Nonlinear two-parameter excitations in an anisotropic ferromagnet

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The two-parameter soliton solutions of the one-dimensional equations for the magnetization in a biaxial ferromagnet are obtained and analyzed. The parameters of the localized wave are its velocity of propagation and the precession frequency ω of the magnetization vector. It is shown that in a certain region of the parameters the obtained solution describes the scattering of two domain walls. The energy and momentum of the solitons are computed, and a quasiclassical quantization of the localized wave is carried out. The quasiclassical energy spectrum of the soliton is found to be identical to the purely quantum spectrum. The possibility of using the sine-Gordon equation to describe magnetic solitons in the limiting cases of near-easy-plane ferromagnets is discussed. It is found that the equations of the magnetization dynamics can be reduced to the sine-Gordon equation only in the case in which $\omega > 0$. For $\omega < 0$ such a reduction is not possible.

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INTRODUCTION

There has in recent years been a significant upsurge in interest in the theoretical study of nonlinear magnetization waves in magnetically ordered structures, and definite success has been achieved in the theory of magnetic solitons.[1,2] The strong (nonlinear) excitations of real three-dimensional magnets are described by quite complicated dynamical equations, the solution of which is extremely difficult even in the particular cases. Therefore, it is important and urgent to analyze the simplest models and, possibly, obtain the exact solutions to the nonlinear dynamical equations, at least in the one-dimensional case.

The system of macroscopic equations describing the long-wave magnetization dynamics of a ferromagnet is the well-known set of Landau-Lifshitz equations. We shall study the spatially one-dimensional solutions to these equations without allowance for the dissipative processes.

We shall describe the state of the ferromagnet with the aid of the magnetization vector **M** treated as a function of the coordinates and the time. Using the usual—for a ferromagnet—condition $\mathbf{M}^2 = M_0^2$, we represent the magnetization vector **M** in the form

$$M_x = M_0 \sin\theta\cos\varphi, \quad M_y = M_0 \sin\theta\sin\varphi, \quad M_z = M_0\cos\theta.$$

Let the ferromagnet possess a magnetic anisotropy characterized by two preferred axes (the x and z axes), the z axis being the axis of easiest magnetization. In the ground state of such a magnet, the vector **M** is oriented along the z axis. A perturbation propagating along the x axis is described by the system of equations

$$l_0^2 \frac{\partial^2\theta}{\partial x^2} - \left[1 + l_0^2\left(\frac{\partial\varphi}{\partial x}\right)^2 + \varepsilon\cos^2\varphi\right]\sin\theta\cos\theta = -\frac{1}{\omega_0}\frac{\partial\varphi}{\partial t}\sin\theta,$$
$$l_0^2\frac{\partial}{\partial x}\left(\sin^2\theta\frac{\partial\varphi}{\partial x}\right) + \varepsilon\sin^2\theta\sin\varphi\cos\varphi = \frac{1}{\omega_0}\frac{\partial\theta}{\partial t}\sin\theta. \qquad (1)$$

This system is characterized by three physical parameters: the length-dimension parameter l_0, the frequency-dimension parameter ω_0, and the dimensionless parameter ε. The parameter l_0 is the magnetic length, the square of which is equal to the ratio of the exchange constant to the constant characterizing the anisotropy along the axis of easiest magnetization. The parameter ε in a one-dimensional ferromagnet characterizes the biaxiality of the anisotropy (it is assumed in this case that $\varepsilon > -1$). If, on the other hand, the system (1) describes the dynamics of magnetization excitations of the plane-wave type in a three-dimensional uniaxial ferromagnet, then the parameter ε allows us to take the magnetic-dipole interaction into account. The parameter ω_0 is the frequency of the homogeneous ferromagnetic resonance for $\varepsilon = 0$. The object of the present paper is to analyze the localized solutions to the system of equations (1). The equations (1) can be derived with the aid of the principle of least action if we introduce the following Lagrangian density of the physical field under investigation (the magnetization field):

$$\mathscr{L} = \frac{\hbar M_0}{\mu_0}(1-\cos\theta)\frac{\partial\varphi}{\partial t} - \frac{\beta M_0^2}{2}\{\sin^2\theta[1+\varepsilon\cos^2\varphi + l_0^2(\nabla\varphi)^2] + l_0^2(\nabla\theta)^2\}.$$

Since the Lagrangian does not explicitly depend on the time, the equations (1) possess an obvious integral of motion, which coincides with the magnetic energy:

$$E = \frac{\beta M_0^2}{2}a^2\int_{-\infty}^{+\infty} dx\left\{l_0^2\left[\left(\frac{\partial\theta}{\partial x}\right)^2 + \sin^2\theta\left(\frac{\partial\varphi}{\partial x}\right)^2\right] + \sin^2\theta(1+\varepsilon\cos^2\varphi)\right\}, \qquad (2)$$

where M_0 is the length of the magnetization vector, β is a constant characterizing the anisotropy along the axis of easiest magnetization ($\beta > 0$), and a is the atomic distance. The frequency ω_0 is connected with β by the relation $\hbar\omega_0 = 2\beta\mu_0 M_0$, where μ_0 is the Bohr magneton. To the ground state of the magnet corresponds either $\theta = 0$, or $\theta = \pi$. We shall call the solutions to the system (1) that behave at infinity in the following manner:

$$\theta = 0 \text{ for } x = \pm\infty,$$

localized solutions (solitons). Since the coefficients in the differential equations (1) do not depend explicitly on x and t, parameters connected with the coordinate origin and zero time may appear in the solution of the equations. But the dependence on these parameters is

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not important, and we shall not make mention of them below.

The simplest localized solutions to the equations (1) are the solutions of stationary-profile wave type

$$\theta=\theta(x-Vt), \quad \varphi=\varphi(x-Vt).$$

Eleonskiĭ *et al.*[3] have investigated in detail the asymptotic forms of, and numerically found, such solutions. The physical parameter on which the wave of stationary profile depend is the velocity V of propagation of the localized perturbation. The solutions to the system of ordinary differential equations to which the Landau-Lifshitz equations (1) for waves of stationary profile reduce may contain additional parameters of the constant-of-integration type. As such a parameter, we can choose, for example, the maximum value of the angular variable θ in the soliton, or φ_m, the value of the angle φ at the point where θ has its maximum value. The dependence of the form of the stationary-profile waves on such a parameter has been discussed by Eleonskiĭ *et al.*[4]

Earlier[5] Ivanov and the present authors had considered solitons whose center of gravity is at rest ($V=0$), but in which the magnetization vector precesses with frequency ω (this frequency is a parameter of the solution). In our short communication[6] we described two-parameter solitons (bions) traveling with velocity V, and possessing a temporal periodicity with period $2\pi/\omega$ in the moving coordinate system.

Thus, the two physical parameters to which we draw attention are V and ω. Somewhat later Bogdan and Kovalev[7] constructed n-parameter localized solutions. The complete integrability of the equations (1) by the method of the inverse problem of scattering theory has now been demonstrated,[8,9] and a rigorous classification of all the soliton states has been carried out. But the methods developed in Refs. 8 and 9 did not allow the construction of the explicit form the two- and many-parameter solutions, the detailed analysis of which is of indisputable interest.

ANALYSIS OF THE TWO-PARAMETER SOLUTIONS

In dimensionless units (i.e., in units of the length l_0 and the time ω_0^{-1}) the two-parameter solutions to the system (1) that are localized in space and periodic in time have the form[6]

$$\operatorname{tg}^2\frac{\theta}{2}=\frac{\varkappa^2/\Omega}{\operatorname{ch}^2\varkappa\xi+B}\left(A-\frac{\varepsilon}{2\Omega}\cos 2\varphi_0\right),$$
$$\operatorname{tg}(\varphi-\psi)=\left(A+\frac{\varepsilon}{2\Omega}\right)\operatorname{tg}\varphi_0, \quad \operatorname{tg}\psi=-\left(\frac{\Omega-\Omega_1}{\Omega+\Omega_1}\right)^{1/2}\operatorname{th}\varkappa\xi, \tag{3}$$
$$\varphi_0=\omega t-K\xi+\gamma, \quad \xi=x-Vt,$$

and

$$\Omega^2=4K^2\varkappa^2+\Omega_1^2. \tag{4}$$

Here γ is an arbitrary constant phase,[1)] while the quantities $\varkappa$, A, B, K, and Ω_1 are given functions of ω and V:

$$A=\left(1+\frac{\varepsilon^2}{4\Omega^2}\right)^{1/2}, \quad B=\frac{1}{2}\left(\frac{\Omega_1}{\Omega}-1\right), \quad \Omega_1=\omega+KV,$$
$$K=V/2A, \quad \varkappa^2=1+\varepsilon/2+K^2-\Omega_1 A. \tag{5}$$

For fixed V and ω, the parameter Ω^2 is determined as the root of the cubic equation (4), (5). Therefore, the analysis of the various specific forms of the localized solutions is tied first and foremost to the study of the various roots of the cubic equation for Ω^2. The choice of the requisite roots of this equation should be based on the requirement that the functions $\theta(x,t)$ and $\varphi(x,t)$ be real, and that the solutions be localized in space (i.e., that $B^2\geq 0$, $\varkappa^2>0$).

It follows from these requirements that the quantity Ω should be either real ($\Omega^2\geq 0$), or pure imaginary ($\Omega^2<0$). In the latter case the condition $B^2\geq 0$ requires that ω and K should also be pure imaginary (i.e., that $\omega^2<0, K^2<0$), and this is possible only if $-\varepsilon^2/4<\Omega^2<0$.

Let us rewrite the equation (4), (5) for Ω^2 in the form

$$\omega^2\left(\Omega^2+\frac{\varepsilon^2}{4}\right)^2=\Omega^2\left[\Omega^2+\frac{\varepsilon^2}{4}-\frac{V_+^2V^2}{4}\right]\left[\Omega^2+\frac{\varepsilon^2}{4}-\frac{V_-^2V^2}{4}\right],$$
$$V_\pm=|(1+\varepsilon)^{1/2}\pm 1|. \tag{6}$$

Let us consider the roots of Eq. (6) as functions of V^2. For $V=0$ these roots are: $\Omega^2=\omega^2$ and $\Omega^2=-\varepsilon^2/4$ (multiple root). For $\omega=0$, to these roots correspond the straight lines (Fig. 1)

$$\Omega^2=0, \quad \Omega^2=V_+^2V^2/4-\varepsilon^2/4, \quad \Omega^2=V_-^2V^2/4-\varepsilon^2/4.$$

For $\omega^2>0$ the roots of Eq. (6) that satisfy the above-formulated requirements lie in the region I in Fig. 1.

It turns out that the existence domain for the localized solutions in the (Ω^2, V^2) plane depends essentially on the sign of the frequency ω. For $\omega\geq 0$ this region corresponds to $0\leq V^2\leq V_+^2$, and lies above the broken line OAB and below the curve d:

$$V^2=(4+\varepsilon^2/\Omega^2)\left[(\Omega^2+\varepsilon^2/4)^{1/2}-(1+\varepsilon/2)\right]. \tag{7}$$

For $\omega<0$ this region lies above the broken line OAB when $V^2\leq V_+^2$ and above the section of the curve d lying to the right of the point B ($V^2>V_+^2$).

The existence domain of the localized solutions look more graphic in the plane, shown in Fig. 2, of the parameters ω and V. The localized solutions correspond to the points lying below the curve $\varkappa(V,\omega)=0$, whose equation is parametrically given by the formulas

$$V=\frac{2K(1+\varepsilon/2+K^2)}{[(1+K^2)(1+\varepsilon+K^2)]^{1/2}}, \quad \omega=\frac{1+\varepsilon-K^4}{[(1+K^2)(1+\varepsilon+K^2)]^{1/2}}, \tag{8}$$

where K runs through all the values from $-\infty$ to $+\infty$.

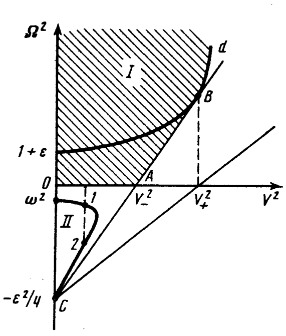

FIG. 1.

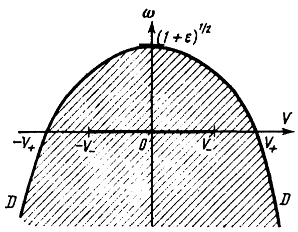

FIG. 2.

This curve for $V>0$ ($K>0$) monotonically decreases:

$$d\omega/dV=-K.$$

It is remarkable, first, that the curve (8) possesses no definite symmetry with respect to the sign of the frequency ω and, secondly, that it contains a nontrivial region of negative frequencies. The latter fact is largely due to the fact that ω is a precession frequency in a reference frame moving with velocity V. In order to grasp the physical meaning of the curve (8), let us consider the spin-wave dispersion law:

$$\tilde{\omega}(k)=[(1+k^2)(1+\varepsilon+k^2)]^{1/2}. \tag{9}$$

Here $\tilde{\omega}(k)$ is the frequency of the spin wave in the laboratory reference frame and k is the spin wave vector. If we introduce the group velocity $V=d\tilde{\omega}/dk$ of the wave and write the dispersion law (9) in the reference frame moving with velocity V:

$$\omega=\tilde{\omega}(k)-kV,$$

then we obtain formulas corresponding to the condition $\varkappa(V,\omega)=0$.

Thus, the curve D in Fig. 2 gives the graph of the dispersion law for the free magnon if we take V to be the magnon group velocity. The hatched region contains a segment of singular points on the $\omega=0$ axis ($|V|<V_-$) that correspond in the limit to nonlocalized solutions. The most interesting limiting solution corresponding to the points of this segment is obtained when $\gamma=n\pi$ (n is an integer):

$$\mathrm{tg}^2\frac{\theta}{2}=\frac{\varkappa^2/|\varepsilon|}{\mathrm{ch}^2\varkappa\xi+B}\{1+[\pm Ft-V\xi]^2\},\quad F=[(V_+^2-V^2)(V_-^2-V^2)]^{1/2},$$
$$\mathrm{tg}(\varphi-\psi)=\pm Ft-V\xi,\quad \Omega_1/\Omega=(\pm F+V^2)/V_+V_-, \tag{10}$$

and the signs ± correspond to $\omega\to\pm0$. We see that the perturbations (10) of the magnetization field do not have the form of a wave of stationary profile, and cannot therefore be analyzed, using the methods proposed in Refs. 3 and 4.

The solution (10) describes the scattering of two domain walls whose center of gravity moves with velocity V, while they themselves have in the center-of-inertia system zero velocities at infinity. Below we shall consider the scattering of walls having nonzero velocities at infinity in the center-of-inertia system (i.e., the case $\omega^2<0$).

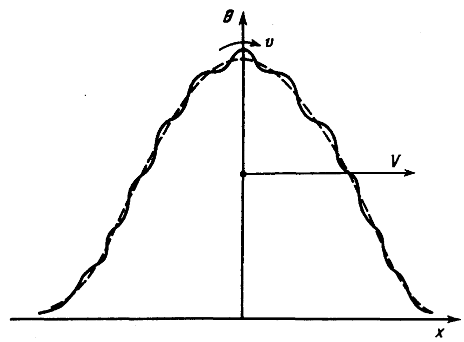

FIG. 4.

In the low-frequency regions around the segment of singular points ($|V|<V_-$, $\omega^2>0$) the solution has the form of a one-dimensional magnon drop whose dimensions vary periodically. The center of gravity of the localized solution moves with velocity V (see Fig. 3).

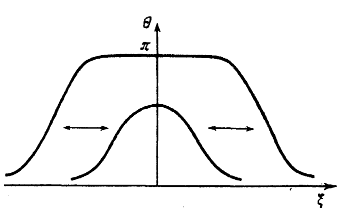

FIG. 3.

To the parameters V and ω at points far from the line of singular points correspond localized magnetization perturbations having the standard soliton form with amplitude modulated by periodic ripples traveling with group velocity

$$v=V+\omega/K(\omega,V)$$

(see Fig. 4). In the limit $\omega\to0$ for $V_-\leqslant V\leqslant V_+$ we obtain soliton solutions of the stationary-profile wave type (which depend only on $\xi=x-Vt$). The analytical description of these solutions is in complete agreement with the results of the analysis performed in Refs. 3 and 4.

For $\omega^2<0$ ($K^2<0$) the roots of Eq. (6) that satisfy the requirement that the solution be localized and be real lie in the region II (in Fig. 1 the region II corresponds to the triangle OAC). If we fix V^2 and ω^2, then this region is found, generally speaking, to contain two different Ω^2 roots corresponding to two temporally aperiodic solutions.

In the general case the aperiodic solutions to the equations (1) have the form

$$\mathrm{tg}^2\frac{\theta}{2}=\frac{\varkappa^2|\varepsilon|/2|\Omega|^2}{\mathrm{ch}^2\varkappa\xi+B}\left[\mathrm{ch}\,2|\varphi_0|-\left(1-\frac{4|\Omega|^2}{\varepsilon^2}\right)^{1/2}\right]$$
$$\mathrm{tg}(\varphi-\psi)=|A+\varepsilon/2\Omega|\,\mathrm{th}\,(|\omega|t-|K|\xi), \tag{11}$$

where Ω^2 corresponds to either the first or the second

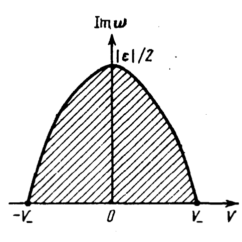

FIG. 5.

root. For $V=0$ these roots are equal to

$$(\Omega^2)_1=\omega^2,\quad (\Omega^2)_2=-\varepsilon^2/4.$$

The solution corresponding to the first root in the case in which $V=0$ is fully described in Ref. 5, and the second solution is given in Ref. 6.

For $\omega\to 0$ one of the solutions in (11), i.e., the one corresponding to the root $(\Omega^2)_1$, coincides with the aperiodic solution (10), while the second solution goes over into a wave of stationary profile:

$$(\Omega^2)_2=-\varepsilon^2/4+V_+^2V^2/4,\quad V^2\leqslant V_-^2.$$

The existence domain of the aperiodic solutions in the case of purely imaginary ω is bounded by the curve (Fig. 5)

$$V=\frac{2\varkappa(1+\varepsilon/2-\varkappa^2)}{[(\varkappa^2-1)(1+\varepsilon-\varkappa^2)]^{1/2}},\quad \omega=i\frac{\varkappa^4-1-\varepsilon}{[(\varkappa^2-1)(1+\varepsilon-\varkappa^2)]^{1/2}},\quad (1+\varepsilon)^{1/2}\leqslant\varkappa^2\leqslant 1+\varepsilon/2\quad(\varepsilon>0). \tag{12}$$

The $\omega=0$ line crossing the region bounded by the curve (12) is singular, since one of the aperiodic solutions degenerates on it into a wave of stationary profile.

QUASICLASSICAL QUANTIZATION OF THE LOCALIZED SOLUTIONS

In a homogeneous magnet the equations (1) always possess two mechanical integrals of motion[2]: the total energy (2) and the total momentum of the magnetization field

$$P=-a^2\int_{-\infty}^{+\infty}\frac{\partial\mathscr{L}}{\partial(\partial\varphi/\partial t)}\frac{\partial\varphi}{\partial\xi}d\xi=-\frac{a^2\hbar M_0}{2\mu_0}\int_{-\infty}^{+\infty}(1-\cos\theta)\frac{\partial\varphi}{\partial\xi}d\xi. \tag{13}$$

In a uniaxial magnet ($\varepsilon=0$) we have an additional integral of motion: the number of spin deviations.[10] But a biaxiality ($\varepsilon\neq 0$) leads to the nonconservation of the magnon number in the localized wave.

In the case of real values of the parameters ω and K the motion of the magnetization is periodic in time with period $T=2\pi/|\omega|$ in the reference frame moving together with the wave ($\xi=x-Vt=\text{const}$). Therefore, we can, using the standard scheme,[11] introduce for the magnetization field the adiabatic invariant:

$$I=\frac{a^2}{2\pi}\int_{-\infty}^{+\infty}dx\int_0^T\frac{\partial\mathscr{L}}{\partial(\partial\varphi/\partial t)}\left(\frac{\partial\varphi}{\partial t}\right)_{\xi=\text{const}}dt=\frac{a^2}{2\pi}\frac{\hbar M_0}{2\mu_0}\int_{-\infty}^{+\infty}dx\int_0^T(1-\cos\theta)\left(\frac{\partial\varphi}{\partial t}\right)_{\xi=\text{const}}dt. \tag{14}$$

It can be verified that the following relation follows from the definitions (2), (13), and (14) (Ref. 10):

$$\delta E=\omega\delta I+V\delta P. \tag{15}$$

The quasiclassical quantization of the magnetic soliton reduces to the requirement that

$$I=\hbar N,\quad N\gg 1, \tag{16}$$

where N is a whole number. But we find in the course of the introduction of the adiabatic invariant I and its subsequent quasiclassical quantization one circumstance seldom encountered in mechanics: the energy of the localized excitation formally turns out to be a periodic function of the adiabatic invariant I. This means that the physically different magnetization states correspond to a finite range of values of the adiabatic invariant.[12] In order to verify this, let us explicitly express the energy in terms of I and P.

The bion energy (2) for real values of ω and K is equal to

$$E=2E_0\varkappa,\quad E_0=2\beta l_0a^2M_0^2, \tag{17}$$

where E_0 is the energy, referred to the area a^2, of the stationary Bloch domain wall. The bion momentum (13) is equal to

$$P=\frac{P_0}{\pi}\arccos\frac{\Omega_1}{\Omega}. \tag{18}$$

The limiting value of the momentum coincides with the limiting momentum value in the case of a uniaxial magnet[10,13]:

$$P_0=\pi\hbar a^2M_0/\mu_0=2\pi s\hbar/a,$$

where s is the spin of the atom.

After determining the explicit form of the adiabatic invariant (14) in terms of the parameters of the solution, let us return to the formula (17) and express the energy in terms of I and P. For $\varepsilon\geqslant 0$ we have for $\varkappa$ the expression

$$\varkappa=\frac{(1+\varepsilon)^{1/2}}{\operatorname{sn}(I/I_0)}\left[\sin^2\left(\frac{\pi P}{2P_0}\right)+k^2\cos^2\left(\frac{\pi P}{2P_0}\right)\operatorname{sn}^2(I/I_0)\right]^{1/2}\times\left[\sin^2\left(\frac{\pi P}{2P_0}\right)+\cos^2\left(\frac{\pi P}{2P_0}\right)\operatorname{sn}^2(I/I_0)\right]^{1/2}. \tag{19}$$

Here $\operatorname{sn}u$ is the Jacobi elliptic sine with modulus $k=(1+\varepsilon)^{-1/2}$ and the quantity I_0 is equal to

$$I_0=2\hbar a^2M_0l_0k/\mu_0=4\hbar skl_0/a. \tag{20}$$

We see that all the values of $\varkappa$ are covered when I is varied in the interval

$$0\leqslant I\leqslant I_{max}=I_0K(k),$$

where $K(k)$ is the complete elliptic integral of the first kind.

For $\varepsilon=0$ the modulus of the elliptic functions becomes equal to unity ($k=1$), while the range of independent I values becomes unbounded from above ($I_{max}=\infty$), and we go over to formulas pertaining to a uniaxial ferromagnet.

For $-1<\varepsilon<0$, when $k>1$, the expression for $\varkappa$ can be obtained from (19) by the usual method of going over to elliptic functions with modulus equal to $1/k$, e.g.,

$$k\operatorname{sn}(u,k)=\operatorname{sn}(ku,1/k).$$

The range of physically different I values for $-1<\varepsilon<0$ should be rewritten in the form

$$0\leqslant I\leqslant I_{max}=I_0K(1/k)/k.$$

Thus, the number N in the quasiclassical-quantization formula (16) is bounded from above ($\hbar N<I_{max}$). In a uniaxial ferromagnet ($\varepsilon=0$) the number N has the meaning of the number of bound magnons, and is infinite.[10,13]

A similar situation (the appearance of a maximum adiabatic-invariant value and the quantum number N) arises in the quasiclassical quantization of the temporally periodic solutions to the sine-Gordon equation.[12] This coincidence is not accidental. As has already been repeatedly pointed out, starting from Enz's paper[14] (see also Refs. 8 and 15), we can directly derive the sine-Gordon equation from the equations (1) in two limiting cases: $1+\varepsilon \ll 1$ and $\varepsilon \gg 1$. Both of these cases correspond to the conversion of the magnet under investigation into a ferromagnet with an anisotropic easy plane. Since the use of the sine-Gordon equation to describe magnetization dynamics is extensively discussed in the literature, we would like to make some remarks apropos of the indicated passages to the limit. An essential condition for the derivation of the sine-Gordon equation from the equations (1) with $\varepsilon \gg 1$ to be possible is that there exist a characteristic length-dimensional parameter of magnitude of the order of l_0. In the presence of such a parameter we can estimate the values of the derivatives

$$\partial\theta/\partial x \sim \theta/l_0, \quad \partial\varphi/\partial x \sim \varphi/l_0,$$

which enables us to simplify the equations.

We have the exact solutions to the equations (1) for all ε, from which it follows that the spatial variation of the magnetization occurs over distances of the order of l_0 only when $\omega > 0$. Consequently, the transition to the sine-Gordon equation is justified in the case in which $\omega > 0$. But the sine-Gordon equation is a second-order differential equation with respect to the time, and the sign of the frequency ω is not important for its solution. Therefore, the question arises whether the equations (1) go over into the sine-Gordon equation in the case in which $\omega < 0$.

In the $\omega < 0$ region, at low values of the velocity V, the characteristic length parameter l determining the spatial variation of the magnetization turns out to be very small (i.e., $l \ll l_0$) when $\varepsilon \gg 1$. Therefore, Enz's[14] procedure for making the transition to the sine-Gordon equation turns out to be unjustified.

Thus, the solutions to the sine-Gordon equation describe only those dynamical magnetization states which correspond to $\omega > 0$. An entire class of states (for $\omega < 0$) found in the present paper, and described by exact solutions to the equations (1) remain, when the sine-Gordon equation is used, unaccounted for and unconsidered.

THE SOLITON ENERGY AND THE QUANTUM ENERGY SPECTRUM OF THE BOUND STATES IN THE *XYZ* MODEL

The quasiclassical quantization yields the correct energy spectrum when $N \gg 1$. If, on the other hand, the quantum number N is not high, then the quantum energy values may differ from the quasiclassical values. To be sure, in the quantum mechanics of an isolated particle we encounter cases in which the quasiclassical energy spectrum coincides exactly with the essentially quantum spectrum. Such a property is possessed, for example, by the spectrum of the one-dimensional harmonic oscillator.

Similar cases are known in the theory of solitons. It was shown at one time[16] that the energy spectrum of the bion (double soliton) in a system described by the sine-Gordon equation coincides with the spectrum for the corresponding quantum system. It has also been established[10,13] that the quasiclassical energy levels of a soliton in a one-dimensional isotropic ferromagnetic coincide with the exact quantum energy levels for spin complexes in an atomic chain with isotropic exchange, which chain is the quantum analog of the isotropic ferromagnet.

It is interesting, in view of the foregoing, to ascertain the existence of a similar correspondence in the system under investigation here. The quantum analog of the above-considered classical magnet is the so-called *XYZ* model (spin $\frac{1}{2}$). The transition from this model to the classical biaxial ferromagnet is well known (it is briefly discussed in Ref. 8). The two-parameter low-lying energy levels of the bound spin states in the *XYZ* model have been obtained by Johnson *et al.*[17] The formula for the indicated energy levels is given in Ref. 17 for certain relations between the exchange integrals J_x, J_y, and J_z. The transition to the classical model requires different relations; therefore, we rewrite this formula in the following form ($J_z > J_y > J_x > 0$):

$$E=\frac{K(k_1)}{K(t')}\frac{(J_z^2-J_x^2)^{1/2}}{\operatorname{sn}(\eta,k_1')}\left[\sin^2\frac{Q}{2}+\cos^2\frac{Q}{2}k_1'^2\operatorname{sn}^2(\eta,k_1')\right]^{1/2} \times\left[\sin^2\frac{Q}{2}+\cos^2\frac{Q}{2}\operatorname{sn}^2(\eta,k_1')\right]^{1/2}, \tag{21}$$

where $K(k)$ is the complete elliptic integral of the first kind; the moduli t and t' of the elliptic integrals are given by the relations

$$t=[(J_z^2-J_y^2)/(J_z^2-J_x^2)]^{1/2}, \quad t'=(1-t^2)^{1/2}, \tag{22}$$

while the modulus k_1 is defined as the root of the equation

$$K(k_1')/K(k_1)=[K(t)-\zeta]/K(t') \quad k_1'=(1-k_1^2)^{1/2}, \tag{23}$$

where ζ is, in its turn, found from the condition

$$\operatorname{cn}(2\zeta, t)=J_x/J_z. \tag{24}$$

The two parameters on which the energy E depends are the continuous quantity Q, which varies in the interval $(-2\pi, 0)$, and the discrete quantity

$$\eta=NK(k_1)\zeta/K(t'), \tag{25}$$

where $N<[K(t)-\zeta]/\zeta$ is a whole number.

The procedure for making the transition[8] to our model corresponds to the assumption that all the exchange integrals are close in magnitude:

$$J_j=4M_0^2 a\beta l_0^2[1+{}^1/_2 J_j^c (a/l_0)^2], \quad j=x,\ y,\ z, \tag{26a}$$

where a is the interatomic distance ($a \ll l_0$) and the βJ_j^c are three constants determining the single-ion anisotropy in the continuous model:

$$1=J_z^c-J_y^c, \quad 1+\varepsilon=J_z^c-J_x^c. \tag{26b}$$

Since $a/l_0 \ll 1$, it follows from (24)–(26) that

$$\zeta^2 = {}^1/_4(1+\varepsilon)(a/l_0)^2 \ll 1.$$

Therefore, in all the formulas (21)–(24) we should, leaving the quantiy ζ in the definition of η, go over to the limit $\zeta \to 0$:

$$\eta = N\zeta = N(1+\varepsilon)^{1/2} a/2l_0. \tag{27}$$

The quantity η in (27) coincides with the quantity I/I_0 [see (20)] when $s=\frac{1}{2}$, i.e., is proportional to the adiabatic invariant ($I = N\hbar$).

From (22) and (23) we find that in the limit $\zeta \to 0$

$$t = 1/(1+\varepsilon)^{1/2} = k_1'.$$

We see that the modulus of the Jacobi sine entering into the expression (21) coincides with the modulus k in the formula (19). The functional dependence of the energy (21) on the two parameters Q and η is equivalent to the dependence of the bion energy (17) and (19) on the momentum P and the adiabatic invariant $I (Q = aP/\hbar - \pi,\ \eta = I/I_0)$. It is not difficult to verify that all the coefficients in these dependences are identical.

Thus, in the case in which the single-ion anisotropy of the continuous model is of an origin described by the asymptotic relations (25) and (26), the quasiclassical energy spectrum of the bion coincides with the quantum energy spectrum of the bound spin states.

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1) In fact γ is a third continuous parameter analogous to the quantity φ_m in Ref. 4. But when $\omega \neq 0$ the dependence of the solutions on γ is trivial. Below we shall see that such a dependence appears only when we go to the $\omega \to 0$ limit.

2) Sklyanin[8] has shown that the equations (1) also possess an additional infinite set of integrals of motion that do not have a simple and obvious physical meaning.

1. A. R. Bishop, Phys. Scr. **20**, 409 (1979).
2. A. M. Kosevich, B. A. Ivanov, and A. S. Kovalev, Physica (Ultrecht) D **3**, 363 (1981).
3. V. M. Eleonskiĭ, N. N. Kirova, and N. E. Kulagin, Zh. Eksp. Teor. Fiz. **74**, 1814 (1978) [Sov. Phys. JETP **47**, 946 (1978)].
4. V. M. Eleonskiĭ, N. N. Kirova, and N. E. Kulagin, Zh. Eksp. Teor. Fiz. **77**, 409 (1979) [Sov. Phys. JETP **50**, 209 (1979)].
5. B. A. Ivanov, A. M. Kosevich, and I. M. Babich, Pis'ma Zh. Eksp. Teor. Fiz. **29**, 777 (1979) [JETP Lett. **29**, 714 (1979)].
6. I. M. Babich and A. M. Kosevich, Pis'ma Zh. Eksp. Teor. Fiz. **31**, 224 (1980) [JETP Lett. **31**, 205 (1980)].
7. M. M. Bogdan and A. S. Kovalev, Pis'ma Zh. Eksp. Teor. Fiz. **31**, 453 (1980) [JETP Lett. **31**, 424 (1980)].
8. E. K. Skylanin, Preprint LOMI, E3, 1979.
9. A. E. Borovik and V. N. Robuk, Teor. Mat. Fiz. **46**, 371 (1981).
10. A. M. Kosevich, B. A. Ivanov, and A. S. Kovalev, in: Nelineĭnye volny (Nonlinear Waves), Nauka, Moscow, 1979, p. 45.
11. D. Hayes, in: Nonlinear Waves (ed. S. Leibovich and A. Seebass), Cornell Univ. Press, Ithaca, 1974 (Russ. Transl., Mir, M., 1977), Chap. 6.
12. L. A. Takhtadzhan and L. D. Fadeev, Teor. Mat. Fiz. **21**, 160 (1974).
13. A. M. Kosevich, B. A. Ivanov, and A. S. Kovalev, Fiz. Nizk. Temp. **3**, 906 (1977) [Sov. J. Low Temp. Phys. **3**, 440 (1977)].
14. V. Enz, Helv. Phys. Acta **37**, 245 (1961).
15. A. R. Bishop, Z. Phys. B **37**, 357 (1980).
16. A. Luther, Phys. Rev. B **14**, 2153 (1976).
17. J. D. Johnson, S. Krinsky, and B. M. McCoy, Phys. Rev. A **8**, 2526 (1973).

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