Theory of optically induced Freedericksz transition (OFT)

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We consider the theory of the optically induced Freedericksz transition (OFT), i.e., the reorientation of the director in a homotropically oriented cell with a nematic liquid crystal (NLC) under the influence of a normally incident light beam, with a threshold dependence on the light power. It is shown that for a beam of width a much larger than the cell thickness L the threshold power is proportional to $1+\xi_l$, where ξ_l is the degree of linear polarization of the beam, $0 \leq \xi_l \leq 1$. For beam with finite transverse dimensions, we discuss the nonplanar director perturbations due to the difference between the Frank constants, and their influence on the polarization of the transmitted radiation. The asymptotic laws of the decrease of the perturbations outside the beam are obtained. Expressions are presented for the stationary amplitude of the perturbation, which is proportional to the square root of the excess above threshold. These laws hold in the general case of transversely bounded beams. The question of OFT in a planar cell is considered. In the Appendices we present geometrical-optics data on inhomogeneous nematics and discuss the question of the correct form of the variational equations for NLC.

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1. INTRODUCTION

The orientational optical nonlinearity of liquid crystals is being intensively investigated of late, see Refs. 1–7. Following the giant optical nonlinearity (GON),[6] an optically induced Freédericksz transition (OFT) was observed.[8] The theory of this phenomenon was later developed in Ref. 9 and has a number of features that distinguish it from the theory of the Freédericksz transition in a static electric or magnetic field. A few other experimental papers on the OFT were published most recently.[10–12] The discussion of the OFT theory in Refs. 11–13 contains may substantial inaccuracies and simplifications.

In this paper we consider a number of new aspects of the OFT theory, which touch principally on the polarization of a light beam and on the three-dimensional character of the perturbations of the director.

2. SYSTEM OF BASIC EQUATIONS

We describe the light beam by a complex amplitude $\mathbf{E}(\mathbf{r})$, which is connected with the real electric field $\mathbf{E}_{\mathrm{real}}(\mathbf{r},t)$ by the relation

$$\mathbf{E}_{\mathrm{real}}(\mathbf{r},t)=\frac{1}{2}[\mathbf{E}(\mathbf{r})e^{-i\omega t}+\mathbf{E}^*(\mathbf{r})e^{i\omega t}]. \tag{1}$$

The free energy per unit volume of a nematic liquid crystal (NLC) in the presence of a field is assumed in the form

$$F\left[\frac{\mathrm{erg}}{\mathrm{cm}^3}\right]=\frac{1}{2}K_{11}(\mathrm{div}\,\mathbf{n})^2+\frac{1}{2}K_{22}(\mathbf{n}\,\mathrm{rot}\,\mathbf{n})^2 +\frac{1}{2}K_{33}[\mathbf{n}\times\mathrm{rot}\,\mathbf{n}]^2-\frac{\varepsilon_a}{16\pi}(\mathbf{nE})(\mathbf{nE}^*)-\frac{\varepsilon_\perp}{16\pi}(\mathbf{EE}^*). \tag{2}$$

Here K_{ii} are Frank's constants. $\mathbf{n}$ is a unit vector in the direction of the director, the dielectric tensor of the NLC at the frequency ω of the light field is given by

$$\varepsilon_{ik}=\varepsilon_\perp\delta_{ik}+(\varepsilon_\parallel-\varepsilon_\perp)n_in_k. \tag{3}$$

In addition, we have introduced in (2) the symbol $\varepsilon_a=\varepsilon_\parallel-\varepsilon_\perp$. The complex amplitude of the quasimonochromatic field $\mathbf{E}(\mathbf{r},t)$ is determined in self-consistent fashion from a solution of Maxwell's equations with $\varepsilon_{ik}(\mathbf{r},t)$ from (3). We emphasize, however, that when variational equations are obtained for the director $\mathbf{n}(\mathbf{r},t)$, the fixed quantity should be taken to be the amplitude of the electric field $\mathbf{E}(\mathbf{r},t)$ and not, e.g., the induction $\mathbf{D}=\hat{\varepsilon}\mathbf{E}$ or some other quantity (see Ref. 14 and Appendix 2 of the present paper concerning this question). The density of the dissipative function R ($\mathrm{erg/cm^3\cdot sec}$) is assumed in the form

$$R={}^1/{}_2\eta(\dot{\mathbf{n}}\dot{\mathbf{n}}), \tag{4}$$

where the relaxation constant η has the dimension of poise ($\eta\sim10^{-2}$–1 P). The variational equations for $\mathbf{n}(\mathbf{r},t)$ are of the form

$$\Pi_{ik}\left[\frac{\partial F}{\partial n_k}-\frac{\partial}{\partial x_l}\frac{\partial F}{\partial(\partial n_k/\partial x_l)}\right]=-\frac{\partial R}{\partial \dot{n}_i},\quad \Pi_{ik}=\delta_{ik}-n_in_k. \tag{5}$$

The operator Π_{ik} projects on a plane perpendicular to the local direction of the director $\mathbf{n}(\mathbf{r},t)$; this ensures satisfaction of the equation $|\mathbf{n}(\mathbf{r},t)|=1$.

The boundary condition for a rigid homotropic orientation of the director on the walls can be assumed in the form

$$\mathbf{n}(x,y,z=0)=\mathbf{n}(x,y,z=L)=\mathbf{e}_z, \tag{6}$$

where L is the cell thickness.

Maxwell's equations can be reduced to the form

$$\mathrm{rot}\,\mathrm{rot}\,\mathbf{E}-\frac{\omega^2}{c^2}\hat{\varepsilon}\mathbf{E}=0. \tag{7}$$

A geometrical-optics approximation for the solution of Eq. (7) is discussed in Appendix 1.

3. DEPENDENCE OF THE OFT ON ε_a AND ON THE POLARIZATION OF THE INCIDENT LIGHT (BROAD BEAMS)

We consider a beam with transverse dimension a much larger than the cell thickness, $a\gg L$, incident from outside strictly along the z axis. The quantities

$n(r)$ and $E(r)$ can then be regarded as dependent only on z. The perturbed state of the director is written in the form

$$\mathbf{n}(z,t)=\mathbf{e}_z(1-\varphi_x^2-\varphi_y^2)^{1/2}+\mathbf{e}_x\varphi_x(z,t)+\mathbf{e}_y\varphi_y(z,t). \tag{8}$$

We then obtain from (5), with accuracy linear in φ inclusive,

$$-\eta\dot{\varphi}_i+K_{33}\frac{\partial^2\varphi_i}{\partial z^2}=-\frac{\varepsilon_a}{16\pi}\{(E_iE_k^*+E_i^*E_k)\varphi_k+(E_iE_z^*+E_i^*E_z)-2(E_zE_z^*)\varphi_i\}, \tag{9}$$

where $i=x,y$. The unperturbed field of the wave incident along the optical axis of the unperturbed NLC has nonzero components E_x and E_y, while $E_z=0$. If we leave out of (9) all the terms $\propto E_z$, the problem of the OFT threshold becomes completely analogous to the problem of the FT in a static magnetic or electric field. An approximation of this type was used in Refs. 11 and 13. Actually, however, in a distorted NLC, a nonzero value of E_z appears even in the first order in φ, and must be taken into account in Eq. (9). To determine this value we note that at $\mathbf{E}=\mathbf{E}(z)$ and $\hat{\varepsilon}=\hat{\varepsilon}(z)$ it follows from (7) that

$$D_z(z)\equiv\varepsilon_{zx}E_x+\varepsilon_{zy}E_y+\varepsilon_{zz}E_z=0, \tag{10a}$$

whence, with accuracy linear in φ, we have

$$E_z=-\frac{\varepsilon_a}{\varepsilon_\parallel}(\varphi_xE_x+\varphi_yE_y)+O(\varphi^3). \tag{10b}$$

The last term of (9) can be omitted, and we obtain as a result

$$\dot{\varphi}_i=\frac{1}{\eta}\left\{K_{33}\frac{\partial^2\varphi_i}{\partial z^2}+\frac{\varepsilon_a}{8\pi}\frac{E_iE_k^*+E_i^*E_k}{2}\varphi_k\left(1-\frac{\varepsilon_a}{\varepsilon_\parallel}\right)\right\}. \tag{11}$$

Thus, allowance for a field perturbation $E_z\propto\varphi$ leads so to speak to a multiplication of the unperturbed field intensity by a factor $1-\varepsilon_a/\varepsilon_\parallel\equiv\varepsilon_\perp/\varepsilon_\parallel$. Under typical conditions, this factor is $\varepsilon_\perp/\varepsilon_\parallel\approx0.65$, i.e., the correction to account for E_z is quite discernible. This correction increases the threshold power by $\varepsilon_\parallel/\varepsilon_\perp\approx1.3$ times compared with the results of the primitive theory. To determine the threshold, it suffices to represent the solution of Eq. (11) with boundary conditions (6) in the form

$$\boldsymbol{\varphi}(z,t)=\sum_{m=1}^{\infty}\sin\frac{m\pi z}{L}(\mathbf{e}_1\varphi_{1m}\exp(-\Gamma_{1m}t)+\mathbf{e}_2\varphi_{2m}\exp(-\Gamma_{2m}t)), \tag{12a}$$

$$\Gamma_{im}=\eta^{-1}\left(\frac{K_{33}\pi^2m^2}{L^2}-\frac{\varepsilon_a}{8\pi}\frac{\varepsilon_\perp}{\varepsilon_\parallel}I_i\right). \tag{12b}$$

Here I_i are the eigenvalues of the symmetrized matrix made up of the x and y components of the unperturbed field:

$$0.5(E_iE_k^*+E_i^*E_k)\varphi=I_{ik}\varphi, \tag{13}$$

and $\varphi_{1,2}$ are the corresponding eigenvectors. It is convenient to represent the polarization density matrix $E^*_iE_k$ in the form

$$\langle E_i^*E_k\rangle={}^1/_2\langle\mathbf{EE}^*\rangle(\hat{1}+\xi_1\hat{\sigma}_1+\xi_2\hat{\sigma}_2+\xi_3\hat{\sigma}_3), \tag{14}$$

where σ_i are Pauli matrices and ξ is the Stokes vector, $|\xi|\leq1$. Then

$$I_{1,2}={}^1/_2\langle\mathbf{EE}^*\rangle(1\pm(\xi_1^2+\xi_3^2)^{1/2}). \tag{15}$$

The quantity I_i corresponds to the intensity of the strongest of the linearly polarized field components. The OFT threshold corresponds to the fact that one of the quantities Γ_{im} reverses sign, i.e., the corresponding perturbation increases in experiment. Since $I_1>I_2$, the OFT threshold for $m=1$ corresponds to a power density inside the medium

$$S_z\left[\frac{\text{erg}}{\text{cm}^2\cdot\text{sec}}\right]=\frac{c\varepsilon_\perp^{1/2}}{8\pi}\langle\mathbf{EE}^*\rangle=S_{\text{thr}}=\frac{c\pi^2K_{33}\varepsilon_\parallel}{\varepsilon_a\varepsilon_\perp^{1/2}L^2}\frac{2}{1+(\xi_1^2+\xi_3^2)^{1/2}}. \tag{16a}$$

So far, the analysis was applicable to the general case of partially polarized radiation. If, however, the radiation is fully polarized, $\xi_1^2+\xi_2^2+\xi_3^2=1$, and then

$$S_{\text{thr}}=\frac{c\pi^2K_{33}\varepsilon_\parallel}{\varepsilon_a\varepsilon_\perp^{1/2}L^2}\frac{2}{1+(1-\xi_2^2)^{1/2}}, \tag{16b}$$

where ξ_2 is a Stokes parameter that characterizes the degree of circularity. In particular, $\xi_2=\pm1$ for purely circular polarization and the threshold is twice as high,[1)] than for linear polarization ($\xi_2=0$). For linearly polarized radiation ($\xi_1^2+\xi_3^2=1$) expression (16) was first obtained in our earlier paper.[9]

We emphasize here the importance of taking into account the E_z components of the field to calculate the threshold, using the following considerations. There are no physical principles that contradict the existence of media with

$$\varepsilon_\perp=\varepsilon_0=\text{const},\quad \varepsilon_a=\varepsilon_\parallel-\varepsilon_\perp\to+\infty$$

(simply speaking, media with $\varepsilon_\parallel/\varepsilon_\perp\gg1$). It is therefore appropriate to consider the dependence of S_{thr} of (16) on ε_a at

$$\varepsilon_\perp=\varepsilon_0=\text{const},\quad 0<\varepsilon_a<+\infty.$$

This dependence is given by curve 1 of Fig. 1. As $\varepsilon_a\to\infty$, S_{thr} assumes a constant value, which is taken in this figure to be unity in the vertical direction, and increases like ε_a^{-1} as $\varepsilon_a\to0$.

If no account is taken of the onset of the E_z component, as was done in Refs. 11 and 13, then the threshold is lowered, in the scale indicated, by exactly unity at any value of ε_a (curve 2). As $\varepsilon_a\to0$ the relative error of curve 2 compared with the correct curve 1 decreases. As $\varepsilon_a\to+\infty$, however, curve 2 (corresponding to the results of Refs. 11 and 13), yields a radically different result $S_{\text{thr}}\propto\varepsilon_a^{-1}\to0$ instead of the correct asymptotic $S_{\text{thr}}\to\text{const}$.

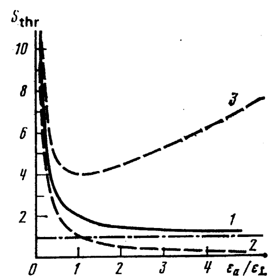

FIG. 1. Dependence of the threshold power S_{thr} (in its units as $\varepsilon_a/\varepsilon_\perp\to\infty$) on the value of $\varepsilon_a/\varepsilon_\perp$ for broad beams. Curve 1 describes the correct behavior of the threshold,[9] curve 2 shows the behavior of the threshold according to the equations of Refs. 11 and 13, while curve 3 shows the same for Ref. 12.

We note also that the value obtained for S_{thr} in Ref. 12 also differs from the corrector by a factor $\varepsilon_\parallel/\varepsilon_\perp$, but now overestimates the threshold. The corresponding curve 3 is shown in Fig. 1.

As $\varepsilon_a \to 0$, the relative difference between all three curves is small, and the threshold itself is obtained by a trivial substitution from the formulas for FT in the static electric or magnetic field, cf. Refs. 16 and 17. At $\varepsilon_a \gtrsim \varepsilon_\perp$, however, these three curves not only yield different numerical results, but have also a radically different asymptotic behavior. Thus, $S \to$ const in Ref. 9, $S \to \varepsilon_a^{-1} \to 0$ in Refs. 11 and 13, and $S \propto \varepsilon_a^{+1} \to \infty$ in Ref. 12. An analysis of the errors of the theoretical part of Ref. 12 is given in Appendix 2.

Above the threshold, an orientational effect on the NLC is produced only by the projection $(\varphi_1 \cdot \mathbf{E})$ of the incident-field vector. The polarization along φ_2 then propagates in the NLC as an ordinary o-wave, and therefore: a) it does not feel the perturbations of the director and b) it produces no GON in the OFT. For a justification of the last statement see Sec. 6. We present here also, for reference, an expression for $\varphi_1^{m=1}$ from Ref. 9 in the case of a small excess above threshold:

$$\varphi_1^{m=1} = \pm\left[2\left(1-\frac{9\varepsilon_a}{4\varepsilon_\parallel}-\frac{K_{33}-K_{11}}{K_{33}}\right)^{-1}\frac{S_z-S_{thr}}{S_{thr}}\right]^{1/2}. \qquad (17)$$

To obtain this expression it is necessary to use the initial nonlinear equation (9) and the solution of Maxwell's equations in the geometrical-optics approximation,[9] see also Sec. 8 and Appendix 1 of the present article.

4. LAWS GOVERNING THE DECREASE OF THE DIRECTOR PERTURBATIONS OUTSIDE THE LIGHT BEAM

We consider the behavior of the cell near the threshold of the OFT, with allowance for the limited transverse dimension of the light beam. Near the OFT (and at any rate, below the threshold) the distortions of the NLC are small and the light beam is deflected little by the perturbations of the director. We shall therefore assume that relation (10b) is satisfied in the entire volume of the cell, and that E_x and E_y represent the unperturbed incident field. Then, in the approximation linear in $\varphi(\mathbf{r},t)$, we obtain from (5)

$$-\eta\dot{\varphi}_i+K_{33}\frac{\partial^2\varphi_i}{\partial z^2}+K_{22}\frac{\partial^2\varphi_i}{\partial x_k\partial x_k}+(K_{11}-K_{22})\frac{\partial^2\varphi_k}{\partial x_i\partial x_k}=-\frac{\varepsilon_a\varepsilon_\perp}{8\pi\varepsilon_\parallel}I_{ik}(\mathbf{r})\varphi_k(\mathbf{r},z,t). \qquad (18)$$

We have separated here explicitly the transverse ($\mathbf{r}$) and longitudinal (z) coordinates, $\partial/\partial x_i$ denotes differentiation with respect to the transverse coordinates, and $I_{ik} = (E_iE_k^* + E_i^*E_k)/2$.

The factor $\varepsilon_\perp/\varepsilon_\parallel$ in the right-hand side of (18) corresponds to allowance for the virtual appearance of the E_z component of the field.

We seek again the solution by separating the variables:

$$\varphi(\mathbf{r},t)=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\varphi_{nm}\exp(-\Gamma_{nm}t)\mathbf{B}_{nm}(\mathbf{r})\sin\frac{m\pi z}{L}. \qquad (19)$$

The possibility of separating the factor $\sin(m\pi z/L)$ is due to the independence of I_{ik} of the coordinate z. We then obtain for the vector function $\mathbf{B}_{nm}(\mathbf{r})$ an equation of the type

$$\left\{K_{22}\frac{\partial^2}{\partial x_l\partial x_l}\delta_{ik}+(K_{11}-K_{22})\frac{\partial^2}{\partial x_i\partial x_k}+\frac{\varepsilon_a\varepsilon_\perp}{8\pi\varepsilon_\parallel}I_{ik}(r)\right\}(\mathbf{B}_{nm})_k = \left[\eta\Gamma_{mn}-K_{33}\left(\frac{m\pi}{L}\right)^2\right](\mathbf{B}_{nm})_k. \qquad (20)$$

Since the threshold is reached earliest for the value $m = 1$, we confine ourselves only to such terms in (19) and (20). We therefore obtain the problem of the eigenfunctions of a Hermitian operator and the corresponding eigenvalues $\eta\Gamma_{1n}+K_{33}(\pi/L)^2$. The threshold means that the smallest of the eigenvalues ($n = 1$) becomes equal to $K_{33}(\pi/L)^2$, i.e., that Γ_{11} vanishes.

Equation (20) is analogous to the two-dimensional Schrödinger equation, the only difference being that the wave function is a two-component vector, and the operators of the potential and kinetic energy have a more complicated tensor form.

Near the threshold and in the stationary state in general, as already stated, we have $\Gamma_{11}\approx 0$. This allows us to find the universal laws that govern the decrease of the perturbations far (at distances $|\Delta\mathbf{r}|\gtrsim L$) from that region where the perturbing action $I_{ik}(\mathbf{r})$ is localized. Namely, putting $I_{ik} = 0$, $m = 1$, and $\Gamma_{11} = 0$ in (20), we obtain the one-dimensional law governing the decrease along the coordinate $(\boldsymbol{\nu}\cdot\mathbf{r})$, where $\boldsymbol{\nu}$ is a real unit vector in the (x,y) plane:

$$\delta\mathbf{n}(\mathbf{r},z)=\mathbf{B}_{11}(\mathbf{r})\sin\frac{\pi z}{L}=\sin\frac{\pi z}{L}\{c_1\boldsymbol{\nu}\exp(-\varkappa_1(\boldsymbol{\nu}\mathbf{r}))+c_2[\mathbf{e}_z\times\boldsymbol{\nu}]\exp(-\varkappa_2(\boldsymbol{\nu}\mathbf{r}))\}, \qquad (21a)$$

$$\boldsymbol{\nu}^2=1,\quad \varkappa_1=\frac{\pi}{L}\left(\frac{K_{33}}{K_{11}}\right)^{1/2},\quad \varkappa_2=\frac{\pi}{L}\left(\frac{K_{33}}{K_{22}}\right)^{1/2}, \qquad (21b)$$

where c_1 and c_2 are constants. Since usually $K_{22} < K_{11}$, K_{33}, we have $\varkappa_1 < \varkappa_2$ and $\pi/L < \varkappa_2$. In the single-constant approximation (i.e., when $K_{11} = K_{22} = K_{33}$) we have $\varkappa_1 = \varkappa_2 = \pi/L$, and the amplitude of the perturbation of the director decreases by a factor $\exp(\pi) \approx 23$ at a distance Δx equal to the thickness L of the cell. In the case when the constants are not equal, the perturbations $\delta\mathbf{n}$ transverse to the $(\boldsymbol{\nu}\cdot\mathbf{r})$ axis decrease even more rapidly, since $\varkappa_2 > \varkappa_1$. In the general case when $m = 1$ and $\Gamma_{11} = 0$, the solution of Eq. (20) can be broken up in the region with $I_{ik}(r) = 0$ into a "longitudinal" (or "potential" V) and "transverse" (or "vortical" W) parts:

$$\delta n_i(\mathbf{r},z)=\sin\frac{\pi z}{L}[\delta_{ik}\nabla_k V(x,y)+e_{ik}\nabla_k W(x,y)], \qquad (22)$$

where $i = x,y$; e_{ik} is a two-dimensional antisymmetrical tensor, $e_{11} = e_{22} = 0$, $e_{12} = -e_{21} = 1$; it follows then from (20) at $I_{ik} = 0$ that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\varkappa_1^2\right)V=0,\quad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\varkappa_2^2\right)W=0. \qquad (23)$$

The one-dimensional solutions of these equations were obtained above—Eqs. (21); their form corresponds to representation of one of the subgroups of the symmetry of Eqs. (23)—the translation subgroup. Interest attaches also to solutions corresponding to the subgroup of the symmetry of Eqs. (23) relative to rotations in the (x,y) plane, e.g., about the origin. Using

$\varkappa_1=\varkappa$ to designate the function V or $\varkappa_2=\varkappa$ for the function W, we obtain in the polar coordinates $x=\rho\cos\varphi$, $y=\rho\sin\varphi$ the solution

$$U(\rho,\varphi)=\sum_{m=-\infty}^{\infty} e^{im\varphi}H_m^{(1)}(i\varkappa\rho)c_m, \tag{24}$$

where $H_m^{(1)}(i\varkappa\rho)$ is a Hankel function. The function $H_m^{(1)}$ has the asymptotic form

$$H_m^{(1)}(i\varkappa\rho)\approx\frac{2^{1/2}}{\pi^{1/2}}\frac{e^{-\varkappa\rho}}{(\varkappa\rho)^{1/2}}\left(1+\frac{4m^2-1}{8\varkappa\rho}+\dots\right) \tag{25}$$

as $\rho\to\infty$.

Thus, the main law governing the decrease of the perturbation at $\rho\gtrsim\varkappa^{-1}$ consists also in the radial case of two exponentials:

$$\delta\mathbf{n}(\rho,\varphi)\approx f_V(\varphi)\frac{e^{-\varkappa_1\rho}}{(\varkappa_1\rho)^{1/2}}\mathbf{e}_\rho+f_W(\varphi)\frac{e^{-\varkappa_2\rho}}{(\varkappa_2\rho)^{1/2}}\mathbf{e}_\varphi, \tag{26}$$
$$\mathbf{e}_\rho=(\cos\varphi,\ \sin\varphi);\quad \mathbf{e}_\varphi=(-\sin\varphi,\ \cos\varphi).$$

To determine the actual constants c_m of (24) or the functions $f_V(\varphi)$ and $f_W(\varphi)$ of (26) we must solve Eqs. (20) in the region with $I_{ik}(r)\neq 0$.

We have assumed so far that the perturbations near the OFT are small. It is easily seen, however, that the equations obtained and their solutions are governed by this assumption only under the following three conditions: 1) $\Gamma_{mn}=0$, 2) $m=1$, and 3) $|\delta\mathbf{n}|\ll 1$. As for the condition $\Gamma_{mn}=0$, it simply corresponds to absence of a time dependence, i.e., it is satisfied for any stationary state. Next, the results with $m=1$ can be generalized in trivial fashion to the case of arbitrary m in the z-dependence of $\sin(m\pi z/L)$ by making the substitution $\varkappa\to m^2\varkappa$. Thus, the only significant condition for the applicability of the foregoing results is the assumption that the perturbations are small, $|\delta\mathbf{n}|\ll 1$. It is clear that there is always a region far from the illuminated part of the beam where this condition is satisfied.

In addition, we have considered so far normal incidence of the light wave on the homotropic cell, when the OFT effect is present, but there is no GON.[6] For the OFT the terms $\propto E^*_i E_k$ enter with a factor δn, and the equations for δn turn out to be homogeneous, i.e., without a right-hand side. In contrast, as indicated in Refs. 2 and 6, in the case of oblique incidence of the extraordinary wave we get GON, and the equations for δn have a right-hand side $\propto E_i E^*_k(\mathbf{n}_0)_l$, where $\mathbf{n}_0$ is the unperturbed direction of the director. In the homotropic case, for the stationary problem, the equations for small perturbations of the director take the form

$$\left\{-K_{33}\left(\frac{m\pi}{L}\right)^2\delta_{ik}+K_{22}\frac{\partial^2}{\partial x_l\,\partial x_l}\delta_{ik}+(K_{11}-K_{22})\frac{\partial^2}{\partial x_i\,\partial x_k}\right\}\delta n_k(\mathbf{r})$$
$$=-\frac{\varepsilon_a}{16\pi}\left[E_i(\mathbf{n}_0\mathbf{E}^*)+E_i^*(\mathbf{n}_0\mathbf{E})-2n_{0i}|\mathbf{n}_0\mathbf{E}|^2\right]. \tag{27}$$

For the same reasons, to consider the GON at inclined incidence, the field $\mathbf{E}$ can be taken from the solution of Maxwell's equations for the unperturbed medium. For the solution of equations of the type (27), and also for a theoretical and experimental investigation of the GON, see Refs. 2, 6, 7, and 18.

5. NON-PLANAR PERTURBATIONS OF THE DIRECTOR IN BOUNDED BEAMS

We consider the problem near the threshold of the OFT for linearly polarized radiation $\mathbf{E}=\mathbf{e}_x I^{1/2}(\mathbf{r})$. The linearized equations for the lowest harmonic of the perturbations

$$\delta\mathbf{n}=(\mathbf{e}_x\varphi_x+\mathbf{e}_y\varphi_y)\sin(\pi z/L)$$

are of the form

$$\left[K_{11}\frac{\partial^2}{\partial x^2}+K_{22}\frac{\partial^2}{\partial y^2}-\left(\frac{\pi}{L}\right)^2K_{33}+\frac{\varepsilon_a}{8\pi}\frac{\varepsilon_\perp}{\varepsilon_\parallel}I(\mathbf{r})\right]\varphi_x=-(K_{11}-K_{22})\frac{\partial^2\varphi_y}{\partial x\,\partial y}, \tag{28a}$$

$$\left[K_{11}\frac{\partial^2}{\partial y^2}+K_{22}\frac{\partial^2}{\partial x^2}-\left(\frac{\pi}{L}\right)^2K_{33}\right]\varphi_y=-(K_{11}-K_{22})\frac{\partial^2\varphi_x}{\partial x\,\partial y}. \tag{28b}$$

We note first that in the single-constant approximation the perturbed director remains in the x,z plane. To elucidate the qualitative features of nonplanar perturbations, it is convenient to turn to the case of relatively broad (compared with a cell thickness) light beams. The right-hand sides of Eqs. (28a) and (28b) can then be treated by perturbation theory. In addition, the differential operator in the left-hand side of Eq. (28b) can also be neglected in the case of broad beams, and then

$$\varphi_y=\frac{K_{11}-K_{22}}{K_{33}}\left(\frac{L}{\pi}\right)^2\frac{\partial^2\varphi_x}{\partial x\,\partial y}=\frac{K_{11}-K_{22}}{K_{33}}\left(\frac{L}{\pi}\right)^2\frac{xy}{a_x^2a_y^2}\varphi_x. \tag{29}$$

The second expression of (29) was obtained under the assumption that $\varphi_x(\mathbf{r})$ can be approximated by the equation

$$\varphi_x(\mathbf{r})=\text{const}\exp(-x^2/a_x^2-y^2/a_y^2).$$

The spatial distribution of the perturbation $\delta\mathbf{n}=\varphi_x\mathbf{e}_x+\varphi_y\mathbf{e}_y$, corresponding to (29), is shown in Fig. 2.

In that part of the (x,y) plane where the perturbation is appreciable, i.e., at $|x|\sim a_x$, $|y|\sim a_y$, there follows from (29) the simple estimate

$$\frac{\varphi_y}{\varphi_x}\sim\frac{K_{11}-K_{22}}{K_{33}}\frac{L^2}{\pi^2a_xa_y}. \tag{30}$$

Thus, for ribbon beams, with either $a_x\to\infty$ or $a_y\to\infty$, the perturbations remain planar. In the case when the light-polarization direction makes an oblique angle with the ribbon direction, the perturbations are again nonplanar.

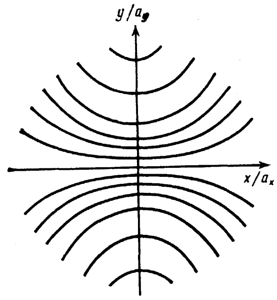

FIG. 2. Spatial distribution of the perturbation of the director $\delta\mathbf{n}\propto\exp(-x^2/a_x^2-y^2/a_y^2)(\mathbf{e}_x+\text{const}(xy/a_xa_y)\mathbf{e}_y)$ near the origin $x=0$, $y=0$. The tangents to the curves show the direction of $\delta\mathbf{n}$.

For a circularly polarized beam, the perturbations $\delta n \propto \mathbf{e}_x$ and $\delta n \propto \mathbf{e}_y$ have equal thresholds, i.e., the system is degenerate with respect to the direction of φ in the (x,y) plane. Under these conditions, the final direction of φ is determined by weak factors that play a minor role for linearly polarized beams.[2)] Thus, e.g., let a circularly polarized beam with transverse dimensions $a \gtrsim L$, $a_y \gtrsim L$ be not axially symmetrical, and for the sake of argument let $a_x > a_y$. It is then easy to show that minimization of the free energy fixes the direction of φ along the x axis, i.e., $\varphi \propto \mathbf{e}_x$.

A curious example is a "ribbon" beam rolled into a "ring" such that the radius of the ring $R \gg L$, and the thickness of the illuminated annular region is of the order of the cell thickness L. Then, owing to the relation $K_{22} \ll K_{11}$, in the case of a depolarized or circularly polarized incident beam, the perturbation of the director should be directed along the tangent to the circle and not along the radius.

6. EFFECT OF NONPLANAR PERTURBATIONS ON THE POLARIZATION OF THE TRANSMITTED LIGHT

The OFT produces a lens with bell-shaped profile in a beam having a finite transverse dimension in a cell. As a result, the plane wavefront of the incident radiation likewise becomes bell-shaped. For the one-dimensional problem, the shape of the wavefront as it emerges from the medium is illustrated in Fig. 3(a). If the phase shift at the center of the beam, compared with its edge, is much larger than the wavelength, then the field in the far zone, in the direction at an angle θ_0 to the z axis [see Fig. 3(b)], is determined by the sum of the contributions of those two points x_1 and x_2, in which the normal to the wavefront is directed at an angle θ_0 to the z axis [see Fig. 3(a)]. The interference of these two contributions produces the characteristic fringe structure shown arbitrarily shaded in Fig. 3(b).

In the two-dimensional problem, the wavefront at the exit from the medium is a two-dimensional bell. Figure 4(a) maps schematically the level lines of the wavefront surface as functions of the transverse coordinates x and y. Just as in the one-dimensional problem, the field at a point with angular coordinates $\boldsymbol{\theta}_0 = (\theta_0, \theta_{0y})$ in the far zone [Fig. 4(b)] is determined by the contribution of two points $\mathbf{r}_1$ and $\mathbf{r}_2$ [Fig. 4(a)] at which the normal to the wavefront surface has the required direction. The interference between these contributions produces in the far zone the characteristic annular structure discussed in self-focusing and defocusing problems, see Ref. 19. For liquid crystals, this annular structure was observed and discussed in the GON-self-focusing regime in Ref. 7, while for CFT self focusing it was discussed in Refs. 18 and 13; in the case of OFT, this question was recently discussed also in Ref. 20.

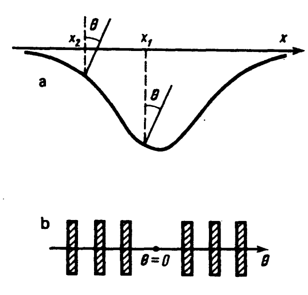

FIG. 3. Wavefront profile (a) and interference pattern in the far zone (b) for the one-dimensional problem.

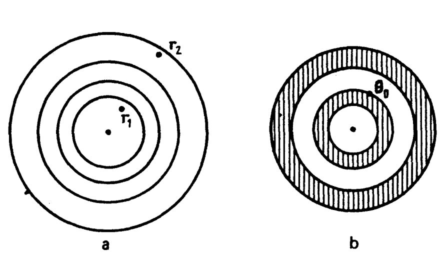

FIG. 4. Constant-depth lines of the wave-front surface (a) and interference pattern in the far zone (b) in the axially symmetrical problem.

We examine now the qualitative results of the nonplanar the director perturbation represented in Fig. 2 (see Fig. 5). It is easily understood that there are no φ_y perturbations of the director on the x and y axes in the (x,y) plane. For this reason, the transmitted wave will have in the far zone on the axes θ_x and θ_y an annular structure only in the same $\mathbf{e}_x$ polarization as of the incident wave. In other words, there should be no $\mathbf{e}_y$ field component in the far zone near the θ_x and θ_y axes. The field in the direction of the middle of the quadrants of the (θ_x, θ_y) plane is determined by the contribution of the points in the middle of the quadrants of the (x,y) plane [cf. Figs. 4(a) and 4(b)]. It is precisely for such $\boldsymbol{\theta}$ that the depolarizing effect of nonplanar perturbations should manifest itself. The order of magnitude of the amplitude of the depolarized field can be roughly estimated from (30) at

$$\frac{E_y}{E_x} \sim \frac{\varphi_y}{\varphi_x} \sim \frac{K_{11}-K_{22}}{K_{33}}\left(\frac{L}{\pi a}\right)^2, \tag{31}$$

and the intensity of the depolarized component can be estimated as the square of this parameter. The annular interference structure of the field in the far zone has in general the same character for the $\mathbf{e}_y$ polarization as for the $\mathbf{e}_x$ polarization.

We assume that the foregoing premises explain the polarizational features, observed in Ref. 8, of light passing through a cell in the case of OFT.[3)] Unfortunately, a quantitative comparison of the theory with ex-

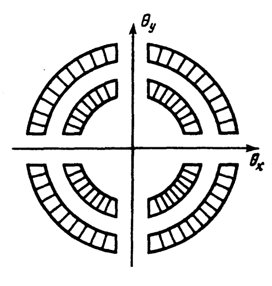

FIG. 5. Schematic picture of the depolarized (e_y) component in the far zone for OFT in a field with e_x polarization.

periment is difficult, since the parameter of (30) and (31) was not small under the conditions of the experiment of Ref. 8.

7. THRESHOLD AND PROFILE OF PLANAR PERTURBATIONS OF THE DIRECTOR IN BOUNDED BEAMS

In the general case of transversely inhomogeneous and nonplanar perturbations of the director, Eqs. (18) cannot be solved even in the linear (i.e., near-threshold) approximation. Particular interest attaches therefore to those cases for which an exact solution of the problem of the OFT threshold problem and of the unstable-mode profile problem can be solved exactly. All these cases pertain to problems in which, by virtue of the symmetry properties, the perturbations remain purely planar. Let us list them in arbitrary sequence.

Let the intensity of the light beam depend only on one transverse coordinate, which we designate by y (ribbon beam). The nonzero perturbation will be in the case $\mathbf{E}(y) = \mathbf{e}_x I^{1/2}(y)$

$$\delta\mathbf{n} \propto \mathbf{e}_x \varphi(y) \sin(\pi z/L),$$

and in the case $\mathbf{E}(y) = \mathbf{e}_y I^{1/2}(y)$,

$$\delta\mathbf{n} \propto \mathbf{e}_y \varphi(y) \sin(\pi z/L).$$

For both cases, Eq. (20) takes the form

$$\left[\eta\Gamma - K_{33}\left(\frac{\pi}{L}\right)^2 + \tilde{K}\frac{\partial^2}{\partial y^2} + \frac{\varepsilon_a}{8\pi}\frac{\varepsilon_\perp}{\varepsilon_\parallel} I(y)\right]\varphi_0(y) = 0, \qquad (32)$$

where $\tilde{K} = K_{22}$ for $\mathbf{E} \propto \mathbf{e}_x$ and $\tilde{K} = K_{11}$ for $\mathbf{E} \propto \mathbf{e}_y$. We now obtain the threshold intensity $I(y = 0)$ and the perturbation profile $\varphi_0(y)$ for a number of specific $I(y)$ distributions.

1) Let $I(y) = I_0$ at $-a \leq y \leq a$ and $I(y) = 0$ at $|y| > a$ (rectangular profile with total width $2a$). The threshold condition can then be written in the form

$$\varkappa = (p^2 - \varkappa^2)^{1/2} \operatorname{tg} (p^2 - \varkappa^2)^{1/2} a,$$

$$\varkappa = \frac{\pi}{L}\left(\frac{K_{33}}{\tilde{K}}\right)^{1/2}, \quad p^2 = \frac{\varepsilon_a \varepsilon_\perp I_0}{8\pi\varepsilon_\parallel \tilde{K}} \qquad (33)$$

(we recall that $S_z[\mathrm{erg/cm^2 \cdot sec}] = c\varepsilon_\perp^{1/2} I_0/8\pi$).

Since $K_{22} < K_{11}$, the threshold for $\mathbf{e}_x$ polarization along the ribbon is somewhat lower than for $\mathbf{e}_y$ polarization.

At $a \gg L$ and $a \ll L$ we have the asymptotic expressions

$$I_0 = \left(\frac{\pi}{L}\right)^2 \frac{8\pi K_3 \varepsilon_\parallel}{\varepsilon_a \varepsilon_\perp}\left[1 + \frac{\tilde{K}}{K_{33}}\left(\frac{L}{2a}\right)^2\right], \quad a \gg L, \qquad (34a)$$

$$I_0 = \left(\frac{\pi}{L}\right)^2 \frac{8\pi K_{33}\varepsilon_\parallel}{\varepsilon_a \varepsilon_\perp}\left[1 + \left(\frac{\tilde{K}}{K_{33}}\right)^{1/2} \frac{L}{\pi a}\right], \quad a \ll L. \qquad (34b)$$

We point out that in a narrow ribbon beam, in the limit $a \ll L$, the threshold value is the "running" power

$$\int_{-\infty}^{\infty} S_z(y)\,dy = \frac{2c\varepsilon_0^{1/2} I_0 a}{8\pi} = \frac{\pi}{L}\frac{2c\varepsilon_\parallel (K_{33}\tilde{K})^{1/2}}{\varepsilon_a \varepsilon_\perp^{1/2}}, \quad a \ll L. \qquad (35)$$

At $a \ll L$ the last statement does not depend at all on the concrete form of $I(y)$. In broad beams, the correction to I_{thr} turns out to be quadratic in the small parameter L/a; this statement is specifically applicable to a flat-top type of distribution.

The unstable mode is of the form

$$\varphi(y) = \begin{cases} \cos(p^2 - \varkappa^2)^{1/2} y, & |y| \leq a \\ \cos[(p^2 - \varkappa^2)^{1/2} a] \exp[-\varkappa(|y| - a)], & |y| > a. \end{cases} \qquad (36a)$$

In particular, at $a \gg L$,

$$\varphi(y) = \begin{cases} \cos\frac{\pi}{2}\frac{y}{a}, & |y| \leq a \\ \frac{\pi}{2\varkappa a}\exp[-\varkappa(|y| - a)], & |y| > a. \end{cases} \qquad (36b)$$

A characteristic feature of (36b) is that for a rectangular intensity distribution the perturbation $\varphi(y)$ has a cosine distribution and falls off practically to zero (more accurately, to the small quantity $\pi/2\varkappa a$) at the beam boundaries $y = \pm a$. For $a \gg L$, even at a relatively small excess over threshold, $(S - S_{\mathrm{thr}})/S_{\mathrm{thr}} \sim (\varkappa a)^{-2}$, modes with higher transverse indices are added to the lower mode (36b), see Sec. 8. As a result, the perturbation inside the illuminated region tends to a constant value typical of an unbounded beam. The law governing the fall-off of $\varphi(y)$ outside the beam remains the same as before, i.e., exponential, see Sec. 4.

2) Let $I(y) = I_0 \cosh^{-2}(y/b)$, with the beam half-width at half-maximum intensity at the maximum $a(HWHM) \approx 0.9b$. The threshold is then determined by the equation

$$I_0 = \left(\frac{\pi}{L}\right)^2 \frac{8\pi\varepsilon_\parallel K_{33}}{\varepsilon_a \varepsilon_\perp}\left[1 + \left(\frac{\tilde{K}}{K_{33}}\right)^{1/2} \frac{L}{\pi b}\right], \qquad (37)$$

and the unstable mode is given by

$$\varphi_0(y) = [\operatorname{ch}(y/b)]^{-\beta},$$

$$\beta = \frac{1}{2}\left\{-1 + \left[1 + \left(\frac{2\pi b}{L}\right)^2 \frac{K_{33}}{\tilde{K}}\left(1 + \left(\frac{\tilde{K}}{K_{33}}\right)^{1/2} \frac{L}{\pi b}\right)\right]^{1/2}\right\}. \qquad (38)$$

At $a \ll L$ everything coincides with the case of a square top provided the integrals $\int I(y)dy$ coincide.

At $b \gg L$, the correction to the threshold is linear in L/b, and the profile of the instability mode takes the form

$$\varphi_0(y) \approx \exp(-y^2/a^2), \quad a = (2b/\varkappa)^{1/2}. \qquad (39)$$

Both profiles, the Gaussian profile of the mode with width $a \propto (Lb)^{1/2}$ and the correction to the threshold in first order in $L/b \ll 1$, are typical of any broad distribution $I(y)$ with a smooth dependence near the origin, $I(y) \approx I_0(1 - y^2/b^2 + \ldots)$. For broad distributions that are smooth at zero it is possible to solve in the single-constant approximation also the two-dimensional problem with $I(x, y)$ in the form

$$I(x, y) = I_0(1 - x^2/b_x^2 - y^2/b_y^2) \qquad (40)$$

the higher terms in x^2 and y^2 have been left out of (40) at a polarization $\mathbf{E} \propto \mathbf{e}_x$. The threshold is determined here by the expression

$$I_0 = \frac{8\pi\varepsilon_\parallel K}{\varepsilon_a \varepsilon_\perp}\left(\frac{\pi}{L}\right)^2\left[1 + \frac{L}{\pi}(b_x^{-1} + b_y^{-1}) + O\left(\left(\frac{L}{b}\right)^2\right)\right] \qquad (41)$$

(we recall that by assumption $b_x \gg L$ and $b_y \gg L$), and the perturbation takes the form

$$\varphi = e_x \exp(-x^2/a_x^2 - y^2/a_y^2),$$
$$a_x = (2b_x L/\pi)^{1/2}, \quad a_y = (2b_y L/\pi)^{1/2}. \tag{42}$$

We note that at $b \gg L$ the half-width of the unstable mode from (38) at the 1/2 level of φ is proportional at the maximum to the geometric mean of the beam width b and the cell thickness L:

$$\Delta x(HWHM\varphi) = (2bL/\pi)^{1/2}. \tag{43}$$

We present for reference an equation for the threshold in the case of a two-dimensional round flat top in the single-constant approximation:

$$I = I_0 \theta(a - |\mathbf{r}|). \tag{44}$$

The profile $\varphi(\mathbf{r})$ is expressed in terms of a Bessel function, see Ref. 9. At $a \gg L$ we have, just for a one-dimensional flat top, a correction of order $(L/a)^2$:

$$I_0 = \left(\frac{\pi}{L}\right)^2 \frac{8\pi\varepsilon_\parallel K}{\varepsilon_a \varepsilon_\perp} \left\{1 + 0.58\left(\frac{L}{a}\right)^2 \left[1 - 0.76\frac{L}{a}\frac{K_0(\pi a/L)}{K_1(\pi a/L)}\right]^2\right\}, \tag{45a}$$

where K_ν is a modified Bessel function of order ν. At $a \ll L$ we have

$$I_0 = \left(\frac{\pi}{L}\right)^2 \frac{8\pi\varepsilon_\parallel K}{\varepsilon_a \varepsilon_\perp}\left[1 + \frac{2}{(\pi a/L)^2 \ln(L/\pi a)}\right]. \tag{45b}$$

Thus, accurate to logarithmic corrections, the threshold value is the total beam power $I_0 a^2$. Unfortunately, we do not know as yet whether this conclusion is valid for narrow beams ($a \ll L$) in the approximation with more than one constant.

8. OFT IN A PLANARLY ORIENTED CELL

We consider in NLC cell with planar orientation of the director and with $\varepsilon_a > 0$. If a wave with polarization of the extraordinary (e) type is incident on such a cell at an angle such that the unit vector $\mathbf{e}_e$ of the electric field of the wave makes an angle with the unperturbed director $\mathbf{n}_0 \equiv \mathbf{e}_x$, then the GON regime is realized (see Refs. 2 and 6). If the e-type wave is normally incident on the cell, then $\mathbf{e}_e \parallel \mathbf{n}_0$ and the free-energy minimum is realized at the same unperturbed position of the director. When the incident wave has a polarization of the ordinary (o) type, there is no GON, since $(\mathbf{e}_0 \cdot \mathbf{n}_0) = 0$ and, in first order in the field, the torque acting on the director is zero:

$$[(\hat{\varepsilon} - 1)\mathbf{E}\times\mathbf{E}^*]/8\pi = 0.$$

It may seem at first glance that under these conditions OFT will take place, i.e., when an o-wave of sufficient intensity is incident the detector $\mathbf{n}$ will be rotated in such a way that $(\mathbf{n}\cdot\mathbf{E}_0) \neq 0$, and as a result the free energy

$$F_E = -\varepsilon_a (\mathbf{nE})(\mathbf{nE}^*)/16\pi$$

is lowered. Actually, as noted in Ref. 9 and later in Ref. 12, when the director varies smoothly in space the wave polarization follows adiabatically the changes of the director (the Mauguin limit). In particular, the o-wave remains an o-wave, inasmuch as $(\mathbf{n}\cdot\mathbf{E})$ is always zero for it.

In the present section we obtain equations for the general case of perturbations of a director, with arbitrary degree of roughness. For such perturbations, the OFT in the o-wave takes place. Its threshold, however, is exceedingly high.

Assume that a linearly polarized wave $\mathbf{E}(z=0) = E_0\mathbf{e}_y$ is normally incident on a planar cell with $\mathbf{n}_0 = \mathbf{e}$. In the absence of perturbations, this is an o-wave. We consider the wave to be homogeneous over the cross section. We assume the director perturbation in the form

$$\mathbf{n}(z) = \mathbf{e}_x\cos\varphi(z) + \mathbf{e}_y\sin\varphi(z) \approx \mathbf{e}_x(1 - 0.5\varphi^2(z)) + \mathbf{e}_y\varphi(z). \tag{46}$$

The equations for $\varphi(z,t)$ and for $\mathbf{E}$ are

$$-\eta\frac{\partial\varphi}{\partial t} + K_{22}\frac{\partial^2\varphi}{\partial z^2} = -\frac{\varepsilon_a}{16\pi}[\sin 2\varphi(|E_y|^2 - |E_x|^2) + \cos 2\varphi(E_y E_x^* + E_y^* E_x)], \tag{47}$$

$$\frac{d^2E_x}{dz^2} + \frac{\omega^2}{c^2}(\varepsilon_\perp + \varepsilon_a\cos^2\varphi)E_x + \frac{\omega^2}{c^2}\varepsilon_a\sin\varphi\cos\varphi E_y = 0, \tag{48}$$

$$\frac{d^2E_y}{dz^2} + \frac{\omega^2}{c^2}\varepsilon_a\sin\varphi\cos\varphi E_x + \frac{\omega^2}{c^2}(\varepsilon_\perp + \varepsilon_a\sin^2\varphi)E_y = 0. \tag{49}$$

Equations (47)–(49) were written without assuming the perturbation φ to be small. They have the trivial solution

$$\varphi(z,t) = 0, \quad E_x(z,t) = 0, \quad E_y(z,t) = E_0\exp(ik_0 z),$$

where $k_0 = \omega\varepsilon_\perp^{1/2}/c$.

To check on the stability, the system (47)–(49) must be linearized, with φ and E_x assumed to be quantities of first order of smallness:

$$-\eta\frac{\partial\varphi}{\partial t} + K_{22}\frac{\partial^2\varphi}{\partial z^2} = -\frac{\varepsilon_a}{16\pi}(2\varphi|E_y|^2 + E_x E_y^* + E_x^* E_y), \tag{50}$$

$$\frac{d^2E_x}{dz^2} + \frac{\omega^2}{c^2}\varepsilon_\parallel E_x = -\frac{\omega^2}{c^2}\varepsilon_a\varphi E_y, \tag{51}$$

$$\frac{d^2E_y}{dz^2} + \frac{\omega^2}{c^2}\varepsilon_\perp E_y = 0. \tag{52}$$

From the last equation we obtain, as before, that at the accuracy sufficient for us

$$E_y(z) = E_0 e^{ik_0 z}. \tag{53}$$

The field $E_x(z)$ is sought in the form

$$E_x(z) = E_0 A(z)\exp(ik_0 z),$$

where $A(z)$ is a function that is slow in the scale $(2k)^{-1}$, but is perhaps not very slow in the scale $(k_e - k_0)^{-1}$, where $k_e = \omega\varepsilon_\parallel^{1/2}/c$. It is then convenient to reduce (50) and (51) to

$$-\frac{\eta}{K_{22}}\frac{\partial\varphi}{\partial t} + \frac{\partial^2\varphi}{\partial z^2} = -\chi^2\left(\varphi + \frac{A(z) + A^*(z)}{2}\right), \tag{54}$$

$$dA/dz - i\mu A = i\mu\varphi E_0, \tag{55}$$

$$\mu = (k_e^2 - k_0^2)/2k_0 = \omega\varepsilon_a/c\varepsilon_\perp^{1/2}, \quad \chi^2 = \varepsilon_0|E_0|^2/8\pi K_{22}. \tag{56}$$

We assume first that at $\varphi(z)$ is a smooth function even in the scale $\mu^{-1} = \lambda\!\!\!^{-}/(n_e - n_0)$. It follows then from (55) that $A(z) \approx -\varphi(z)$. Thus, in the case of very small perturbations of $\varphi(z)$ the o-wave field follows adiabatically the inclinations of the director, and the right-hand side of (56) vanishes identically. This corresponds to the statement that there is no OFT in the adiabatic Mauguin limit.

If we forgo the assumption of such a smooth z-dependence of the angle φ, then the solution of (55) can be written in the form

$$A(z)=i\mu\int_0 \varphi(z')\exp(i\mu(z-z'))\,dz' \tag{57}$$

and the equation for $\varphi(z,t)$ takes the form

$$\frac{\eta}{K_{22}}\frac{\partial\varphi}{\partial t}=\frac{\partial^2\varphi}{\partial z^2}+\chi^2\left\{\varphi(z,t)-\mu\int_0^z \varphi(z',t)\sin\mu(z-z')\,dz'\right\}. \tag{58}$$

Unfortunately, the integral operator in the right-hand side of (58) is not self-adjoint. Therefore the instability of the trivial solution $\varphi\equiv 0$, even if it exists, can be more complicated than a simple exponential growth of one eigenfunction, as was the case of OFT in a homotropic cell.

We consider now limiting cases. Let the cell thickness be small, $L\lesssim\lambda_0/(n_e-n_0)\approx 2\pi/\mu$; here λ_0 is the light wavelength in vacuum; a typical value is $\lambda_0/(n_e-n_0)\sim(2.5-5)\cdot 10^{-4}$ cm. At such a small thickness, the field does not manage to follow the rotation of the director. In the language of Eq. (58) this means that the integral operator can be neglected, and then we have OFT in its classical form

$$\varphi\propto\sin(\pi z/L),\quad \chi^2_{\text{thr}}=\frac{\varepsilon_a|E_{\text{thr}}|^2}{8\pi K_{22}}=\left(\frac{\pi}{L}\right)^2\gtrsim\frac{(n_e-n_0)^2}{\lambda_0^2}. \tag{59}$$

Since the parameter $\lambda_0/(n_e-n_0)$ is smaller by approximately two orders than the customarily employed cell thickness, the OFT threshold from (59) turns out to be higher by about four orders (in terms of the power density per cm^2).

At $\mu L\ll 1$ the action of the integral operator in (58) can be treated by perturbation theory. As a result we obtain that the true OFT threshold is higher than given by (59):

$$\frac{\varepsilon_a|E_{\text{thr}}|^2}{8\pi K_{22}}=\left(\frac{\pi}{L}\right)^2\left[1+\left(\frac{\mu L}{\pi}\right)^2+O((\mu L)^4)\right]. \tag{60}$$

It must be remembered that Eq. (60) was obtained only for $\mu L\ll 1$.

At a larger thickness, $L\gtrsim\lambda_0/(n_e-n_0)$, the smooth perturbations of $\varphi(z)$, as already mentioned, do not cause instability. As for perturbations with the small scale Δz, we have here two quite strong effects described by the system (59) and (60), or even by the initial system (50)–(52). Foremost is the lattice optical nonlinearity (LON), which consists in the following.[3] In the zeroth approximation, the waves E_x and E_y propagate each with its own wave vector

$$E_x=E_1\exp(ik_ez),\quad E_y=E_2\exp(ik_0z).$$

Their interference $\propto E_1E_2^*\exp(i\mu z)$ + c.c. produces in the medium a perturbation

$$\delta\varphi(z)\approx\varepsilon_aE_1E_2^*e^{i\mu z}/16\pi K_{22}\mu^2+\text{c.c.} \tag{61}$$

The scattering of the initial wave by such perturbations [the right-hand sides in Eq. (51) and (52)] gives rise to a positive increment in the refractive index of the E_x wave; this increment is proportional to $|E_y|^2$; conversely, δk for the E_y wave is proportional to $|E_x|^2$. These effects cause mutual focusing of the waves with the $\mathbf{e}_x$ and $\mathbf{e}_y$ polarizations.

In addition, if the waves E_1 and E_2 have frequencies that differ little, by $E_1E_2^*\propto\exp(-i\Omega t)$, then

$$\delta\varphi(z,t)\approx\varepsilon_aE_1E_2^*\exp(i\mu z-i\Omega t)/16\pi K_{22}\mu^2(1+i\Omega/\Gamma), \tag{62}$$

where $\Gamma=K_{22}\mu^2/\eta$ is the damping constant in reciprocal seconds. The same effect of rescattering of the E_2 wave by the perturbations (62) leads at $\Omega>0$ to an exponential amplification of the E_1 wave:

$$|E_1(z)|^2=|E_1(0)|^2e^{gz},\quad g\approx\frac{\omega}{c}\frac{\Omega/\Gamma}{1+(\Omega/\Gamma)^2}|E_2|^2. \tag{63}$$

For NLC, this process was first considered in Ref. 1 and is called stimulated scattering of light.

It is easy to verify that at a power density corresponding to the OFT threshold in a thin cell [Eq. (59)] an increase of the cell thickness leads to an increase of the role of the LON self-focusing and of stimulated scattering. We note that the LON and stimulated scattering should take place in the case of normal incidence of an e-wave on a planar cell. It is necessary in this case to make in (54), (55), and (58) the substitutions $\mu\to\mu k_0k_e^{-1}$ and $\chi^2\to-\chi^2$. The nonlinear stage of development of this process will not be considered here.

9. CONCLUSION

From our point of view, great interest attaches to an experimental verification of the deductions drawn in this paper. These include: an exact quantitative experimental determination of the threshold in broad beams, dependence of the threshold on the dimension, shape, and polarization of the beam, the nonplanar character of the perturbations, and the law governing their decrease outside the beam, and many others. It is also of interest to observe the LON and OFT in other types of liquid crystals, such as smectics and cholesterics.

The authors are deeply grateful to E.I. Katz and Yu.S. Chiligaryan for discussions.

APPENDIX 1

GEOMETRICAL OPTICS OF AN INHOMOGENEOUS NEMATIC

Maxwell's equations for a monochromatic field $\mathbf{E}(r)\exp(-i\omega t)$ of frequency in a medium with a dielectric tensor $\hat\varepsilon=\varepsilon_{ik}(\mathbf{r})$ will be written in the form

$$\text{grad div}\,\mathbf{E}-\Delta\mathbf{E}-\frac{\omega^2}{c^2}\hat\varepsilon\mathbf{E}=0. \tag{A1.1}$$

To obtain the equations we need, we represent the field $\mathbf{E}(\mathbf{r})$ in the form

$$\mathbf{E}(\mathbf{r})=e^{i\frac{\omega}{c}\psi}\mathbf{E}^0(\mathbf{r}). \tag{A1.2}$$

It is convenient to introduce the vector

$$\mathbf{p}=\nabla\psi, \tag{A1.3}$$

which coincides for a homogeneous medium with wave vector k divided by (ω/c), i.e., the length of the vector p is equal to the "phase refractive index." In the zeroth approximation in (ω/c) it follows then from (A1.1) that

$$(p_i p_k - p^2\delta_{ik} + \hat{\varepsilon}_{ik}(\mathbf{r}))E_i^0(\mathbf{r}) = 0. \quad (A1.4)$$

Equating to zero the determinant of the linear homogeneous system (A1.4), we get the known Fresnel equation for $\mathbf{p}(\mathbf{r})$ for the local value of the tensor $\varepsilon_{ik}(\mathbf{r})$. We are interested in the case of a nematic in which the director lies in the (x, z) plane:

$$n_x = \sin\varphi(z), \quad n_y = 0, \quad n_z = \cos\varphi(z), \quad (A1.5)$$

and the inclination angle $\varphi(z)$ does not depend on the coordinate x and y; the z axis is chosen here normal to the planes of the cell.

The dielectric constant is

$$\varepsilon_{ik}(z) = \varepsilon_\perp\delta_{ik} + \varepsilon_a n_i(z) n_k(z), \quad (A1.6)$$

where $\varepsilon_a = \varepsilon_\parallel - \varepsilon_\perp$. The Fresnel equation takes the form

$$\left(\frac{p^2}{\varepsilon_\perp} - 1\right)\left[\frac{(p_x\sin\varphi + p_z\cos\varphi)^2}{\varepsilon_\perp} + \frac{(p_x\cos\varphi - p_z\sin\varphi)^2}{\varepsilon_\parallel} - 1\right] = 0. \quad (A1.7)$$

We are interested in the second of the roots of this equation, namely the one corresponding to the extraordinary wave. Assume that an e-polarized wave is incident on the cell from the air at an angle $\alpha_{\rm air}$ to the normal in the (x, z) plane. By virtue of the translational invariance of the problem to displacements along the x axis, we can seek a solution inside the medium in the form

$$\psi(x, z) = sx + \psi_1(z), \quad s = \sin\alpha_{\rm air}. \quad (A1.8)$$

It follows then from the Fresnel equation (A1.7) that

$$\psi_1(z) = \int_0^z p_z(z')\,dz', \quad p(z) = \frac{(\varepsilon_\parallel\varepsilon_\perp)^{1/2}(\varepsilon_\perp - s^2 + \varepsilon_a\cos^2\varphi)^{1/2} - s\varepsilon_a\sin\varphi\cos\varphi}{\varepsilon_\perp + \varepsilon_a\cos^2\varphi}, \quad (A1.9)$$

where $\varphi = \varphi(z)$.

The direction of the field vector $\mathbf{E}(\mathbf{r})$ is determined from the equation (A1.4):

$$\frac{E_z}{E_x} = \frac{sp_z(\varphi) + \varepsilon_a\sin\varphi\cos\varphi}{s^2 - \varepsilon_\perp - \varepsilon_a\cos^2\varphi}. \quad (A1.10)$$

The Poynting vector $\mathbf{S} = c\mathbf{E}\times\mathbf{H}^*/8\pi$ can be expressed in terms of the vector $\mathbf{E}$ with the aid of the equation $\mathbf{H} = \mathbf{p}\times\mathbf{E}$; this yields

$$\mathbf{S} = \frac{c}{8\pi}\{\mathbf{p}(\mathbf{E}\mathbf{E}^*) - \mathbf{E}^*(\mathbf{p}\mathbf{E})\}. \quad (A1.11)$$

In the problem with $\hat{\varepsilon} = \hat{\varepsilon}(z)$, for a propagating plane wave, the Poynting vector $\mathbf{S}(z)$ is itself far from constant. This statement can be verified in trivial fashion with a scalar medium having $\varepsilon = \varepsilon(z)$ as the example. In the absence of absorption, however, $\operatorname{div}\mathbf{S} = 0$, and for the problem homogeneous in x, y it follows therefore that $S_z(z)$ is constant. It is easy to verify that a field in the form

$$E_x(x,z) = A(\varepsilon_\perp - s^2 + \varepsilon_a\cos^2\varphi)^{1/4}\exp\left[i\frac{\omega}{c}(sx + \psi_1(z))\right],$$

$$E_z(x,z) = -A\frac{s(\varepsilon_\parallel\varepsilon_\perp)^{1/2} + \varepsilon_a\sin\varphi\cos\varphi(\varepsilon_\perp - s^2 + \varepsilon_a\cos^2\varphi)^{1/2}}{(\varepsilon_\perp + \varepsilon_a\cos^2\varphi)(\varepsilon_\perp - s^2 + \varepsilon_a\cos^2\varphi)^{1/4}} \times\exp\left[i\frac{\omega}{c}(sx + \psi_1(z))\right] \quad (A1.12)$$

is the solution, of interest to us, of Maxwell's equations in the geometrical-optics approximation. In this case A is constant and

$$S_z\left[\frac{\text{erg}}{\text{cm}^2\cdot\text{sec}}\right] = \frac{c}{8\pi}(\varepsilon_\parallel\varepsilon_\perp)^{1/2}|A|^2. \quad (A1.13)$$

We shall need in what follows double the value of the contribution made to the free energy by the interaction with the field:

$$q_E = 2F_E = -2\varepsilon_{ik}E_iE_k^*/16\pi = -\varepsilon_\parallel\varepsilon_\perp|A|^2/8\pi(\varepsilon_\perp - s^2 + \varepsilon_a\cos^2\varphi)^{1/2} = -(\varepsilon_\parallel\varepsilon_\perp)^{1/2}S_z/c(\varepsilon_\perp - s^2 + \varepsilon_a\cos^2\varphi)^{1/2}. \quad (A1.14)$$

We recall once more that $s = \sin\alpha_{\rm air} = \text{const}$ and $S_z = \text{const}$.

APPENDIX 2

HOW TO USE THE VARIATIONAL PRINCIPLE

It is known (see, e.g., the book by Landau and Lifshitz[21]) that the forces acting on a mechanical subsystem in an electric field can be obtained by varying the free energy

$$\Phi_E\left[\frac{\text{erg}}{\text{cm}^2\cdot\text{sec}}\right] = -\int\frac{\varepsilon_{ik}E_iE_k}{8\pi}dV, \quad \delta\Phi_E = -\int\frac{E_iE_k}{8\pi}\delta\varepsilon_{ik}\,dV \quad (A2.1)$$

at a fixed value of the field $\mathbf{E}$ (but not, e.g., the induction $D_i = \varepsilon_{ik}E_k$). As shown by Pitaevskiĭ,[14] for high-frequency fields $\mathbf{E}_{\rm real} = 0.5(\mathbf{E}e^{-i\omega t} + \mathbf{E}^*e^{i\omega t})$, with account taken of the frequency dispersion $\varepsilon_{ik}(\omega)$, Eq. (A2.1) retains the same form apart from the natural substitution $E_{i\,\rm real}E_{k\,\rm real} \to 0.5E_iE_k^*$. This result of Ref. 14, which by itself is not trivial, was obtained with account taken of the action of both the dispersion and of the contribution of the energy of the magnetic field that accompanies the time-varying electric fields. We emphasize once more that the correct expression

$$\Phi_E = -\int\frac{\varepsilon_{ik}E_iE_k^*}{16\pi}dV, \quad \delta\Phi_E = -\int\frac{E_iE_k^*}{16\pi}\delta\varepsilon_{ik}\,dV \quad (A2.2)$$

corresponds, as it were to account taken of only the contribution of the electric part of the energy. In problems involving the physics of liquid crystals, $\delta\varepsilon_{ik}$ is usually taken to mean

$$\delta\varepsilon_{ik} = [\partial\varepsilon_{ik}(\omega, n)/\partial n_l]\delta n_l, \quad (A2.3)$$

where $\mathbf{n}$ is the liquid-crystal-director vector.

It is easy to verify that the following relation holds in a plane electromagnetic wave in a general anisotropic medium

$$\varepsilon_{ik}E_iE_k^*/16\pi = (\mathbf{H}\mathbf{H}^*)/16\pi. \quad (A2.4)$$

It is tempting therefore to choose in place of Φ_E the quantity

$$\int q_E\,dV = Q_E = -\int\left(\frac{\varepsilon_{ik}E_iE_k^*}{16\pi} + \frac{(\mathbf{H}\mathbf{H}^*)}{16\pi}\right)dV = 2\Phi_E \quad (A2.5)$$

and use it for the calculations. As shown in Ref. 14, the use of the expression for Q_E at fixed fields $\mathbf{E}$ yields a result that is twice as large (and is by the same token incorrect!).

In nonlinear optics of liquid crystals, the situation is additionally complicated by the fact that in the case of propagation through a liquid crystal the complex-amplitude vector $\mathbf{E}(\mathbf{r})$ itself can be strongly changed as a result of distortions of the field of the director $\mathbf{n}(\mathbf{r})$ com-

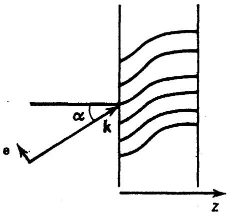

FIG. 6. NLC cell with perturbed director orientation, on which a light wave is incident from air at an angle α.

pared with the unperturbed field $E_0(\mathbf{r})$. For a Freédericksz' transition in a static field E, these effects are well known; they modify the expressions for the above-threshold stationary amplitude of the perturbation, see Ref. 17. Both in the static and in the optical field the expansion for $E(\mathbf{r}) - E_0(\mathbf{r})$ begins with terms $\propto\varepsilon_a$, therefore as $\varepsilon_a \to 0$ the difference between $E(\mathbf{r})$ and $E_0(\mathbf{r})$ can as a rule be neglected. In the visible band, however, $\varepsilon_a \simeq 1$ at $\varepsilon_\perp \approx 2$, i.e., the parameter $\varepsilon_a/\varepsilon_\perp$ is far from small. Therefore, besides using the correct expression for Φ_E and the correct method of its variation (A2.2) and (A2.3), it is necessary to substitute in the equations obtained for the director the fields $E(\mathbf{r})$ from the solutions of Maxwell's equations in the distorted structure.

In a large number of problems Maxwell's equations admit of simple explicit integrals of motion. Thus, e.g., when an infinite plane wave is incident at an arbitrary angle α (in air) on a cell with $\hat{\varepsilon} = \hat{\varepsilon}(z)$ (see Fig. 6), the transverse component of the wave vector is preserved (Snell's law) and the quantity S_z is concerned, where $\mathbf{S} = \mathrm{Re}(\mathbf{E} \times \mathbf{H}^* c/8\pi)$ is the Poynting vector.[4)]

Under these conditions there is one more temptation: to express the tensor $E_i(\mathbf{r})E_k(\mathbf{r})$ in terms of $\mathbf{n}(\mathbf{r})$ and conserved quantities, after which the obtained expression for $\Phi_E(\varepsilon_{ik}(\mathbf{n}), \mathbf{E}(\mathbf{n}))$ is varied with respect to n:

$$\delta\Phi_E \overset{?}{=} -\int \frac{1}{16\pi}\delta\varepsilon_{ik}E_i(\mathbf{n})E_k^*(\mathbf{n})\,dV - \frac{\varepsilon_{ik}}{16\pi}\int\left(\frac{\partial E_i}{\partial n_l}E_k^* + \frac{\partial E_k^*}{\partial n_l}E_i\right)\delta n_l\,dV. \tag{A2.6}$$

It is possible to make also another mistake, namely vary by the methods (A2.6) not the quantity Φ_E but the twice-as-large Q_E from (A2.5).

Both expression (A2.6) and its doubled value yield in the general case incorrect results. We present the following illustrative example. Assume that oblique incidence of the extraordinary wave produces in a cell with an NLC, as a result of reorientation of the director by the optical field, external self-focusing of light—see the theory and the direct experiment in Refs. 6 and 7. Let a sphere with a refractive index $\varepsilon_1^{1/2} > 1$ be placed in the region $\mathbf{r} \approx \mathbf{r}_1$ of the focus produced in air, see Fig. 7. Owing to the refraction by the inhomogeneities of the director, the field $\mathbf{E}(\mathbf{r}_1)$ at the point $\mathbf{r}_1$ is a functional of the entire distribution of $\mathbf{n}(\mathbf{r})$ in the cell:

$$\mathbf{E}(\mathbf{r}_1) = \mathscr{F}[\mathbf{n}(\mathbf{r})]. \tag{A2.7}$$

Thus, perturbations $\delta\mathbf{n}$ of one type cause an increase of the field (focusing), while other perturbations weaken the field $\mathbf{E}(\mathbf{r}_1)$ (cause defocusing), etc. The free energy Φ_E contains a contribution of the region of the sphere, which we write arbitrarily in the form

$$\Phi_E = -V_1\varepsilon_1|\mathbf{E}(\mathbf{r}_1)|^2/16\pi. \tag{A2.8}$$

Thus, there is a nonzero variational derivative of the free energy Φ_E with respect to $\delta\mathbf{n}(\mathbf{r})$, due to the presence of the sphere. At the same time, it is perfectly obvious that placing a dielectric sphere in the transmitted beam does not influence in any way the reorientation of the director in the cell! On the other hand, if $\delta E/\delta\mathbf{n}(\mathbf{r})$ is not taken into account, and the electric field $\mathbf{E}(\mathbf{r})$ is assumed fixed in the derivation of the variational equations, then no such paradoxes arise, as expected.

It is possible, of course, to regard as fixed not the field $\mathbf{E}(\mathbf{r})$ but the induction $\mathbf{D}(\mathbf{r})$. To this end, however, we must write

$$\delta\Phi = -\frac{1}{8\pi}\int(\delta\varepsilon\mathbf{E}^2 - 2\mathbf{D}\delta\mathbf{E})\,dV, \tag{A2.9}$$

where we assume the fields to be static in order to make (A2.9) more compact. A Legendre transformation from the independent variables n and E to the variables n and D yields then

$$\delta\tilde{\Phi} = -\frac{1}{8\pi}\int\left(\frac{\mathbf{D}^2}{\varepsilon^2}\delta\varepsilon - 2\mathbf{E}\delta\mathbf{D}\right)dV, \tag{A2.10}$$

where $\tilde{\Phi}_E = \Phi_E - \mathbf{E}\cdot\mathbf{D}/4\pi$. Of course, all the equations for n obtained with the aid of (A2.10) coincide at $\delta\mathbf{D} = 0$ with the equations that follows from (A2.9) at $\delta\mathbf{E} = 0$.

We have dwelled on these (generally known) statements in so much detail because many theoretical papers on optical nonlinearities of liquid crystal contain some or other of the errors discussed above.

In Ref. 13 the field $\mathbf{E}(\mathbf{r})$ was assumed to have the same value as in the unperturbed medium. For the calculation of the GON this would be a legitimate approximation. However, it is precisely for the OFT problem considered in Ref. 13 that this procedure is incorrect, see our paper[9] and Sec. 3 of the present paper. As a result the OFT threshold in Ref. 13 is underestimated by a factor $\varepsilon_\parallel/\varepsilon_\perp$ compared with the correct calculation.

In Ref. 12, apparently, the following procedure was used. The field was expressed in terms of the Poynting vector and the local orientation of the director, characterized by the angle $\varphi(z)$. The obtained expression was substituted in the doubled free energy

$$2F_E = q_E(\varphi, \mathbf{E}(\mathbf{n}(z)), \mathbf{E}^*(\mathbf{n}(z))).$$

[our q_E corresponds to $K_{33}G(\theta)$ in Ref. 12]. The equations for the equilibrium of the director were then obtained by varying $F_{\mathrm{Fr}} + q_E$:

$$\frac{\partial}{\partial z}\frac{\partial F_{\mathrm{Fr}}}{\partial(\partial\varphi/\partial z)} - \frac{\partial F_{\mathrm{Fr}}}{\partial\varphi} - \frac{dq_E}{d\varphi}\bigg|_{\mathbf{E}} = 0, \tag{A2.11}$$

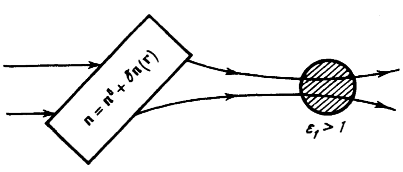

FIG. 7. Illustrating the discussion of the variational principle.

where F_{Fr} is the Frank energy and

$$\frac{dq_z}{d\varphi}=\frac{\partial q_z}{\partial\varphi}+\frac{\partial q_z}{\partial \mathbf{E}}\frac{\partial \mathbf{E}}{\partial\varphi}+\frac{\partial q_z}{\partial \mathbf{E}^*}\frac{\partial \mathbf{E}^*}{\partial\varphi}. \quad (A2.12)$$

In Appendix 1 we presented expression (A1.14) for q_E. Direct differentiation shows that

$$\frac{\delta F_z}{\delta\varphi}\bigg|_{E=\mathrm{const}}=-\frac{\varepsilon_a S_z}{c}(\varepsilon_\perp+\varepsilon_a\cos^2\varphi)^{-2}(\varepsilon_\perp-s^2+\varepsilon_a\cos^2\varphi)^{1/2}$$
$$\times\{(\varepsilon_\parallel\varepsilon_\perp)^{1/2}\sin\varphi\cos\varphi(\varepsilon_\perp+\varepsilon_a\cos^2\varphi-2s^2)$$
$$-s(\varepsilon_\perp-s^2+\varepsilon_a\cos^2\varphi)^{1/2}(\varepsilon_a\cos^2\varphi+\varepsilon_\perp\cos 2\varphi)\}, \quad (A2.13)$$

$$\frac{dq_z}{d\varphi}=2\frac{\partial F_z}{\partial\varphi}\bigg|_{E=\mathrm{const}}+2\frac{dF_z}{d\mathbf{E}}\frac{d\mathbf{E}}{d\varphi}+2\frac{dF_z}{d\mathbf{E}^*}\frac{d\mathbf{E}^*}{d\varphi}$$
$$=-\frac{(\varepsilon_\parallel\varepsilon_\perp)^{1/2}S_z}{c}\frac{\varepsilon_a\sin\varphi\cos\varphi}{(\varepsilon_\perp-s^2+\varepsilon_a\cos^2\varphi)^{1/2}}. \quad (A2.14)$$

That (A2.4) is in error can be seen already from the following considerations. At oblique incidence ($s=\sin\alpha\neq 0$) the correct expression (A2.13) gives a nonzero torque $\partial F_E/\partial\varphi$ in the first nonvanishing order in $|E|^2$ and as $\varphi\to 0$ in the right-hand side. The last result was most unambiguously confirmed by experimental observation of the GON.[6,7] In contrast, $dq_E/d\varphi$ in (A2.14) vanishes as $\varphi\to 0$ independently of the value of $s=\sin\alpha$, i.e., according to (A2.14) there should be no GON?!

In the OFT problem the angle $\alpha=0$, and by accident the different approaches (A2.13) and (A2.14) yield the same result in the root. By the same token, the authors of Ref. 12, using erroneous reasoning, would be able to obtain accidentally the correct result.

This, however, did not occur, since in Ref. 12 they made one additional error. Namely, they assumed that when a plane wave propagates in a medium with z-dependent properties, the length $|\mathbf{S}|$ of the Poynting vector is preserved, whereas the conservation equation $\operatorname{div}\mathbf{S}=0$ requires only that $S_z(Z)$ be constant. As a result, the equation obtained in Ref. 12 overestimates the OFT threshold by $\varepsilon_\parallel/\varepsilon_\perp$ times.

We note that if we take F in the form

$$F=\frac{1}{2}[K_{11}(\operatorname{div}\mathbf{n})^2+K_{22}(\mathbf{n}\times\operatorname{rot}\mathbf{n})^2+K_{33}[\mathbf{n}\times\operatorname{rot}\mathbf{n}]^2]-\frac{\varepsilon_{ik}E_iE_k^*}{16\pi}$$
$$+\frac{c^2}{\omega^2}\frac{1}{16\pi}\left[\frac{\partial E_i}{\partial x_k}\frac{\partial E_i^*}{\partial x_k}-\frac{\partial E_i}{\partial x_k}\frac{\partial E_k^*}{\partial x_i}\right], \quad (A2.15)$$

then the variation of (A2.15) with respect to $\mathbf{n}(\mathbf{r})$ at constant $\mathbf{E}$ yields the correct equations for $\mathbf{n}$, and variation with respect to $\mathbf{E}(\mathbf{r})$ yields Maxwell's equations (7). It is curious that for a field $\mathbf{E}(\mathbf{r})$ that satisfies exactly the variational equation (7) the sum of the contributions $-\varepsilon E^2$ (from the electric field) and

$$+[(\partial E_i/\partial x_k)(\partial E_i^*/\partial x_k)-(\partial E_i/\partial x_k)(\partial E_k^*/\partial x_i)]$$

(from the magnetic field) yields identically zero upon integration over all of space. The use of a single Lagrangian (A2.15) makes it possible to use in the problem the Noether theorem for finding nontrivial conservation laws.

1) After the conclusion of the present study, a paper was published[15] reporting experimental observation of doubling of the threshold and a number of rather interesting features of OFT for circularly polarized light.

2) These questions were investigated experimentally in Ref. 15.

3) See also the discussion of this question in Ref. 15.

4) It is appropriate to recall in this connection the trivial circumstances that neither $\alpha(z)$ nor $|\mathbf{S}(z)|$ are by themselves conserved.

1. B. Ya. Zel'dovich and N. V. Tabiryan, Pis'ma Zh. Eksp. Teor. Fiz. **30**, 510 (1979) [JETP Lett. **30**, 478 (1979)].
2. N. V. Tabiryan, B. Ya. Zel'dovich, Mol. Cryst. Liquid Cryst. **62**, 237 (1980); FIAN Preprint No. 63, 1980.
3. N. V. Tabiryan and B. Ya. Zel'dovich, Mol. Cryst. Liquid Cryst. **69**, 19 (1981); FIAN Preprint No. 62, 1980.
4. N. V. Tabiryan and B. Ya. Zel'dovich, Mol. Cryst. Liquid Cryst. **69**, 31 (1981); FIAN Preprint No. 61, 1980.
5. B. Ya. Zel'dovich and N. V. Tabiryan, Kvant. Elektron. (Moscow) **7**, 770 (1980) [Sov. J. Quantum Electron. **10**, 440 (1980)].
6. B. Ya. Zel'dovich, N. F. Pilipetskiĭ, A. V. Sukhov, and N. V. Tabiryan, Pis'ma Zh. Eksp. Teor. Fiz. **31**, 287 (1980) [JETP Lett. **31**, 263 (1980)].
7. N. F. Pilipetskii, A. V. Sukhov, N. V. Tabiryan, and B. Ya. Zel'dovich, Opt. Commun. **37**, 280 (1981).
8. A. S. Zolot'ko, V. F. Kitaeva, N. Kroo, N. I. Sobolev, and L. Csilag, Pis'ma Zh. Eksp. Teor. Fiz. **32**, 170 (1980) [JETP Lett. **32**, 158 (1980)].
9. B. Ya. Zel'dovich, N. V. Tabiryan, and Yu. S. Chilingaryan, Zh. Eksp. Teor. Fiz. **81**, 72 (1981) [Sov. Phys. JETP **54**, 32 (1981)].
10. S. R. Galstyan, O. V. Garibyan, N. V. Tabiryan, and Yu. S. Chilingaryan, Pis'ma Zh. Eksp. Teor. Fiz. **33**, 454 (1981) [JETP Lett. **33**, 437 (1981)].
11. I. C. Khoo, Phys. Rev. **A23**, 2077 (1981).
12. S. D. Durbin, S. M. Arakelian, and Y. R. Shen, Phys. Rev. Lett. **47**, 1411 (1981).
13. A. S. Zolot'ko, V. F. Kitaeva, N. I. Sobolev, and A. P. Sukhorukov, Zh. Eksp. Teor. Fiz. **81**, 933 (1981) [Sov. Phys. JETP **54**, 496 (1981)].
14. L. P. Pitaevskiĭ, *ibid.* **39**, 1450 (1960) [**12**, 1008 (1961)].
15. A. S. Zolot'ko, V. F. Kitaeva, N. Kroo, N. N. Sobolev, and L. Csillag, Pis'ma Zh. Eksp. Teor. Fiz. **34**, 263 (1981) [JEPT Lett. **34**, 250 (1981)].
16. P. G. De Gennes, Physics of Liquid Crystals, Oxford, 1974.
17. L. M. Blinov, Electro- i magnitooptika zhidkikh kristallov (Electro- and Magneto-optics of Liquid Crystals), Nauka, 1978.
18. B. Ya. Zel'dovich and N. V. Tabiryan, Zh. Eksp. Teor. Fiz. **79**, 2388 (1980) [Sov. Phys. JETP **52**, 1210 (1980)].
19. A. P. Sukhorukov *et al.*, Teoriya voln (Wave Theory), Nauka, 1979.
20. S. D. Durbin, S. M. Arakelian, and Y. R. Shen, Opt. Lett. **6**, 411 (1981).
21. L. D. Landau and E. M. Lifshitz, Electrodynamics of Continuous Media, Pergamon, 1959.

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