Semiclassical theory of superradiance in one-dimensional crystalline structures

A. V. Karnyukhin, R. N. Kuz'min, and V. A. Namiot

Moscow State University
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A semiclassical theory of collective spontaneous emission in a one-dimensional system is developed, which takes account of the forward and backward waves of the radiation field in the active medium. The basic equations of the theory are derived without the assumption that the space derivatives of the radiation field amplitude and the atomic inversion are small. The process of collective spontaneous decay of nuclei in a periodic lattice is investigated under Bragg-diffraction conditions and for arbitrary deviations from these conditions. The superradiance of samples of dimensions of the order of the wavelength of the radiation field is considered for the limiting case of a continuous medium. The property of a superradiant system to retain part of its energy in the form of a localized collective excitation of the medium is discovered. The criterion for transition from the regime of pure (single-pulse) superradiance to the oscillatory regime differs fundamentally from the criterion given by the quantum mode theory of superradiance.

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I. INTRODUCTION

The possibility in principle of the collectivization of the spontaneous decay of an assemblage of excited atoms was first demonstrated by Dicke,[1] using as an example a system of N closely spaced atoms ($R \ll \lambda$; R is the dimension of the region occupied by the atoms and λ is the wavelength of the radiation). In this case the interaction between the atoms via their radiation self-field leads to the result that the intensity of the spontaneous radiation emitted by such a system is proportional to N^2, while the lifetime $\sim N^{-1}$. Dicke's paper[1] became the starting point for further theoretical and experimental investigations of this phenomenon, which is called superradiance. In its general formulation this problem is a multiplanned one; for it is necessary to answer a number of questions: the question of the formation of macroscopic correlations from an initially uncorrelated state, of the role of the geometry of the active medium, of the allowance for the nonlinearity of the system, etc. Also diverse are the methods used in the theoretical investigation.[2] Thus, the superradiance of extended systems, i.e., systems with $R \gg \lambda$, is investigated by two entirely different methods: the quantum mode approach, in which the radiation field is quantized and one or several important modes of the quantized field are considered, and the semiclassical approach, in which the field is considered to be classical. The advantage of the first approach is that it can lead to the correct description of the initial phase of the spontaneous decay, when there is no correlation between the atoms and the quantum fluctuations of the field are substantial. But the limitation of the number of modes negates the description of the spatial inhomogeneities of the field and the atomic characteristics. The semiclassical approach, on the other hand, allows us to take the inhomogeneity of the decay process into account, but is not capable of correctly describing the initial phase of the process, and therefore requires the specification of the atomic currents initiating the decay at zero time. Let us also note that the normal semiclassical approach is essentially based on the assumption that the space derivatives of the field amplitude and the atomic characteristics are small: $\partial F/\partial x \ll F/\lambda$, where F is any function describing the state of the system (the radiation-field amplitude, the polarization, or the inversion). It is clear from the foregoing that it is expedient to use the normal semiclassical approach in the $R \gg \lambda$ case, in which the inhomogeneity of the decay process is slight in the sense of the above condition; the quantum description, on the other hand, is effective only for sufficiently short samples, in which no inhomogeneity occurs at all. In the present paper we investigate the question: What is the greatest active-sample length for which the collective-decay process will still be homogeneous? We propose a semiclassical method that takes account of inhomogeneities of arbitrary scale, and goes over in the limit of long-wave inhomogeneities into the method proposed by MacGillivray and Feld.[3]

Thus far, what has been considered is essentially the continuous-medium case, which obtains in the optical wavelength region. Let us now turn to the problem of collective spontaneous emission in crystalline structures, when the radiant objects are nuclei (the x- and γ-ray regions) and the wavelength of the radiation is comparable to the radiator spacing. The effect of the discreteness of the active medium on the possibility of collective spontaneous decay was first discussed in Refs. 4–7. The assertion is made in Refs. 5 and 6 that the spontaneous decay process in a crystalline structure can become collective only when the condition $\mathbf{k} = 2\pi\mathbf{b}$, where $\mathbf{k}$ is the wave vector of the radiation field and $\mathbf{b}$ is a reciprocal-lattice vector of the crystal, is fulfilled. But subsequent more detailed investigations[7,8] have shown that this condition is not a necessary one. Thus, Afanas'ev and Kagan[7] have shown that the superradiance effect can be realized in an arbitrary crystal with the aid of a special coherent excitation of the crystal. But let us note that in Ref. 4 the real nuclei are modeled by damped linear oscillators, while in Refs. 5–7 the results are obtained in first-order perturbation theory; therefore, there arises the question of the analysis of more realistic nonlinear models. The main

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purpose of the present paper is to consider this problem. Let us note that the results of the nonlinear treatment differ in many respects from the results obtained in the linear approximations. Let us discuss the main features of the model in question and the assumptions made.

First, the analysis is carried out in the approximation of a dipole interaction between the nuclei and the radiation field. The problem for the periodic structure then turns out to be equivalent to some problem for a continuous sample of a definite length. It is precisely this equivalence that allows a parallel consideration of the above-discussed two outwardly different problems: the superradiance of a sample with $R\sim\lambda$ and the superradiance of a crystal, the first problem being characteristic of the infrared region and the second of the x- and γ-ray regions.

Second, the study of the one-dimensional model is expedient now because the anisotropic emission process is of greatest interest in the real three-dimensional case, and in such a process the state functions of the active medium depend essentially on one space coordinate. As the possible causes of the anisotropy we can cite the geometry of the active medium, the discrete periodic structure of the medium, or the special initial conditions imposed on the radiators. The one-dimensional radiators are actually planes in three-dimensional space, each filled with identical atoms (nuclei) whose states (inversion, polarization) are identical at all moments of time. The problem of the spontaneous decay of one such radiator is solved in Ref. 9.

Third, the analysis is limited to the case of a nonabsorbing rigid crystal. The neglect of the absorption can be justified if we choose a crystal of sufficiently small length in comparison with the absorption length, and also note that, in view of the higher probability for radiative decay in the collective process, the relative weight of the internal-conversion effect occurring in the nuclei is smaller.[7] But the neglect of the lattice dynamics is apparently a major simplification of the real situation, in view of the recoil that accompanies the collective emission by a group of neighboring nuclei, and can be advisable only at the first stage of the investigation. Another argument for the necessity of the consideration of the nuclear motion will be adduced in the final section.

II. DERIVATION OF THE BASIC EQUATIONS

1. *The equations for the nucleus.* We shall treat the nucleus as a two-level quantum system with ground-state energy $E_1\equiv\hbar\omega_1$ and excited-state energy $E_2=\hbar\omega_2$; $\omega\equiv\omega_2-\omega_1$ is the transition frequency. The state of the nucleus is described by the two-component wave function

$$\Psi(t)=\begin{pmatrix}C_2(t)\\ C_1(t)\end{pmatrix},\quad |C_1|^2+|C_2|^2=1,$$

the Schrödinger equation for which has, in the dipole approximation, the form[10]

$$\begin{aligned} i\hbar dC_1/dt&=-E(t)dC_2\exp(-i\omega t),\\ i\hbar dC_2/dt&=-E(t)dC_1\exp(i\omega t),\end{aligned}\tag{1}$$

where $E(t)$ is the electromagnetic field intensity and d is the matrix element of the dipole-moment operator. Let us introduce the new variables θ, φ_1, and φ_2 with the aid of the following relations:

$$C_1=\cos(\theta/2)e^{i\varphi_1},\quad C_2=\sin(\theta/2)e^{i\varphi_2}.$$

Then from the system (1) we obtain for the variables θ and $\varphi=\varphi_2-\varphi_1$ in the case of a prescribed external field $E(t)$ a closed system of two equations, which it is convenient to represent in the form of one complex equation

$$\frac{1}{\cos\theta}\frac{d}{dt}(\sin\theta e^{i\varphi})=i\frac{2Ed}{\hbar}e^{i\omega t}.\tag{2}$$

If $E(t)=\mathscr{E}(t)e^{-i\omega t}$, where $d\mathscr{E}/dt\ll\omega\mathscr{E}$, we obtain from (2) the equation

$$\frac{1}{\cos\theta}\frac{d}{dt}(\sin\theta e^{i\varphi})=i\frac{2\mathscr{E}d}{\hbar}.\tag{2'}$$

2. *The equations for the electromagnetic field.* Let us express the field intensity $E(x,t)$ in terms of the currents of all the nuclei of the system. To do this, let us proceed from the one-dimensional Maxwell equation for the intensity

$$\left(\frac{\partial^2}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)E(x,t)=\frac{4\pi}{c^2}\frac{\partial J(x,t)}{\partial t},\tag{3}$$

where J is the transverse-current density ($J\perp e_x$). The retarded solution to Eq. (3) has the form

$$E(x,t)=-\frac{2\pi}{c}\int J\left(y,t-\frac{|x-y|}{c}\right)dy.\tag{4}$$

The current density in the system of plane radiators is given by the expression

$$J(x,t)=n\sum_k\delta(x-x_k)\frac{\partial}{\partial t}d(x_k,t),\tag{5}$$

where n is the surface density of the nuclei in the plane radiator and $d(x_k,t)$ is the dipole moment of the nucleus at the point x_k, given by the expression

$$d(x,t)=d(C_1(x,t)C_2^*(x,t)e^{i\omega t}+\text{c.c.}).$$

In differentiating the last equation, we should use the equations (1). We obtain in the variables θ and φ the expression

$$\frac{\partial}{\partial t}d(x,t)=\omega d\sin\theta(x,t)\sin(\varphi(x,t)-\omega t).\tag{6}$$

Substituting successively (6) into (5) and then (5) into (4), we express the field intensity in terms of the states of all the radiators in the variables θ and φ:

$$E(x,t)=-\frac{2\pi nd}{c}\sum_k\omega_k\sin\theta_k(t_k)\sin(\varphi_k(t_k)-\omega_kt_k),\tag{7}$$

where $t_k\equiv t-|x-x_k|/c$. In (7) it is assumed that the ω_k differ from each other (inhomogeneous broadening). We shall limit ourselves below to the case $\omega_k=\omega$.

3. *Closed equations for the nuclei in the Bloch variables.* The substitution of the expressions (7) for the intensity into Eq. (2) allows us to obtain an equation in which only the nuclear variables θ and φ figure:

$$\frac{1}{\cos\theta_m}\frac{d}{d\tau}(\sin\theta_m\exp(i\varphi_m))=-2i\exp(i\omega t)\sum_k\sin\theta_k(t_k)\sin(\varphi_k-\omega t_k),\tag{8}$$

where $\tau \equiv 2\pi n\omega d^2 t/\hbar c$. Let us note that the variables θ and φ are slowly varying quantities in comparison with the quantity $e^{i\omega t}$ [this can be seen from the equations (1): $dC_{1,2}/dt \sim Ed/\hbar \ll \omega$]. Therefore, it is expedient to separate in (8) the rapidly- and slowly-varying—in time—quantities, and then discard the rapidly-oscillating terms, i.e., use the "rotary-wave" approximation.[11] The final equation into which only the "slow" quantities enter has the form

$$\frac{1}{\cos\theta_m}\frac{d}{d\tau}(\sin\theta_m\exp(i\varphi_m)) = -\sum_k \sin\theta_k(t_k)\exp\left[i\left(\varphi_k(t_k)+\frac{\omega}{c}|x_m-x_k|\right)\right]. \qquad (9)$$

It should be borne in mind that allowance for the term with $k=m$ in Eq. (9) corresponds to allowance for the self-action of the radiator via the self-field of the radiation. In the semiclassical description this self-action is responsible for the spontaneous decay of a radiator. The relatively slow variation in time of the quantities θ and φ allows us to simplify Eq. (9): specifically, when the condition $2\pi n\omega d^2 N/\hbar c \ll c/R$ is fulfilled, the functions θ_k and φ_k change little during the time $\Delta t = R/c$ it takes the field to propagate through the system; therefore, the retardation in the right member of (9) can be neglected:

$$\frac{1}{\cos\theta_m}\frac{d}{d\tau}(\sin\theta_m\exp(i\varphi_m)) = -\sum_k \sin\theta_k\exp[i(\varphi_k+k_0|x_k-x_m|)]. \qquad (10)$$

Let us emphasize that the transition from (8) to (10) is made under the condition that $\omega\to\infty$, $c\to\infty$, and $\omega/c = k_0 = \text{const}$.

III. THE CASE OF A CONTINUOUS MEDIUM

For a continuous medium Eq. (10) should be represented in the form

$$\frac{1}{\cos\theta}\frac{d}{d\tau}(\sin\theta\exp(i\varphi)) = -\int \sin\theta(y,t)\exp[i(\varphi(y,t)+k_0|x-y|)],$$

where $\tau \equiv 2\pi\rho d^2 t/\hbar$ and ρ is the volume density of the atoms, or in the equivalent differential form

$$\left(\frac{\partial^2}{\partial x^2}+k_0^2\right)\left[\frac{1}{\cos\theta}\frac{\partial}{\partial\tau}(\sin\theta\, e^{i\varphi})\right] = -2ik_0^2\sin\theta\, e^{i\varphi}. \qquad (11)$$

Let us consider some properties of Eq. (11).

1. *The traveling wave*

$$E(x,t)=\mathscr{E}\exp[i(k_0x-\omega t)],\quad \partial\mathscr{E}/\partial t\ll\omega\mathscr{E},\quad \partial\mathscr{E}/\partial x\ll k_0\mathscr{E}.$$

In this case $\varphi(x,t) = \pi/2 + k_0 x$, as follows from Eq. (2'). Substituting this expression in (11), we obtain the equation

$$\partial^2\theta/\partial\eta\partial\tau = -\sin\theta \quad (\eta = k_0 x), \qquad (12)$$

which coincides with the equation of the standard single-wave semiclassical theory developed in Ref. 3. On the other hand, this is the so-called sine-Gordon equation, which is well known in the theory of nonlinear waves (see, for example, Ref. 12), and has soliton solutions, which we easily obtain by seeking the solution in the from $\theta(\eta,\tau) = \theta(\eta - v\tau)$, $v > 0$. The solution with zero boundary conditions at infinity

$$\theta = 2\arccos\left(\text{th}^2\left(\frac{\eta - v\tau}{v^{1/2}}\right)\right)$$

describes the propagation in the medium of a pulse of self-induced transparency (SIT).[11] For $v<0$ we obtain another steady-state solution:

$$\theta = 2\arcsin\left(\text{th}\left(\frac{\eta - v\tau}{-v^{1/2}}\right)\right)$$

with the following condition at infinity: $\theta(\pm\infty)=\pi$. This solution describes the motion of the de-excitation region through the completely inverted medium, the direction of propagation of this region being opposite to the direcection of propagation of the photons in it ($V_{ph} = \omega/k_0 > 0$), i.e., such a soliton reminds us of the property of a "hole:" its motion in one direction is accompanied by the transport of energy in the opposite direction. The interpretation of this solution is the following: during the motion of the de-excitation region to the left (Fig. 1a), photons are emitted backwards (i.e., to the right) at the leading edge of this region, and subsequently absorbed by the trailing edge of the region. As a result, the shape of the inversion distribution at the next instant (it is indicated by the dashed curve) will shift to the left. This apparently curious property of an invert-

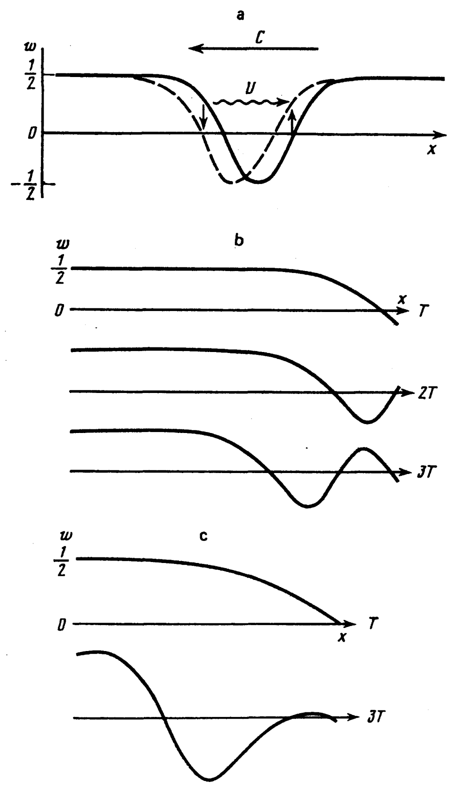

FIG. 1. a) Propagation of an inverted soliton; w is the level population inversion. The vertical arrows indicate the variation in time of the inversion at the soliton fronts, and the horizontal arrows indicate: *U*) direction of radiation and *C*) direction of soliton motion. b) Inversion distribution in an extended continuous medium under the condition of the oscillatory regime of superradiance [the solution to Eq. (12)]; T is an arbitrary time interval. c) Inversion distribution in a crystalline structure in the case of coherent initial conditions (the right wave predominates at zero time); $L = \lambda/4$ and T is an arbitrary time interval. The relaxation of the inversion at the left end of the system is due to the presence of a weak backward radiation wave.

ed medium to produce inverted solitons, has, however, a fundamental effect on the collective decay process.[1)] Let us turn to Fig. 1b, which illustrates the results of the numerical solution of Eq. (12) (these results were first obtained by MacGillivray and Feld[3]). De-excitation regions with parameters close to the parameters of the inverted solitons successively appear at the boundary of the medium in the course of the collective decay, and the appearance of each such region is accompanied by the emission of a radiation pulse. Consequently, it is natural to interprete the oscillatory regime of superradiance as the decay of the nonlinear system through the production of a succession of elementary nonlinear excitations—solitons (no soliton interpretation of the oscillatory regime is given in Ref. 3). Let us note that the nonlinear essence of the oscillatory regime of superradiance, that is manifested in the semiclassical description, does not fit into the quantum mode treatment of this phenomenon, since in the quantum treatment the oscillations of the radiation intensity appear in the linear problem. Below we shall see that the parameters characterizing the transition to the oscillatory regime in these two approaches also differ.

3. *The collective spontaneous decay of a sample of length* $R\sim\lambda$. This problem is of special interest in view of the fact that occurrence in it of a spatial inhomogeneity whose dimensions are "dictated" by the dimensions of the active medium, i.e., are of the order of λ, is to be expected. Under such conditions the semiclassical method proposed in Ref. 3 is inapplicable, and Eq. (11) must be used. The results presented below were obtained by solving this equation numerically with a computer. Before proceeding to analyze the results, let us discuss the question of the choice of the initial conditions in the system of radiators. As stated in the Introduction, the semiclassical method has the disadvantage that it is not capable of describing correctly the initial phase of the spontaneous decay, since there is no agent impelling the atom (nucleus) to leave the steady state. We must specify the value $\theta_0=\theta(0)$, thus prescribing the initial current. An adequate value has been obtained for the semiclassical parameter θ_0 in a number of papers (see, for example, Ref. 14) from the quantum analysis of the initial phase of the decay: $\theta_0\approx 2/N^{1/2}$. To obtain an uncorrelated initial state, it is natural to prescribe the phase function $\varphi(x,t=0)$ in the form of a random function of the coordinate x. If, on the other hand, the initial state is obtained by means of a coherent excitation of the medium, then the phase at zero time should be given in the form of the function $\varphi=kx$, where k is the wave number of the field produced by the pump.

Let us now consider the results of the numerical analysis of Eq. (11). These results are shown in Fig. 2, which depict the decrease of the energy in the active medium in the course of the collective decay. The fixed parameter is the total number of radiators in the sample, the sample length varying from $R\ll\lambda$ to $R=25\lambda$. The coherent initial conditions were chosen in the form $\varphi(x)=k_0x$, where k_0 is the wave number of the radiation field. The de-excitation rate has its highest value at the initial stage when the wave number k is chosen in this way. It can be seen from Fig. 2 that, when the sample length is significantly shorter than the wavelength of the radiation field, the behavior of the system totally corresponds to the regime of pure superradiance: the emission time is inversely proportional to the number N of radiators, while the radiation intensity $\sim N^2$. As the sample length increases, the picture significantly changes: a "plateau" is formed in the plot of the time dependence of the energy of the system, and the radiation intensity depends on the time according to a complicated law. As the sample length increases further, this complicated law gradually goes over into the oscillatory regime considered by us earlier. Figure 3 illustrates the phenomena that occur in the sample when the energy of the system reaches the "plateau:" there is formed a solitary inversion hump, the nature of the evolution of which is, at the initial stage, characterized by an almost periodic variation of its shape as it moves in a stepwise fashion through the medium. As the energy becomes exhausted, the hump comes to a stop, further releasing energy from itself

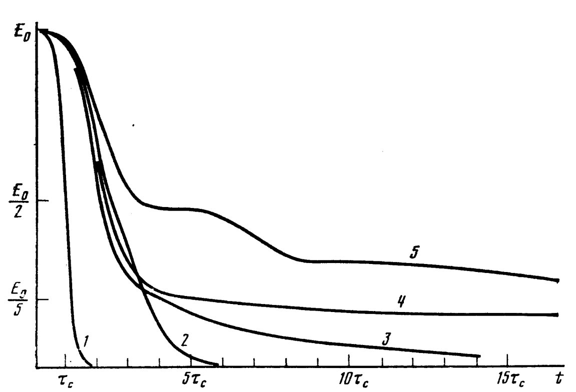

FIG. 2. The relaxation in time of the energy of an active medium as a function of the sample length. The numbers on the curves correspond to the following parameters: 1) $R\ll\lambda$; 2) $R=0.5\lambda$; 3) $R=0.575\lambda$; 4) $R=\lambda$; 5) $R=25\lambda$ and (relevant to Sec. IV) $\delta L=\lambda/4$.

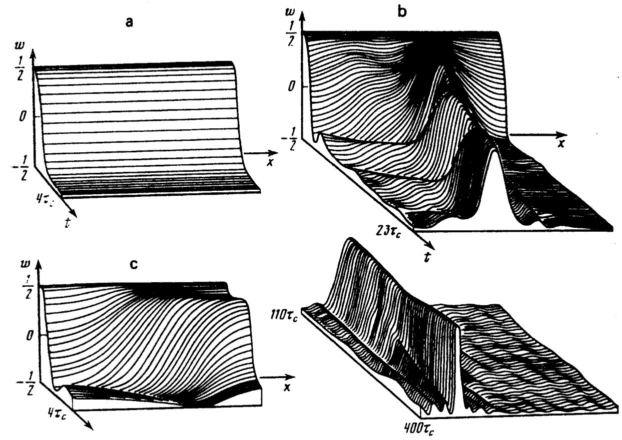

FIG. 3. Inversion distribution in an active medium during the collective decay (solitary excitation). The sample length is equal to: a) $R\ll\lambda$, b) $R=0.5\lambda$, c) $R=\lambda$; $\tau_c=\tau/N$.

in the form of very small-scaled medium excitations successively breaking away from it. The formation of such a solitary excitation evolving in such a nontrivial manner has been found to occur in all the cases considered (the sample length R reached a value of 25λ in the numerical solution). As regards the experimental investigation of this phenomenon, let us mention Ref. 15, which reports experiments with the following parameters: $\lambda = 1.5$ mm, $R = 5$ mm. Apparently, it is precisely in such experiments that the decrease of the energy production during the emission is most likely to be detected. We can, on the basis of the results obtained by us, give a rough estimate for the energy that will remain in the medium in the form of a localized excitation: $E' \sim E_0\lambda/5R$, where E_0 is the total pump energy. Finally, let us point out the definite analogy between the above-obtained solution and the solution given in Yuen and Lake's paper in Ref. 12 for a solitary wave on the surface of deep water, and characterized by a periodic (in space and time) variation of its shape. This similarity indicates the advisability of carrying out a detailed study of Eq. (11), which, however, is outside the scope of the present paper.

IV. THE CASE OF A PERIODIC STRUCTURE

1. *The exact solutions to Eq.* (10). Let us consider the case $L = \lambda n$, where L is the lattice constant, λ is the wavelength of the radiation field, and n is an integer. In this case, as is easy to see, Eq. (10) does not depend on the specific value of n; therefore, let us set $n = 0$. The spatial dependence then disappears from Eq. (10):

$$\frac{1}{\cos\theta_m}\frac{d}{d\tau}(\sin\theta_m \exp(i\varphi_m)) = -\sum_k \sin\theta_k \exp(i\varphi_k). \tag{13}$$

Let us introduce the following collective functions:

$$E = -\frac{1}{2}\sum_k \cos\theta_k, \quad P = \sum_k \sin\theta_k \exp(i\varphi_k).$$

The three-dimensional vector $\mathbf{S} = \{2E, \operatorname{Re}P, \operatorname{Im}P\}$ is the total-energy-spin vector of the system of radiators, its modulus being conserved in the present case. Indeed, from (13) we obtain for the functions E and P the system of equations

$$dE/d\tau = -|P|^2/2, \quad dP/d\tau = 2EP, \tag{14}$$

from which follows the vector-length conservation law:

$$\frac{d}{d\tau}[(2E)^2 + |P|^2] = 0.$$

We also obtain from the system (14) an equation for the energy of the system:

$$d^2E/d\tau^2 = 2dE^2/d\tau,$$

the solution to which has the form

$$E(\tau) = \frac{N}{2}\operatorname{th}[N(\tau_0 - \tau)], \quad \tau_0 = \operatorname{arcth}\left(1 - \frac{2}{N}\right). \tag{15}$$

Here we have used the fact that $\theta_0 = 2/N^{1/2}$ and that $N \gg 1$. It can be shown that the radiation intensity is connected with the energy E of the system of radiators by the simple relation $I = -dE/dt$, which means that the total energy inside the sample is made up of only the radiator energy, and does not include the energy of the field. This circumstance is due to the assumption that $c = \infty$. Thus, we obtain from (15) an expression for the radiation intensity:

$$I(\tau) = N^2 \frac{\pi n \omega d^2}{\hbar c} \frac{1}{\operatorname{ch}[N(\tau - \tau_0)]}. \tag{16}$$

This solution is similar to the solution obtained in the Dicke model for systems with dimensions much smaller than the wavelength of the radiation field.[2]

The second case allowing the exact analytical solution of Eq. (10) corresponds to the condition $L = \lambda/2 + \lambda n$, where n is a whole number. Making the change of variables $\theta_k \to \tilde{\theta}_k$, $\varphi_k \to (-1)^k \tilde{\varphi}_k$, we obtain Eq. (10) for the new variables $\tilde{\theta}_k$ and $\tilde{\varphi}_k$. Thus, when the Bragg conditions are fulfilled, Eq. (10) admits of analytic solutions, and the superradiance effect manifests itself most characteristically: as will be established below, the collective-decay time for the system is shortest under this condition, while the radiation intensity is highest, and is described by the normal superradiant emission law (16). Let us note that the apparent disagreement with Ref. 7 in the dependence of the maximum intensity on the number of radiators (here this intensity is $\sim N^2$, while in Ref. 7 it is $\sim N^{4/3}$) is due to the one dimensionality of our problem.

The property of a Bragg crystal to conserve the total energy spin vector has an interesting consequence. Let us consider a system in which only one radiator is initially excited. Then the evolution of the state of this radiator is described by the following equation, which follows from (13):

$$d\theta/d\tau = -\sin\theta - (N-1)\sin(\theta - \theta_0), \tag{17}$$

where θ_0 is the initial angle of the excited radiator. As can be seen from (17), the rate of its decay at zero time coincides with the rate of spontaneous decay of an isolated radiator. But the process of emission from the crystal stops after a time period of the order of $\tau_c = \tau/N$, where τ is the independent decay time, and the initially excited radiator loses in the course of the reconstruction of the system into a collective state energy roughly equal to $(\sin^2\theta_0)/2N$, after which it "freezes" in its state of excitation. Such behavior of the process contradicts the results obtained in Ref. 5, where the increase of the decay rate under Bragg-diffraction conditions is predicted. The nonlinear treatment shows that the radiation emitted by the excited radiator is screened off in the crystal, and that the system goes over into the collective state, which, in turn, decays apparently at the independent decay rate. The latter assertion follows from the fact that the energy spin vector, as the quantum treatment shows, is conserved only over time periods significantly shorter than the spontaneous decay time. It is possible though that such a collective state will turn out to be stable in the exact quantum treatment of the present problem.

2. *Small deviations from the Bragg condition.* Let $L = \lambda n + \delta L$, where $\delta L \ll \lambda$. As has already been noted, the value of the integral parameter n does not affect the solution to Eq. (10); therefore, let us, as before, choose $n = 0$. Let us now note that the radiator spacing

is then significantly smaller than the wavelength of the radiation, and the case under consideration is essentailly the case of a continuous medium whose length is equal to $R=\delta L(N-1)$, where N is the number of radiators. For a quasi-Bragg crystal this quantity is precisely the important parameter determining the course of the collective decay. The numerical computer analysis shows that the solution for a crystalline structure coincides to within 1% with the solution for a continuous medium of equivalent length $R=\delta L(N-1)$ over the entire duration of the decay process provided $\delta L/\lambda<0.025$. We specially investigated with the aid of a computer the possibility that a continuous sample with a regular disposition of the radiators and a continuous sample in which the disposition of the radiators is random will behave differently, but no differences were found within the limits of the indicated error. Thus, we in fact have solutions, shown in Figs. 2 and 3, for crystalline structures. It is only necessary to remember that the length of the continuous sample in the case of a crystal corresponds to the quantity $R=\delta L(N-1)$, the actual length of the crystalline system being $R_{cr}=L(N-1)\gg\lambda$. Another important difference between a crystalline structure and a continuous medium is the fact that, if in the continuous medium the dimension of the localized excitation $\mathscr{L}\sim\lambda/5$, in a crystal the dimension of this localization can significantly exceed the radiation wavelength, i.e., $\mathscr{L}\sim 1/5\lambda L/\delta L\gg\lambda$. In other words, the localized excitation in a crystal has macroscopic dimensions and, consequently, a relatively large amount of energy that is not released from the crystal in the form of a radiation pulse.

3. *Arbitrary deviations from the Bragg condition*. Let us note that the basic equation (10) for the case of a crystalline structure is invariant under the following transformation:

$$\{\theta_k\to\theta_k;\ \varphi_k\to-\varphi_k;\ L\to-L\},$$

which can be verified directly. From this it follows, in particular, that the deviations $+\delta L$ and $-\delta L$ from the Bragg diffraction conditions are equivalent. In this sense the criterion for maximum deviation from the optimum (Bragg) conditions for superradiance can be represented in the form $L=\lambda(\frac{1}{4}+n/2)$. Let us note that, when a traveling wave propagates through such a lattice, the radiation emitted in the backward direction by neighboring radiators cancel each other because of a phase difference equal to π; therefore, the intensity of the backward wave is significantly lower, and the problem can be considered in a single-wave approximation. For the purpose of illustrating the foregoing, we present Fig. 1c. which shows the results of the numerical solution of the exact two-wave problem in the indicated lattice in the case in which coherent initial conditions are imposed on the phase function: $\varphi_m=k_0Lm$. Under these conditions the radiation wave traveling to the right predominates at the initial moment of time. As can be seen from Fig. 1c, the behavior of such a system is almost identical to the behavior of a continuous medium in the single-wave theory (see Ref. 3 and Fig. 1a). In Fig. 3 we have inserted for comparison with the Bragg case the variation in time of the energy in the structure under consideration. This comparison shows that, in spite of the fact that the spontaneous decay is speeded up in both cases, on the whole the superradiance effect is more strongly pronounced in the Bragg case. The present conclusion differs from the conclusion drawn in Ref. 7 on the basis of the result obtained in first order perturbation theory, which is that these two cases are identical.

V. SUMMARY

1. The phenomenon of collective spontaneous emission in one-dimensional continuous and discrete periodic systems of radiators has been theoretically investigated within the framework of the semiclassical approach and in the dipole electromagnetic field—atom (nucleus) interaction approximation. The absorption of the radiation in the sample, the inhomogeneous line broadening, and the lattice dynamics were neglected in the analysis. Basic equations, (10) and (11), containing only the state functions of the atoms (nuclei) have been obtained for the collective decay process. The indicated equations take account of the two opposite radiation-field waves, and are valid in the presence of arbitrary, and not just long-scaled, radiation-field and atomic-inversion inhomogeneities inside the active medium. The proposed method coincides with the method given in Ref. 3 in the long-scaled-inhomogeneity and single-wave limit.

2. The problem of superradiant emission by a continuous sample of dimension of the order of the wavelength of the radiation field is solved on the basis of the proposed method. It is shown that, as the sample length R increases from $R\ll\lambda$ to $R\sim\lambda$, the single-pulse regime of superradiance goes over into the multipulse regime, which is accompanied by the formation of a localized collective medium excitation distinguished by a relatively small scale $\sim\lambda/5$ and a complicated character of its evolution, during which the excitation energy is gradually dissipated.

3. In the semiclassical approach the transition from the single-pulse superradiance regime to the multipulse regime is determined by the parameter R/λ, where R is the length of the active medium and λ is the wavelength of the radiation field: when $R/\lambda\ll 1$ the single-pulse emission regime obtains; when $R/\lambda\sim 1$, the multipulse emission regime; and when $R/\lambda\gg 1$, the oscillatory emission regime. The analogous parameter in the quantum-mode treatment[2] is the quantity τ/τ_c, where τ is the time of flight of a photon through the medium and τ_c is the duration of the collective process. In our treatment $\tau=0$, since $c=\infty$, but the emission regime is not the single-pulse regime. This inconsistency indicates that it is necessary to seek a more satisfactory approach to the description of the spatial evolution of the superradiant emission process, and this approach should possess the advantages of both methods.

4. A series of spontaneous-emission problems for one-dimensional periodic structures are solved. It is established that the optimum conditions for superradiance are the Bragg-diffraction conditions: $L=\lambda(n$

$+1)/2$, where n is a whole number, L is the lattice constant, and λ is the radiation wavelength. Exact analytic solutions are obtained for this case. Under these conditions the radiation-pulse length is shortest and proportional to N^{-1}, while the maximum radiation intensity is proportional to N^2, where N is the number of radiators. The superradiance of near-Bragg crystals, i.e., crystals satisfying the condition $\delta L \ll \lambda$, is accompanied by the local excitation of a crystal of dimension $\sim 1/5\lambda L/\delta L$. The energy of such an excitation formation is approximately equal to $E_0\lambda/5N\delta L$, where E_0 is the nuclear-transition energy.

5. The problem of the decay of a single excited radiator in an unexcited Bragg crystal is solved. The energy emitted in such a decay is equal to E_0/N, i.e., almost all the energy remains inside the system. Hence it can be inferred that the spontaneous radiation emitted by an excited nucleus in a three-dimensional crystal can be highly anisotropic; for the diffraction condition may be fulfilled in one of the directions in the crystal. Let us emphasize that, for a completely excited system, this direction will, on the contrary, be the direction of maximum intensity. These questions are currently being investigated.

6. In view of the fact that even a small deformation, $\delta R = N\delta L \sim \lambda$, of the Bragg crystal leads to a significant change in the picture of the collective process, there arises the question of the role in this process of the thermal and quantum fluctuations of the positions of the nuclei.

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[1)]Reference 13, which is devoted to the quantum theory of self-induced transparency, contains a soliton solution for the pulse shape, but because of an incorrect formulation of the problem, the soliton turns out to be inverted, a fact which was not noticed by the authors of the paper. Without going into a detailed analysis of the paper, we point out that the incorrectness of the paper is indicated by the fact that the equations obtained there do not possess solutions having zero boundary conditions at infinity, i.e., describing normal self-induced transparency pulses.

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