of an inhomogeneous phase of this kind can be rigorously proved for thin plates.

[1]K. P. Belov, A. M. Kadomtseva, A. K. Zvezdin, and R. Z. Levitin, Orientatsionnye perekhody v redkozemel'nykh magnetikakh (Orientational Transitions in Rare-Earth Magnets), Nauka, 1979.

[2]M. I. Kaganov, Zh. Eksp. Teor. Fiz. **79**, 1554 (1980) [Sov. Phys. JETP **52**, 779 (1980)].

[3]L. D. Landau and E. M. Lifshitz, Statistical Physics, Part 1, Pergamon, 1980 [§150 of Russian original].

[4]M. I. Kaganov and I. S. Karpinskaya, Zh. Eksp. Teor. Fiz. **76**, 2143 (1979) [Sov. Phys. JETP **49**, 1083 (1979)].

[5]E. A. Turov and V. G. Shavrov, Fiz. Tverd. Tela (Leningrad) **7**, 217 (1965) [Sov. Phys. Solid State **7**, 166 (1965)].

[6]V. L. Ginzburg and A. A. Sobyanin, Usp. Fiz. Nauk **120**, 153 (1976) [Sov. Phys. Usp. **19**, 773 (1976)].

Translated by J. G. Adashko