Perturbation theory analysis of the hydromagnetic dynamo problem

Yu. A. Brodskiĭ

Institute of Terrestrial Magnetism, the Ionosphere, and Radio-wave Propagation of the Academy of Sciences of the USSR

(Submitted 30 March 1981)
Zh. Eksp. Teor. Fiz. 82, 709–724 (March 1982)

We construct the integral equations which describe the problem of the kinematic planetary dynamo. Using these equations we propose a method for calculating the characteristics of the field, using perturbation theory. To do this we construct the vector Green function for the unperturbed equation that includes axisymmetrical differential rotation. When analyzing the perturbation-theory series we point out the special case of degeneracy of the damping of the axisymmetric toroidal and poloidal dipoles. We estimate within the framework of the theory developed here, for a number of models, the generation conditions and the values of the observed characteristics of the geomagnetic field.

PACS numbers: 91.25.Cw

1. INTRODUCTION

There is at the present time a generally accepted idea that the planetary magnetic field is the consequence of the motion of the conducting medium in the liquid core.[1-5] The basic equation for the analysis of the terrestrial (and also of the planetary) dynamo is the induction equation for the magnetic field in a frame of reference fixed at the Earth's surface, in the form

$$\frac{d\mathbf{H}}{dt}=-\operatorname{rot} D \operatorname{rot}\mathbf{H}+\operatorname{rot}[\mathbf{V}\times\mathbf{H}]. \tag{1}$$

Here $\mathbf{H}$ is the magnetic field strength, D the magnetic diffusion coefficient, $\mathbf{V}$ the velocity of the conducting medium, and the magnetic permeability is taken to be unity. For a complete solution of the problem Eq. (1) must be supplemented by a dynamic equation for the velocity $\mathbf{V}$ of the conducting medium.

Up to recently in most papers were restricted in the analysis of the equations for the planetary dynamo to a simplified kinematic statement of the problem, in which the velocity field of the conducting medium was assumed to be given while the field generation conditions were that Eq. (1) have solutions that did not vanish as $t\to\infty$. Even in this simple kinematic form the solution of the problem met with considerable difficulties. A large number of so-called anti-dynamo theorems[1] were found which showed the impossibility to generate the observed magnetic field at velocities subject to very simple restrictions (for instance, when axial symmetry was conserved). The problem of the general necessary and sufficient conditions for generation then remained, as was shown in Ref. 6, unsolved even in the kinematic formulation. A natural way to analyze Eqs. (1) for the case of arbitrary velocities is the use of perturbation theory. Braginskiĭ[7] was the first to suggest such an approach; he considered the problem with axisymmetric velocities of the conducting medium as the unperturbed problem. One is not allowed to choose an unperturbed problem corresponding to a very high degree of spherical symmetry in the velocities, as it is known from paleomagnetic data and from data about other planets that the magnetic axis is always close to the axis of rotation. For the same reason the non-axisymmetric motions cannot be large although their presence is necessary to violate the conditions for the anti-dynamo theorem. Further analysis of various variants of the dynamo theory, performed basically by using numerical computer calculations, led either to contradictory results[8,9] or started from conditions which are certainly not satisfied in the Earth.[10]

Considerable difficulties arise from the necessity to satisfy the restrictions which are imposed on the dynamo mechanism by the presence of an upper limit, allowable independent considerations, on the velocities in the liquid core. It is also necessary to have agreement between the results and the observed spatial-temporal structure of the magnetic field, namely the angle of inclination of the magnetic axis, the ratio of the amplitudes of the higher field harmonics to that of the dipole, the presence of inversions, and so on. It turned out that it was far from trivial to satisfy simultaneously the restrictions and the conditions for generation. In particular, in Ref. 11 this was attained by imposing rather artificial conditions on the functional form of the motions in the conducting medium.

In connection with what has been said above we consider in the present paper consistently a method for analyzing the induction equation, using perturbation theory based upon a study of integral equations. We have included here in the unperturbed equation a term with axisymmetric differential rotation. We show in the second section that the unperturbed vector Green function in the integral equation can be expressed in terms of two scalar Green functions which satisfy ordinary differential equations. When analyzing the spectral properties of the latter Green functions it turns out to be possible to use the efficient apparatus of the theory of ordinary differential equations.

2. BASIC EQUATIONS

The induction equation for the field $\mathbf{H}$ in the reference frame which is fixed in the Earth's surface can be written as follows:

$$\frac{d\mathbf{H}(\mathbf{r},t)}{dt}+\operatorname{rot} D(r)\operatorname{rot}\mathbf{H}(\mathbf{r},t)-\operatorname{rot}[[\Omega(r)\mathbf{e}_z\times\mathbf{r}]\mathbf{H}\times(\mathbf{r},t)]$$
$$-(\hat{K}\mathbf{H})(\mathbf{r},t)=0,\quad (\hat{K}\mathbf{H})_i(\mathbf{r},t)=\operatorname{rot}_i[\mathbf{V}(\mathbf{r})\times\mathbf{H}(\mathbf{r},t)]. \tag{2}$$

We shall call Eq. (2) with $\hat{K}=0$ the unperturbed equation. We have included in the unperturbed equation a term that takes into account differential rotation around the third axis e_3 with a radius-dependent angular velocity $\Omega(r)$. We denote by $D(r)$ the magnetic diffusion coefficient, which is inversely proportional to the conductivity. We assume for simplicity that $D(r)$ depends solely on the radius. The velocity $\mathbf{V}(\mathbf{r})$ of the conducting medium is contained in the perturbation; we write it in the form

$$\mathbf{V}(\mathbf{r})=\mathbf{V}^T(\mathbf{r})+\mathbf{V}^p(\mathbf{r}),$$
$$\mathbf{V}^T(\mathbf{r})=\operatorname{rot}[\hat{\mathbf{r}}f^T(\mathbf{r})],\quad \mathbf{V}^p(\mathbf{r})=\operatorname{rot}\operatorname{rot}[\hat{\mathbf{r}}f^p(\mathbf{r})], \tag{3}$$

where $f^p(\mathbf{r})$ and $f^T(\mathbf{r})$ are some scalar functions. Following the usual definitions in dynamo theory we call $\mathbf{V}^T$ the toroidal and $\mathbf{V}^p$ the poloidal component of the velocity. The choice (3) corresponds to a zero divergence of the velocity. This condition is not an important restriction as, according to Vaĭnshteĭn's theorem[12] a velocity which has the form of a gradient cannot lead to generation.

We construct for the unperturbed induction equation a retarded Green function $G_{ij}(\mathbf{r},\mathbf{r}',t-t')$ which satisfies the equation

$$\frac{d}{dt}G_{ij}(\mathbf{r},\mathbf{r}',t-t')+e_{isk}e_{kmn}\nabla_s D(r)\nabla_m G_{nj}(\mathbf{r},\mathbf{r}',t-t')$$
$$-e_{ipq}e_{qsl}\nabla_p[\Omega(r)\mathbf{e}_3.\,\mathbf{r}]_s G_{lj}(\mathbf{r},\mathbf{r}',t-t')=\delta_{ij}\delta(t-t')\delta^3(\mathbf{r}-\mathbf{r}'),$$

where e_{ijk} is a completely antisymmetric tensor. Summation over repeated indices is implied. We take the Fourier transform with respect to t and expand in terms of the vector spherical harmonics of the field $\mathbf{H}(\mathbf{r},t)$:

$$\mathbf{H}(\mathbf{r},t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}d\omega\Big\{e^{-i\omega t}\sum_{JM}\frac{1}{r}\Big[H^T(r,J,M,\omega)\mathbf{Y}_{JM}^{(m)}(\hat{\mathbf{r}})$$
$$+\frac{D^{1/2}(r)}{r}H^p(r,J,M,\omega)\mathbf{Y}_{JM}^{(0)}(\hat{\mathbf{r}})+\frac{d}{dr}\Big(\frac{D^{1/2}(r)}{[J(J+1)]^{1/2}}H^p(r,J,M,\omega)\mathbf{Y}_{JM}^{(e)}(\hat{\mathbf{r}})\Big)\Big]\Big\}. \tag{4}$$

In (4) we have allowed for the fact that the divergence of the magnetic field vanishes and we denote, respectively, by $\mathbf{Y}_{JM}^{(m)}(\hat{\mathbf{r}})$, $\mathbf{Y}_{JM}^{(0)}(\hat{\mathbf{r}})$, $\mathbf{Y}_{JM}^{(e)}(\hat{\mathbf{r}})$ the magnetic, longitudinal, and electric vector harmonics in Newton's notation (Ref. 13).[1)] The scalar functions $H^{p,T}(r,J,M,\omega)$ give, respectively, the poloidal and toroidal field components. We shall drop in what follows completely or partially the arguments of the functions where this does not cause confusion. At $\hat{K}=0$, Eq. (2) takes, after substituting the expansion (4), the form of a set of equations for $H^p(r,J,M)$ and $H^T(r,J,M)$:

$$D^{1/2}(r)\frac{d^2}{dr^2}D^{1/2}(r)H^p(J,M)$$
$$-\frac{D(r)J(J+1)}{r^2}H^p(J,M)-iM\Omega(r)H^p(J,M)+i\omega H^p(J,M)=0, \tag{5a}$$

$$\frac{d}{dr}D(r)\frac{d}{dr}H^T(J,M)-\frac{D(r)J(J+1)}{r^2}H^T(J,M)-iM\Omega(r)H^T(J,M)$$
$$+i\omega H^T(J,M)+D^{1/2}(r)\frac{d\Omega(r)}{dr}\Big\{H^p(J-1,M)\Big[\frac{(J+1)(J^2-M^2)}{J(4J^2-1)}\Big]^{1/2}$$
$$-H^p(J+1,M)\Big[\frac{J[(J+1)^2-M^2]}{(4(J+1)^2-1)(J+1)}\Big]^{1/2}\Big\}=0. \tag{5b}$$

The set (5) has a characteristic triangular structure and accordingly no components of the toroidal field occur in the equations for the poloidal components. We introduce scalar Green functions $G_{J,M}^{(e)}(r,r',\omega)$ and $G_{JM}^{(m)}(r,r',\omega)$ which satisfy the following equations:

$$\Big[D^{1/2}(r)\frac{d^2}{dr^2}D^{1/2}(r)-\frac{D(r)J(J+1)}{r^2}-iM\Omega(r)+i\omega\Big]G_{JM}^{(e)}(r,r',\omega)=-\delta(r-r')$$
$$\Big[\frac{d}{dr}D(r)\frac{d}{dr}-\frac{D(r)J(J+1)}{r^2}-iM\Omega(r)+i\omega\Big]G_{JM}^{(m)}(r,r',\omega)=-\delta(r-r'). \tag{6}$$

The functions $G_{JM}^{(e)}(\omega)$ and $G_{JM}^{(m)}(\omega)$ are constructed in the standard way from a pair of fundamental solutions satisfying the zero boundary conditions at $r=0,\infty$, respectively. It is important that $G_{JM}^{(m,e)}(\omega)$ are meromorphic functions of $i\omega$ with poles for values of $i\omega$ equal to the eigenvalues of the operators on the left-hand side of (6). One shows easily that all these eigenvalues $i\omega^{(e)}(J,M)$ and $i\omega^{(m)}(J,M)$ satisfy the condition

$$\operatorname{Re} i\omega^{(e,m)}(J,M)>0, \tag{7}$$

that corresponds to impossibility of generation. One can directly check that the total retarded Green function $G_{ij}(\mathbf{r},\mathbf{r}',t-t')$ of the unperturbed Eq. (3) can be written in the form

$$G_{ij}(\mathbf{r},\mathbf{r}',t-t')=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}d\omega\{e^{-i\omega(t-t')}G_{ij}(\mathbf{r},\mathbf{r}',\omega)\}$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}d\omega\, e^{-i\omega(t-t')}\sum_{JM}\Big\{\Big[\frac{D^{1/2}(r)}{r^2}\mathbf{Y}_{JM}^{(0)}(\hat{\mathbf{r}})$$
$$+\frac{1}{r}\frac{d}{dr}\Big(\frac{D^{1/2}(r)}{[J(J+1)]^{1/2}}\Big)\mathbf{Y}_{JM}^{(e)}(\hat{\mathbf{r}})\Big]_i G_{JM}^{(e)}(r,r',\omega)D^{-1/2}(r')[\mathbf{Y}_{JM}^{(0)*}(\hat{\mathbf{r}}')]_j$$
$$-\sum_{J'}\frac{1}{r}[\mathbf{Y}_{JM}^{(m)}(\hat{\mathbf{r}})]_i\int dr''\,G_{JM}^{(m)}(r,r'',\omega)$$
$$\times\frac{d\Omega(r'')}{dr''}\Big(\Big[\frac{J((J+1)^2-M^2)}{(J+1)(4(J+1)^2-1)}\Big]^{1/2}\delta_{JJ'-1}$$
$$-\Big[\frac{(J+1)(J^2-M^2)}{(4J^2-1)J}\Big]^{1/2}\delta_{JJ'+1}\Big)G_{J'M}^{(e)}(r'',r',\omega)D^{-1/2}(r')[\mathbf{Y}_{J'M}^{(0)*}(\hat{\mathbf{r}}')]_j$$
$$+\frac{1}{r}[\mathbf{Y}_{JM}^{(m)}(\hat{\mathbf{r}})]_i G_{JM}^{(m)}(r,r',\omega)\frac{1}{r'}[\mathbf{Y}_{JM}^{(m)*}(\hat{\mathbf{r}}')]_j\Big\}. \tag{8}$$

When writing down (8) we have taken into account the fact that the Green function $G_{ij}(\mathbf{r},\mathbf{r}',t-t')$ acts only on vector functions with zero divergence. As in Eq. (5), there are no transitions from toroidal to poloidal field components in (8). That $G_{ij}(\mathbf{r},\mathbf{r}',t-t')$ vanishes when $t<t'$ (initial condition) follows from the fact, noted above, that the functions $G_{JM}^{(e)}(r,r',\omega)$ and $G_{JM}^{(m)}(r,r',\omega)$ have no singularities in the lower complex half-plane of ω. Using (8) one can replace the complete induction equation together with the boundary conditions by an integral equation of the following form:

$$\mathbf{H}_i(\mathbf{r},t)=\mathbf{H}_i^0(\mathbf{r},t)+\int d^3\mathbf{r}'\int_{-\infty}^{+\infty}dt'\,G_{ij}(\mathbf{r},\mathbf{r}',t-t')(\hat{K}\mathbf{H})_j(\mathbf{r}',t'). \tag{9}$$

Here $\mathbf{H}_i^0(\mathbf{r},t)$ is an arbitrary solution of the unperturbed equation.

We consider now the shift of the eigenvalues of the induction equation when the perturbation is switched on. We shall in this case start from Eq. (9) in which we substitute for the first term on the right-hand side the solution $\mathbf{H}^0(\mathbf{r},\omega_n)e^{-i\omega_n t}$ of the unperturbed Eq. (5) corresponding to a definite damping rate (eigenvalue) $i\omega_n$.

We shall be especially interested in the solution $\mathbf{H}^0(\mathbf{r}, \omega_0)e^{-i\omega_0 t}$ with the smallest value of the real part $i\omega_0$. For this solution there is the largest probability for a change in sign of the real part after the perturbation is switched on and, indeed, for the occurrence of magnetic-field generation. It is natural to assume that $i\omega_0$ corresponds to the damping rate of a solution corresponding to a poloidal dipole. In that case the solution of (5) corresponding to an axisymmetric poloidal dipole has the form

$$\mathbf{H}_1^0(\mathbf{r}, t) = \mathbf{H}_1^0(\mathbf{r}) e^{-i\omega_0 t} = e^{-i\omega_0 t}\left\{\frac{D^{1/2}(r)}{r^2} H^{p0}(r, 1, 0)\mathbf{Y}_{10}^{(0)}(\mathbf{r}) + \frac{1}{r}\frac{d}{dr}\left[\left(\frac{D(r)}{2}\right)^{1/2} H^{p0}(r, 1, 0)\right]\mathbf{Y}_{10}^{(e)}(\hat{\mathbf{r}}) - D^{1/2}(r)\mathbf{Y}_{20}^{(m)}(\hat{\mathbf{r}})\int dr'\left[G_{20}^{(m)}\frac{d\Omega(r')}{dr'}\sqrt{\frac{2}{5}} H^{p0}(r', 1, 0)\right]\right\}. \tag{10}$$

In (10) the function $H^{p0}(r, 1, 0)$ is the solution of the first of Eqs. (5) for $\omega = \omega_0$. It is then convenient to choose for $H^{p0}(r)$ the normalization

$$\int H^{p0*}(r) H^{p0}(r)\, dr = 1. \tag{11}$$

Following the usual scheme we transform the integral Eq. (9) in such a way that we eliminate its secular terms. We then get the set

$$\mathbf{H}_i(r, \omega) = \mathbf{H}_i^0(r, \omega_0) + \int d^3\mathbf{r}'\{G_{ij}'(\mathbf{r}, \mathbf{r}', -i\omega_0 - \Delta_0^p)(\hat{K}\mathbf{H})_j(r')\},$$
$$i\omega = i\omega_0 + \Delta_0^p, \quad \Delta_0^p = -\int d\mathbf{r}^3\{\mathbf{H}_i^*(r)(\hat{K}\mathbf{H})_i(r)\}. \tag{12}$$

We have denoted in (12) by $G_{ij}'(\omega)$ the modified Green function of the unperturbed equation, obtained from expression (8) for $G_{ij}(\omega)$ by eliminating the terms which are singular when $\omega = \omega_0$. To do this it is sufficient to replace (regularize) the function $G_{JM}^{(e)}(\omega)$ contained in $G_{ij}(\omega)$ by

$$G_{JM}^{(e)}(\omega) \to G_{JM}^{(e)\prime}(\omega) = G_{JM}^{(e)}(\omega) - \frac{H_i^{p0}(r) H_i^{p0*}(r')}{i\omega - i\omega_0}. \tag{13}$$

The expression for Δ_0^p in (12) determines the eigenvalue shift which specifies the time dependence of the field. The set (12) will be the basis for constructing the perturbation-theory series in the non-degenerate case. Recognizing that $\operatorname{div}\mathbf{H} = 0$ and also the symmetry of the unperturbed problem, we shall use in what follows an expansion in a spherical basis and write $\mathbf{H}$ as a sum of a poloidal, $\mathbf{H}^p$, and a toroidal, $\mathbf{H}^T$, components.

One can see the meaning of the notation used in what follows from the equations:

$$(\hat{K}\mathbf{H})^\alpha(J, M, r) = \sum_{J'M'}\int_0^\infty dr'[\hat{K}^{\alpha p}(r, r', J, J', M, M') \times H^p(r', J', M') + \hat{K}^{\alpha T}(r, r', J, J', M, M') H^T(r', J', M')], \quad \alpha = p, T. \tag{14}$$

The operators $\hat{K}^{pp}(r, r')$, $\hat{K}^{pT}(r, r')$, $\hat{K}^{TT}(r, r')$, and $\hat{K}^{Tp}(r, r')$ introduced in (14) have the following meaning:

$$\hat{K}^{pp}(r, r', J, J', M, M') = -\int d\varphi\, d\cos\theta\left\{D^{-1/2}(r)[J(J+1)]^{1/2} \times \mathbf{Y}_{JM}^{(m)*}(\hat{\mathbf{r}})\left[\mathbf{V}(\mathbf{r}), \frac{D^{1/2}(r')}{r'}\mathbf{Y}_{J'M'}^{(0)}(\hat{\mathbf{r}}') + \frac{d}{dr'}\left[\left(\frac{D(r')}{J'(J'+1)}\right)^{1/2}\right]\mathbf{Y}_{J'M'}^{(e)}(\hat{\mathbf{r}}')\right]\delta(r - r')\right\}, \tag{15}$$

$$\hat{K}^{pT}(r, r', J, J', M, M') = -\int d\varphi\, d\cos\theta\{D^{-1/2}(r)[J(J+1)]^{1/2}\mathbf{Y}_{JM}^{(m)*}(\hat{\mathbf{r}}) \times [\mathbf{V}(\mathbf{r}), \mathbf{Y}_{J'M'}^{(m)}(\hat{\mathbf{r}}')]\delta(r - r')\},$$

$$\hat{K}^{TT}(r, r', J, J', M, M') = \int d\varphi\, d\cos\theta \times\left\{\frac{1}{r}\mathbf{Y}_{JM}^{(m)*}(\hat{\mathbf{r}})\operatorname{rot}\left[\mathbf{V}(\mathbf{r}), \frac{1}{r'}\mathbf{Y}_{J'M'}^{(m)}(\hat{\mathbf{r}}')\right]\delta(r - r')r'^2\right\},$$

$$\hat{K}^{Tp}(r, r', J, J', M, M') = \int d\varphi\, d\cos\theta\left\{\frac{1}{r}\mathbf{Y}_{JM}^{(m)*}(\hat{\mathbf{r}}) \times \operatorname{rot}\left[\mathbf{V}(\mathbf{r}), \frac{D^{1/2}(r')}{r'^2}\mathbf{Y}_{J'M'}^{(0)}(\hat{\mathbf{r}}') + \frac{1}{r}\frac{d}{dr}\left[\left(\frac{D(r)}{J'(J'+1)}\right)^{1/2}\right]\mathbf{Y}_{J'M'}^{(e)}(\mathbf{r}')\right]\delta(r - r')r'^2\right\}.$$

The basic quantity Δ_0^p can be expressed as follows in terms of the operators $\hat{K}^{\alpha\alpha'}$:

$$\Delta_0^p = -\sum_{J',M'}\int dr'\,dr''\{H^{p0}(r'')^*[\hat{K}^{pp}(r'', r', 1, J', 0, M') H^p(r', J', M', \omega) + \hat{K}^{pT}(r'', r', 1, J', 0, M') H^T(r', J', M', \omega)]\}. \tag{16}$$

The discussion given here must be modified when the eigenvalue $i\omega_0$ is degenerate. In the problem considered, greatest interest attaches to the possibility of exact or approximate equality of two eigenvalues corresponding to symmetric poloidal and toroidal dipoles. Accordingly we assume in what follows that the eigenvalue $i\omega_0$ corresponds simultaneously to two solutions of the unperturbed equation: the solution $\mathbf{H}_1^0(\mathbf{r}, t)$ of the form (10) and also the solution

$$\mathbf{H}_2^0(\mathbf{r}, t) = \mathbf{H}_2^0(\mathbf{r}) e^{-i\omega_0 t} = e^{-i\omega_0 t}\frac{H^{T0}(r, 1, 0)}{r}\mathbf{Y}_{10}^{(m)}(\mathbf{r}). \tag{16a}$$

We denoted in (16a) by $H^{T0}(r, 1, 0)$ the solution of Eq. (5b) at $J = 1$, $M = 0$, and the terms proportional to $d\Omega(r)/dr$ set equal to zero. ' In the case of degeneracy we must, by analogy with the usual perturbation theory at the degenerate level, choose special combinations

$$\mathbf{H}_s^0 = \alpha_s\mathbf{H}_1^0(\mathbf{r}) + \beta_s\mathbf{H}_2^0(\mathbf{r}),$$

where α_s and β_s are constants to be determined ($s = 1, 2$). One can use these combinations, taking them as the first terms of the right-hand side of (12), to construct two solutions $\mathbf{H}_s(\mathbf{r})$ which depend linearly on α_s and β_s. The quantities α_s, β_s and Δ_s are determined from the conditions that the secular terms be excluded. These conditions reduce to the requirement of satisfying at $s = 1, 2$ the relations

$$\sum_{J'M'}\int dr\,dr'\{H^{p0*}(r, 1, 0)[\hat{K}^{pp}(r, r', 1, J', 0, M') H_s^p(r', J', M', \omega) + \hat{K}^{pT}(r, r', 1, J', 0, M') H_s^T(r', J', M', \omega)]\} + \alpha_s\Delta_s = 0, \tag{17}$$

$$\sum_{J'M'}\int dr\,dr'\left\{H^{T0*}(r, 1, 0)\left[\hat{K}^{Tp}(r, r', 1, J', 0, M') H_s^p(r', J', M', \omega) + \hat{K}^{TT}(r, r', 1, J', 0, M') H_s^T(r', J', M', \omega) - \int dr''\left\{\frac{d\Omega(r'')}{dr''}\sqrt{\frac{2}{5}} G_{20}^{(e)}(r, r', \omega_0)[\hat{K}^{pp}(r', r'', 2, J', 0, M') \times H_s^p(r'', J', M', \omega) + \hat{K}^{pT}(r', r'', 2, J', 0, M') H_s^T(r'', J', M', \omega)]\right\}\right]\right\} + \beta_s\Delta_s = 0.$$

3. PERTURBATION THEORY

In the analysis of the integral equations the principal role is played by the operators $\hat{K}^{\gamma\gamma'}(r, r')$, with $\gamma, \gamma' = p, T$, introduced in (15). They satisfy the following relations:

$$\hat{K}^{pT}(r, r', J, J', 0, 0)=0, \tag{18a}$$

$$\hat{K}^{pT}(r, r'\ J, J', M, M')=0 \text{ when } \mathbf{V}(\mathbf{r})=\sum_{JM} v(r)\mathbf{Y}_{JM}^{(m)}(\hat{\mathbf{r}}), \tag{18b}$$

$$\hat{K}^{pp}(r, r', J, J', 0, 0)=\hat{K}^{TT}(r, r', J, J', 0, 0)=0 \text{ when } \mathbf{V}(\mathbf{r})=\sum_{J} v(r)\mathbf{Y}_{J0}^{(m)}(\hat{\mathbf{r}}). \tag{18c}$$

These must be understood to be operator equations, i.e., e.g., (18a) corresponds to vanishing of an integral of the form

$$\int dr dr' \{N_1^*(r)\hat{K}^{pT}(r, r', J, J', 0, 0)N_2(r')\}=0,$$

where $N_1^*(r)$ and $N_2(r)$ are functions of the radius which vanish when $r=0, \infty$. Precisely such integrals enter in all the formulae which we consider. The proof of (18a) and (18b) follows from (15) if it is recognized that in the axisymmetric case (i.e., when the angular-momentum components vanish) the spherical vector functions $\mathbf{Y}_{J0}^{(0)}$, $\mathbf{Y}_{J0}^{(e)}$, $\mathbf{Y}_{J0}^{(m)}$ for all J are directed, respectively, along the orthogonal unit vectors $\hat{\mathbf{r}}$, $\hat{\theta}$, and $\hat{\varphi}$. The proof of (18c) follows from the vanishing of the triple product of vectors lying in one plane. It is important that it does not follow from (18a) that $\hat{K}^{Tp}(r, r', J, J', 0, 0)$ vanishes.

It is important for what follows to know the properties of the operators $\hat{K}^{\gamma\gamma'}(r, r')$ for the case when they contain only poloidal or only toroidal velocity components. In that case we get from the condition that the velocity be real and from the rule for the behavior of spherical vector functions, when the complex the conjugate is taken,

$$\begin{aligned}
\hat{K}^{pp*}(r, r', J, J', M, M')&=(-1)^f\hat{K}^{pp}(r, r', J, J', -M, -M'),\\
\hat{K}^{TT*}(r, r', J, J', M, M')&=(-1)^f\hat{K}^{TT}(r, r', J, J', -M, -M'),\\
\hat{K}^{pT*}(r, r', J, J', M, M')&=(-1)^{f+1}\hat{K}^{pT}(r, r', J, J', -M, -M'),\\
\hat{K}^{Tp*}(r, r', J, J', M, M')&=(-1)^{f+1}\hat{K}^{Tp}(r, r', J, J', -M, -M').
\end{aligned} \tag{19}$$

Here $f=0$ for poloidal velocities and $f=1$ for toroidal velocities.

In the non-degenerate case we get from (12) and (16) a perturbation-theory series for Δ_0^p in the form

$$\Delta_0{}^p=\Delta_0{}^{p1}+\Delta_0{}^{p2}+\ldots,\quad \Delta_0{}^{p1}=-H^{p0*}(r, 1, 0)\hat{K}^{pp}(r, r', 1, 1, 0, 0)H^{p0}(r', 1, 0),$$

$$\begin{aligned}
\Delta_0{}^{p2}=&-H^{p0*}(r, 1, 0)\hat{K}^{pp}(r, r'\ 1, J, 0, M)G_{JM}^{(e)}(r, r'', \omega_0)\\
&\times\Big[\hat{K}^{pT}(r'', r''', J, 2, M, 0)G_{20}^{(m)}(r''', r^{IV}, \omega_0)D^{1/2}(r^{IV})\\
&\times\frac{d\Omega(r^{IV})}{dr^{IV}}\left(-\sqrt{\frac{2}{5}}\right)H^{p0}(r^{IV}, 1, 0)+\hat{K}^{pp}(r'', r''', J, 1, M, 0)\\
&\times H^{p0}(r''', 1, 0)\Big]+H^{p0*}(r, 1, 0)\hat{K}^{pT}(r, r', 1, J, 0, M)\\
&\times G_{JM}^{(m)}(r, r'', \omega_0)\Big\{\hat{K}^{Tp}(r'', r''', J, 1, M, 0)H^{p0}(r''', 1, 0)\\
&+\hat{K}^{TT}(r'', r''', J, 2, M, 0)G_{20}^{(m)}(r''', r^{IV}, \omega_0)D^{1/2}(r)\left(-\sqrt{\frac{2}{5}}\right)\\
&\times\frac{d\Omega(r^{IV})}{dr^{IV}}H^{p0}(r^{IV}, 1, 0)-\frac{d\Omega(r'')}{dr''}\delta_{MM'}G_{J'M'}^{(e)}(r'', r''', \omega_0)\\
&\times\left(\left[\frac{J((J+1)^2-M^2)}{(4(J+1)^2-1)(J+1)}\right]^{1/2}\delta_{JJ'-1}-\left[\frac{(J+1)(J^2-M^2)}{J(4J^2-1)}\right]^{1/2}\right.\\
&\left.\times\delta_{JJ'+1}\right)\Big[\hat{K}^{pp}(r''', r^{IV}, J', 1, M, 0)H^{p0}(r^{IV}, 1, 0)\\
&+\hat{K}^{pT}(r''', r^{IV}, J', 2, M', 0)\left(-\sqrt{\frac{2}{5}}\right)G_{20}^{(m)}(r^{IV}, r^{V}, \omega_0)D^{1/2}(r)\\
&\times\frac{d\Omega(r^{V})}{dr^{V}}H^{p0}(r^{V}, 1, 0)\Big]\Big\}.
\end{aligned} \tag{20}$$

We imply in (20) summation over the indexes J, M, J', M', and integration over all variables $r, \ldots, r^V$ from 0 to ∞.

The expression for the shift in the lowest eigenvalue corresponding to a toroidal field is also of interest. Repeating the reasoning leading to (16) we get for the eigenvalue corresponding to a toroidal dipole the expression

$$\Delta_0{}^T=-\sum_{JM}\int dr dr' \{H^{T0*}(r, 1, 0)[\hat{K}^{Tp}(r, r')H^p(r', J, M)+\hat{K}^{TT}(r, r')H^T(r', J, M)]\}. \tag{21}$$

It is important that when there is a "large" axisymmetric toroidal motion (e.g., differential rotation) present, Δ_0^T can be less than Δ_0^P. This follows from the fact that, according to (18), only the operator $\hat{K}^{Tp}(r, r')$ is non-zero for that kind of motion.

By considering the signs of the terms in the perturbation theory series for Δ_0^p we can prove the main antidynamo theorems: Cowling's theorem,[6] Backus' theorem,[12] and Vaĭnshteĭn's theorem.[12]

Of most interest is the elucidation of the necessary conditions under which generation occurs. In order to advance in that direction we use the pole approximation for the scalar Green functions in (20). This approximation corresponds to replacing the kernel of the integral equation (12) by a finite-rank operator.[13] We then retain in the spectral representation for the scalar Green functions, corresponding to the unperturbed equations, only terms[2)] corresponding to the lowest eigenvalues $i\omega_n$, for which $|i\omega_n - i\omega_0| \leq i\omega_0$. These terms give the main contribution in the case $|\Delta_0^p| \approx |i\omega_0|$ which is of interest to us when we consider generation. In this approximation the series can be summed[3)] (see the Appendix). In that case (as in the case when one sums a geometric series) the sign and the order of magnitude of Δ_0^p are determined by the first terms of the perturbation-theory series, evaluated up to the order which is equal to the number of terms retained in the spectral representation of the Green function, provided that the displaced higher eigenvalues $i\omega_n + \Delta_n$ remain larger than $i\omega_0 + \Delta_0^p$. If, however, this last condition is violated for a finite number of eigenvalues, one must go over to a calculation scheme which takes degeneracy into account.

We consider the properties of the perturbation-theory series (20). First of all we note that the terms of the series for Δ_0^p which contain only toroidal velocities make a positive contribution to $\mathrm{Re}\,\Delta_0^p$, as can be verified in agreement with Backus' theorem,[14] and can therefore not lead to magnetic-field generation. We further show that as a consequence of (19) the terms in the series for Δ_0^p are real. As an example we consider the first term on the right-hand side of Eq. (20) for Δ_0^{p2} corresponding to a total angular momentum J and to a z-component M. It equals

$$\Theta(J, M)=H^{p0*}(r)\,\tilde{K}^{pp}(r, r', 1, J, 0, M)\,G^{(e)}_{JM}(r', r''; \omega_0)$$

$$\times\tilde{K}^{pT}(r'', r''', J, 2, M, 0)\sqrt{\frac{2}{5}}G^{(m)}_{20}\frac{d\Omega(r^{\mathrm{IV}})}{dr^{\mathrm{IV}}}D^{1/2}(r^{\mathrm{IV}})\,H^{p0}(r^{\mathrm{IV}}). \tag{22}$$

Here, as in (20), we assume integration over all variables $r, \ldots, r^{\mathrm{IV}}$. We use the obvious relation

$$G^{(e,m)*}_{JM}(r, r', \omega_0)=G^{(e,m)}_{J-M}(r, r', \omega_0),$$

Eqs. (19), and the fact that the velocity of the conducting medium is real, and easily find that

$$\Theta^*(J, M)=\Theta(J, M). \tag{23}$$

It follows from (23) that the imaginary part of the term corresponding to (22) in the series for Δ_0^p vanishes.

We now turn to the degenerate case. Degeneracy of the lowest eigenvalues in Eqs. (5) is directly possible only for a special form of $D(r)$. More realistic is the possibility of dynamic degeneracy caused by the effect of axisymmetric motions. It follows from the discussion following Eq. (21) that axisymmetric motions must cause the lowest eigenvalue $i\omega_0^p$ corresponding to a poloidal dipole to approach closer the lowest eigenvalue $i\omega_0^T$ corresponding to a toroidal dipole. For axisymmetric velocities for which $i\omega_0^p$ and $i\omega_0^T$ are approximately equal it is necessary to use the perturbation theory at the degenerate level. In order not to complicate the kinematics, we take into account in the exposition which follows the effect of degeneracy-causing axisymmetric velocities by introducing additional shifts $\tilde{\Delta}_0^p$ and $\tilde{\Delta}_0^T$ of the eigenvalues in the unperturbed equations. After this addition the equations of the type (5) take at $J=1$ and $M=0$ the form

$$\begin{gathered}\left[D^{1/2}(r)\frac{d^2}{dr^2}D^{1/2}(r)-\frac{\sqrt{2}D(r)}{r^2}+(i\omega_0{}^p+\tilde{\Delta}_0{}^p)\right]H^p(1,0)=0,\\ \left[\frac{d}{dr}D(r)\frac{d}{dr}-\frac{\sqrt{2}D(r)}{r^2}+(i\omega_0{}^T+\tilde{\Delta}_0{}^T)\right]H^T(1, 0)=0,\\ i\omega_0{}^p+\tilde{\Delta}_0{}^p=i\omega_0{}^T+\tilde{\Delta}_0{}^T.\end{gathered} \tag{24}$$

The shift of the eigenvalue in the degenerate case is found by setting the secular determinant corresponding to (17) equal to zero. It follows from (18a) that in first approximation the quantities $\Delta_s^{(1)}(s=1, 2)$ are the same as Δ_0^{p1} and Δ_0^{T1}, respectively. After the second iteration the system of secular equations takes the form

$$\begin{gathered}\alpha_s\Delta_s^{(2)}-a_1\alpha_s-b_1\beta_s=0,\\ \beta_s\Delta_s^{(2)}-a_2\alpha_s-b_2\beta_s=0.\end{gathered} \tag{25}$$

From the condition that the determinant of the set (25) be equal to zero we get for $\Delta_s^{(2)}$ and the ratio β_s/α_s expressions of the following form:

$$\begin{gathered}\Delta_{1,2}^{(2)}=\{a_1+b_2\pm[(a_1+b_2)^2-4(a_1b_2-a_2b_1)]^{1/2}\}/2,\\ \beta_{1,2}/\alpha_{1,2}=(2b_1)^{-1}\{(b_2-a_1)^2\pm[(b_2-a_1)^2+4b_1a_2]\}^{1/2}.\end{gathered} \tag{26}$$

Expressions for the quantities $a_{1,2}$ and $b_{1,2}$ can be obtained by comparing (25) and (17) after substituting there the second iteration instead of H_s^p and H_s^T.

4. CONCLUSION

We consider those consequences from the formulae given above for the terrestrial dynamo problem which can be obtained without performing numerical calculations with specific expressions for the velocities. We dwell first of all upon the degenerate case. Starting from the eigenvalues for the freely damped magnetic field modes, which are, e.g., indicated for a specific model of the Earth in Ref. 1, one can reach the conclusion that even the terms with the lowest eigenvalues $J=1, 2$ in the spectral expansion of the scalar Green functions should yield reasonable answer in the search for the necessary conditions for generation. It then turns out to be sufficient in the non-degenerate case to retain terms in second order of the perturbation theory. The axisymmetric poloidal motions must be small, since they shift $i\omega_0^p$ in the positive direction according to Cowling's theorem and thereby inhibit generation. We shall therefore assume in what follows that the first term in the series (20) for Δ_0^p, which is not zero only when the velocity has a component $v(r)\mathbf{Y}_{20}^{(0)}(\hat{\mathbf{r}})$, is not important.

We make some model estimates of the velocities necessary for generation. To fix the ideas we assume that the velocity that enters in the perturbation can be written in the form

$$\begin{gathered}\mathbf{V}(\mathbf{r})=\mathbf{V}^p(\mathbf{r})+\mathbf{V}^T(\mathbf{r}),\quad \mathbf{V}^T(\mathbf{r})=v_1{}^T(r)\,\mathbf{Y}_{11}^{(m)}(\hat{\mathbf{r}})\\ +v_1{}^{T*}(r)\,\mathbf{Y}_{11}^{(m)*}(\hat{\mathbf{r}})+v_2{}^T(r)\,\mathbf{Y}_{20}^{(m)}(\hat{\mathbf{r}}),\\ \mathbf{V}^p(\mathbf{r})=\sum_{J=1,2}\Big\{v_J{}^p(r)\,\mathbf{Y}_{J1}^{(0)}(\hat{\mathbf{r}})+v_J{}^{p*}(r)\,\mathbf{Y}_{J1}^{(0)*}(\hat{\mathbf{r}})\\ +\frac{1}{r[J(J+1)]^{1/2}}\left[\frac{d}{dr}(r^2v_J{}^p(r))\,\mathbf{Y}_{J1}^{(e)}(\hat{\mathbf{r}})+\frac{d}{dr}(r^2v_J{}^{p*}(r))\,\mathbf{Y}_{J1}^{(e)*}(\hat{\mathbf{r}})\right]\Big\}.\end{gathered} \tag{27}$$

The choice of the velocities in the form (27) (with the smallest angular momenta that allow the necessary transitions) corresponds, in the estimates of the perturbation-theory series, to retaining the largest terms so that adding velocities with larger angular momenta does not change qualitatively the conclusions that follow. Moreover, one can give a direct physical meaning to velocities of the form (27). In particular, $\mathbf{V}^T(r)$ in (27) can include a differential-rotation component that arises when the rotation axis of the mantle deviates from the rotation axis of the core by an angle γ. According to independent data[15] this angle has an upper limit of the order[4)] of 10^{-5} radians. For a model estimate it is sufficient to retain in the expression (20) for Δ_0^{p2} only the first two terms. After evaluating the angular integrals, which are of order unity, we can express the quantity Δ_0^{p2} in the form

$$\begin{gathered}\Delta_0{}^{p2}=\int dr'dr''dr'''\Big\{H^{p0*}(r)\,v_1{}^T(r)\,G_{11}^{(e)}(r, r', \omega_0)\\ \times v_2{}^p(r')\,G_{20}^{(m)}(r', r'', \omega_0)\frac{d\Omega(r'')}{dr''}H^{p0}(r'')\Big\}+\int dr'dr''\\ \times\{H^{p0*}(r)\,v_1{}^T(r)\,G_{11}^{(e)}(r, r', \omega_0)\,v_1{}^T(r')\,H^{p0}(r')\}.\end{gathered} \tag{28}$$

It turns out that the first term in Eq. (28) is the main one in the limit of large differential rotation, while the second one is the main one in the limit of small differential rotation. Using the spectral representation of the Green functions (see the second footnote) and replacing the radial integrals of the velocities and their derivatives by average values (marked by bars) we arrive at the following expression:

$$\frac{\Delta_0^{p2}}{i\omega_0} \approx \frac{c_1 R_m(v_1^T) R_m(v_2^p) R_m^\Omega}{(R_m^\Omega)^2+c_2} + \frac{c_3 (R_m(v_1^T))^2}{(R_m^\Omega)^2+c_4}, \quad c_{2,3,4}>0,$$

$$R_m^\Omega = \frac{\overline{\Omega(r) R_2^2}}{\overline{D(r)}}, \quad R_m(v_2^p) = \frac{\overline{v_2^p(r) R_2}}{\overline{D(r)}}, \tag{29}$$

$$R_m(v_1^T) = \frac{\overline{v_1^T(r) R_2}}{\overline{D(r)}}.$$

The dimensionless constants c_1, c_2, c_3, c_4 in (29) are of order unity. That c_2, c_3, c_4 are positive is a consequence of the antidynamo theorems listed in the Introduction. The possibility of negative c_1 follows from the fact that the corresponding term contains the independent velocities $v_1^T(r)$ and $v_2^p(r)$ to the first degree. When c_1 is positive, generation cannot proceed, notwithstanding the violation of axial symmetry. The violation of axial symmetry is thus in the general case insufficient for the appearance of generation. In writing down (29) we made the following estimates of the frequency $i\omega_0$ and the frequencies $\omega^T(2,0)$ and $\omega(1,|1|)$ corresponding, respectively, to the eigenvalues in the spectral expansions of $G_{20}^{(m)}$ and $G_{11}^{(m,e)} + G_{1,-1}^{(m,e)}$:

$$\omega^T(2,0) - \omega_0 \sim \omega_0 \sim \overline{D(r)} R_2^{-2},$$
$$\omega(1,|1|) - \omega_0 \sim \overline{\Omega(r)}^2 R_2^2 / \overline{D(r)}. \tag{30}$$

These estimates follow from the form of the equations satisfied by the corresponding combinations of Green functions. It is clear from (29) that the condition for generation

$$\Delta_0^{p2}/i\omega_0 \sim -1 \tag{31}$$

can be satisfied for negative c_1 only if

$$R_m^\Omega > 1, \quad R_m(v_1^T) R_m(v_2^p) > R_m^\Omega. \tag{32}$$

If we use for R_m^Ω the estimate 10^4 based on the westerly drift of the magnetic field[11] we get for $R_m(v_1^T) \sim R_m(v_2^p)$ a value 10^2 which corresponds to average velocities of the order of 10^{-2} cm/s, which are admissible from geophysical considerations. The second of the inequalities (32) is a reflection of the condition that according to Backus' theorem generation is impossible when $|\overline{\Omega(r)} R_2|$ is much larger than all other quantities with dimensions of velocity. Retention of two terms in (29) corresponds to an interpolation between the cases $R_m^\Omega \gg 1$ and $R_m^\Omega \ll 1$. In the range of values $R_m^\Omega \sim 1$ a more complicated behavior of Δ_0^{p2} is possible, which depends on the detailed structure of the velocities. It is important that in the case considered the ratio of the energies of the toroidal and the poloidal fields is of the order $(R_m^\Omega)^2$, i.e., very large.

We now turn to estimates in the degenerate case. The quantities a_i and b_i which enter here and in terms of which $\Delta_{1,2}^2$ in (26) can be expressed are very unwieldy. We therefore confine ourselves to two limiting variants of physical interest. We assume in the first variant that $\hat K^{Tp}(r, r', 1, 1, 0, 0) = 0$ or, which is the same, that $v_2^T(r) = 0$ in (27). The simplest estimate for $\Delta_{1,2}^2$ will then be the same as in the non-degenerate case; however, a detailed analysis of the radial integrals shows that the restrictions on the velocity of the conducting medium, which enter in the perturbation, are weaker. We consider in more detail the second variant, assuming the existence of a velocity that ensures the non-vanishing of $\hat K^{Tp}(r, r', 1, 1, 0, 0)$. In this variant the transition from the poloidal to the toroidal symmetric dipole becomes important; this transition is determined by the quantity

$$\int dr dr' \{H^{T0*}(r) \hat K^{Tp}(r, r', 1, 1, 0, 0) H^{p0}(r')\}$$
$$\approx \int \frac{d\ln D(r)}{d\ln r} \frac{v_2^T(r) D^{1/2}(r) H^{p0}(r) H^{T0}(r)}{r^2} dr \tag{33}$$
$$\approx \overline{\frac{d\ln D(r)}{d\ln r}}\, \overline{D^{1/2}(r)}\, \frac{\overline{v_2^T(r)}}{R_2}.$$

In deriving (33) we used (15) and Eqs. (24). Starting from (26) and (33) and using arguments similar to those leading to (32) we get for generation in the degenerate case the following condition:

$$\frac{\mathrm{Re}\,\Delta_s^2}{i\omega_0} \approx -\left(\overline{\frac{d\ln D(r)}{d\ln r}}\, \frac{R_m(v_1^p) R_m(v_1^T) R_m(v_2^T)}{c_1' + c_2' (R_m^\Omega)^2}\right), \tag{34}$$

where c_1' and c_2' are dimensionless constants of order unity. When writing down (34) we assumed that the inequality

$$(R_m(v_1^T))^2 < R_m(v_2^T) R_m(v_1^p) (R_m^\Omega)^2, \tag{35}$$

which is equivalent to the condition $\mathrm{Re}(b_2 a_1 - a_2 b_1) < 0$ for the quanities in (26), is satisfied. When the inequality which is the opposite of (35) is satisfied, generation is utterly impossible. When (35) is satisfied we get for the ratio of the energies of the toroidal and poloidal dipoles in the degenerate case, according to (26), the relation

$$\left(\frac{\beta_1}{\alpha_1}\right)^2 \approx \frac{R_m(v_2^T)}{R_m(v_1^T) R_m(v_1^p)} [c_1'' + c_2'' (R_m^\Omega)^2], \tag{36}$$

where c_1'' and c_2'' are dimensionless constants of order unity.

We note the following four important differences from the non-degenerate case considered earlier. Firstly, generation becomes possible for small differential rotation. Secondly, the energy ratio of the toroidal and poloidal fields need not be necessarily so large as in the non-degenerate case. This is particularly important because for a large toroidal field the balance of the energy dissipated due to ohmic losses is worsened. Thirdly, in the expression for Δ_s^2 in the degenerate case the r-dependence of $D(r)$ may become important. For large $d\ln D(r)/d\ln r$ the velocities which are critical for the generation threshold will be significantly lower. Fourthly, in the degenerate case the appearance of imaginary Δ_s is possible, and thus magnetic field oscillations are possible. The physical cause of these oscillations are beats corresponding to energy transfer between the toroidal and poloidal dipoles.

Both in the degenerate case and in the non-degenerate case, due to the appreciable contribution to (27) from non-axisymmetric velocities, the observed total poloidal dipole must be deflected from the rotational axis by an angle γ. If γ is small, we get for if the estimate:

$$\gamma \sim \frac{R_m(v_1^T)}{c_1''' + c_2''' (R_m^\Omega)^2}. \tag{37}$$

The relation (37) limits considerably the values of the non-axisymmetric velocity components.

In conclusion I must express my gratitude for useful discussions to S. I. Braginskiĭ, A. A. Ruzmaĭkin, V. M. Fishman, and A. P. Anufriev.

APPENDIX

We consider a partial summation of the perturbation-theory series in the pole approximation and indicate simultaneously a method for proving a number of antidynamo theorems corresponding to the case of an anti-Hermitean perturbation operator $\hat{K}$. In symbolic (Dirac) notation we can write (12), when there is no degeneracy in the form

$$|\mathbf{H}\rangle=|\mathbf{H}_0\rangle+G'\hat{K}|\mathbf{H}\rangle,\quad \Delta_0=-\langle\mathbf{H}_0|\hat{K}|\mathbf{H}\rangle. \tag{A.1}$$

Retaining a finite number N of eigensolutions in the spectral representation of the Green function G' we can write it in the form

$$G'=\sum_{n=1}^{N}\frac{|\mathbf{H}_n\rangle\langle\mathbf{H}_n^*|}{i\omega_n-i\omega_0}. \tag{A.2}$$

Such an approximation is valid when $|\Delta_0|<i\omega_n-i\omega_0$, attainable in the dynamo problem[5] at sufficiently large N by virtue of the structure of the operator $\hat{K}_0$ indicated in the text. After substituting (A.2) into (A.1), the determination of Δ_0 reduces to solving an algebraic set of $N+1$ linear equations in $\langle \mathrm{H}_n|\hat{K}|\mathbf{H}\rangle$, $(n=0,1,\ldots,N)$:

$$\langle\mathrm{H}_n|\hat{K}|\mathrm{H}\rangle=\langle\mathrm{H}_n|\hat{K}|\mathrm{H}_0\rangle+\sum_{m=1}^{N}\frac{\langle\mathrm{H}_n|\hat{K}|\mathrm{H}_m\rangle\langle\mathrm{H}_m|\hat{K}|\mathrm{H}\rangle}{i\omega_m-i\omega_0}. \tag{A.3}$$

The solution of (A.3) is equivalent to a partial summation of the perturbation theory series in which the transitions between $\mathrm{H}_0,\ldots,\mathrm{H}_N$ are retained. Assuming[6] that $\langle\mathrm{H}_n|\hat{K}|\mathrm{H}_n\rangle=0$ for all n, we get for Δ_0 the following expression in the form of the ratio of two determinants:

$$\Delta_0=\det\begin{vmatrix}0 & \frac{\langle\mathrm{H}_0|\hat{K}|\mathrm{H}_1\rangle}{(i\omega_1-i\omega_0)^{1/2}} & \cdots & \frac{\langle\mathrm{H}_0|\hat{K}|\mathrm{H}_N\rangle}{(i\omega_N-i\omega_0)^{1/2}}\\ \frac{\langle\mathrm{H}_1|\hat{K}|\mathrm{H}_0\rangle}{(i\omega_1-i\omega_0)^{1/2}} & 1 & \cdots & B_{1N}\\ \vdots & \vdots & & \vdots\\ \frac{\langle\mathrm{H}_N|\hat{K}|\mathrm{H}_0\rangle}{(i\omega_N-i\omega_0)^{1/2}} & B_{N1} & \cdots & 1\end{vmatrix}\times\det{}^{-1}\begin{vmatrix}1 & 0 & 0 & \cdots & 0\\ 0 & 1 & B_{12} & \cdots & B_{1N}\\ 0 & B_{21} & 1 & \cdots & B_{2N}\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & B_{N1} & B_{N2} & \cdots & 1\end{vmatrix} \tag{A.4}$$

where the B_{mn} denote the quantities

$$\langle\mathrm{H}_m|\hat{K}|\mathrm{H}_n\rangle/[(i\omega_m-i\omega_0)(i\omega_n-i\omega_0)]^{1/2}.$$

$(n, m=1,\ldots,N)$. The numerator of (A.4) has, as can easily be checked, the structure of the N-th term of perturbation theory (without repeated factors). The matrix in the denominator of (A.4) has the form $1+A$, where A vanishes when $\hat{K}=0$. When $\|A\|<1$ the sign of Δ_0 is determined by the sign of the numerator of (A.4). For an anti-Hermitean $\hat{K}$ the expression for Δ_0 is found to be positive.

The operator $\hat{K}$ can be reduced to anti-Hermitean form when the conditions for Cowling's, Backus', and Vaĭnshteĭn's theorems are satisfied (the reduction of the induction equation to the necessary form when the conditions for the generalized Vaĭnshteĭn theorem are satisfied was performed in Ref. 17). We demonstrate this using as an example Cowling's theorem, which is formulated as follows: The generation of an axisymmetric magnetic field by an axisymmetric velocity is impossible when the diffusion coefficient is constant. Under the conditions considered, the induction equation reduces to a set of scalar equations for scalar functions describing the behavior of the poloidal and the toroidal components of the field. In the axisymmetric case the toroidal component of the magnetic field and the vector potential of the poloidal component are directed along the unit vector $\hat{\varphi}$. We can therefore express the magnetic field and the velocity field in the form

$$\begin{gathered}\mathbf{H}=\mathbf{H}^P(r,\theta)+\mathbf{H}^T(r,\theta)=\mathrm{rot}\{A^P(r,\theta)\,\mathrm{grad}\,\varphi\}+H^T(r,\theta)\,\mathrm{grad}\,\varphi\\ =[\mathrm{grad}\,A^P(r,\theta)\,\mathrm{grad}\,\varphi]+H^T(r,\theta)\,\mathrm{grad}\,\varphi,\\ V=V^P+V^T,\end{gathered}$$

and get the following scalar equations:

$$\frac{dA^P}{dt}=D\Delta A^P-2D([\mathrm{rot}\,\hat{\varphi},\hat{\varphi}]\,\mathrm{grad}\,A^P)-(V^P\,\mathrm{grad}\,A^P), \tag{A.5}$$

$$\frac{dH^T}{dt}=D\Delta H^T-2D([\mathrm{rot}\,\hat{\varphi},\hat{\varphi}]\,\mathrm{grad}\,H^T)-(V^P\,\mathrm{grad}\,H^T)+(\hat{\varphi}\,\mathrm{rot}[V^T\,\mathrm{rot}\{A^P(r,\theta)\,\mathrm{grad}\,\varphi\}]). \tag{A.6}$$

We consider first Eq. (A.5). Splitting the second term on the right-hand side of Eq. (A.5) into its Hermitean and anti-Hermitean parts, one shows easily by direct integration that the Hermitean part is negative-definite. Choosing then as the perturbation operator $\hat{K}$ the anti-Hermitean part of the second term plus the third term on the right-hand side of (A.5) (which is also anti-Hermitean), and using the arguments given above, we find that $A^P(r,\theta)\to 0$ as $t\to\infty$. A similar study of Eq. (A.6) shows that also $H^T(r,\theta)\to 0$ as $t\to\infty$, thus proving Cowling's theorem.

1)In the notation used the following equations hold:

$$\mathbf{Y}_{JM}^{(0)}=\hat{\mathbf{r}}Y_{JM},\quad \mathbf{Y}_{JM}^{(m)}=i[(J+1)J]^{-1/2}\hat{\mathbf{L}}Y_{JM},\quad \mathbf{Y}_{JM}^{(e)}=[\mathbf{Y}_{JM}^{(m)}\hat{\mathbf{r}}],$$

where the Y_{JM} are the usual spherical harmonics and $\hat{\mathbf{L}}=-i[\mathbf{r}\times\nabla]$ is the angular-momentum operator. The vector spherical harmonics have then the following properties under complex conjugation, which are used below:

$$\mathbf{Y}_{JM}^{(m,0,e)*}(\hat{\mathbf{r}})=(-1)^{J+M}\mathbf{Y}_{J-M}^{(m,0,e)}(\hat{\mathbf{r}}).$$

2)It is impossible to guarantee for the scalar Green functions $G_{JM}^{(e,m)}(r,r',\omega)$ with $M\neq 0$ the existence of spectral representations, as they satisfy Eqs. (6) with non-Hermitean operators on the left-hand side. However, the Green-function combination

$$\overset{+}{G}_{JM}(r,r',\omega)=[G_{JM}^{(e,m)}(r,r',\omega)+G_{J-M}^{(e,m)}(r,r',\omega)]/2,$$

which occurs in the expression for $\Delta_0^{p,T}$ satisfies an equation with Hermitean operators on the left-hand side, in the symbolical form

$$[G_{J_0}^{(e,m)-1}+M^2\Omega G_{J_0}^{(e,m)}\Omega]\overset{+}{G}_{JM}(r,r',\omega)=\delta(r-r').$$

3)A similar method of partial summation of the perturbation-theory series is used in the quantum theory of scattering (see, e.g., Ref. 13) and is called the separable-kernel method. We assume below in the text that the Fredholm determinant does not vanish under the conditions considered.

4)In the model in which the motion in the liquid core is represented as the motion of a viscous fluid between two rotating concentric spheres we have[16]

$$v_1^T(r)=R_1^3R_2^3(R_2^3-R_1^3)^{-1}\left(\frac{1}{r^3}-\frac{1}{R_2^3}\right)\Omega_0\,\mathrm{tg}\,\gamma.$$

Here R_1 and R_2 are, respectively, the inner and outer radii of the liquid core.

5)Specific peculiarities, which for the sake of brevity have not been considered, arise only when an M-th shifted eigenvalue "overtakes" the $M+1$st value. In that case it is necessary to use a scheme of calculations which is similar to the one described in the text for the degenerate case.

6)This condition is not a restriction, as the terms proportional to $\langle \mathbf{H}_n|\hat{K}|\mathbf{H}_n\rangle$ can simply be included in the unperturbed equations.

[1]H. K. Moffatt, Magnetic Field Generation in Electrically Conducting Fluids, Cambridge University Press, London, New York, Melbourne, 1978.

[2]S. I. Braginskiĭ, Izv. Akad. Nauk SSSR Ser. Fiz. Nr **10**, 3, (1972).

[3]P. H. Roberts, Philos. Trans. R. Soc. London Ser. A **272**, 663 (1972).

[4]P. H. Roberts and A. M. Soward, Ann. Rev. Fluid Mech. **4**, 117 (1972).

[5]E. H. Levy, Ann. Rev. Earth Planet. Sci. **4**, 159 (1976).

[6]S. I. Braginskiĭ in Itogi nauki i tekhniki (Results of Science and Technology) (Ed. N. P. Ben'kov) VINITI, Moscow, 1980, Vol. 5.

[7]S. I. Braginskiĭ, Zh. Eksp. Teor. Fiz. **47**, 1084 (1964) [Sov. Phys. JETP **20**, 726 (1965)].

[8]E. C. Bullard and H. Gellman, Philos. Trans. R. Soc. London Ser. A **247**, 213 (1954).

[9]F. E. M. Lilley, Proc. R. Soc. London Ser. A **136**, 153 (1970).

[10]D. Lorts, Plasma Phys. **10**, 967 (1968).

[11]S. I. Braginskii, Geomagn. Aèronomiya **4**, 732 (1964).

[12]S. I. Vaĭnshteĭn, Zh. Eksp. Teor. Fiz. **50**, 818 (1966) [Sov. Phys. JETP **23**, 542 (1966)].

[13]R. G. Newton, Scattering Theory of Waves and Particles, McGraw-Hill, New York, 1966.

[14]G. E. Backus, Ann. Phys. (N.Y.) **4**, 372 (1958).

[15]J. A. Jacobs, The Earth's Core, Academic Press, London, New York, San Francisco, 1975, p. 293.

[16]L. D. Landau and E. M. Lifshitz, Mekhanika sploshnykh sred (Fluid Mechanics), Gostekhizdat, Moscow, 1954 [English translation published by Pergamon Press, Oxford].

[17]Yu. A. Brodskiĭ, Geomagn. Aèronomiya No. 3, 531 (1981).

Translated by D. ter Haar